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# Bounds on Dominating Energy of Graph

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**Abstract:**
The Minimum Dominating Energy (MDE) of a graph$\mathfrak{A}$, denoted by $E_{M\,D}$ ($\mathfrak{A}$), is nothing but the sum of absolute values of all minimum dominating eigenvalues of $\mathfrak{A}$. In this study, few upper and lower constraints on the minimum dominating energy are obtained.

**Keywords:** Minimum dominating matrix, minimum dominating eigenvalues, minimum dominating energy.

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## 1. Introduction

Let $\mathfrak{E}$ be the edge set and $\mathfrak{V}$, set of vertices of a simple graph $\mathfrak{A}$. Let $|\mathfrak{E}| = m$ and $|\mathfrak{V}| = n$.

$$\mathcal{E}(\mathfrak{A}) := \sum_{k=1}^{n} |\mathcal{V}_k|$$

where $\mathcal{V}_k,\ k = 1, 2, 3, \dots, n$ are the eigenvalues (characteristic roots) of the Adjacency matrix ($AM$) of a graph $\mathfrak{A}$, Ivan Gutman [12] conducted this study on $\mathfrak{A}$ for the first time in 1978, and named it as Energy of a Graph $\mathfrak{A}$, from then numerous research has been conducted on $AM$, with inspiration drawn by this, different matrix types for a graph $\mathfrak{A}$ [18, 17, 13, 2] are defined and studied. For basic mathematical properties of the theory of graph energy including its upper and lower bounds one can see [ 20, 21, 22]. Erich Huckle [3], employed the energy of graphs technique in the early 1930s to develop approximations solutions for a family of organic molecules known as conjugated hydro carbons.

Let $\mathcal{D} \subseteq \mathfrak{V}$ ($\mathfrak{A}$), if every vertex of $\mathfrak{V} - \mathcal{D}$ is adjacent to some vertex in $\mathcal{D}$, then $\mathcal{D}$ is referred to as a dominating set of $\mathfrak{A}$. A minimum dominating set (MDS) $\mathcal{D}$ of $\mathfrak{A}$ is a dominating set of $\mathfrak{A}$ with minimum cardinality.

Let $\mathcal{D}$ be a MDS of $\mathfrak{A}$. The following kind of matrix, known as the minimum dominating matrix (MDM) of $\mathfrak{A}$, introduced by M.R.Rajesh Kanna et.al. in [22]: The $n \times n$ matrix $M_{\mathcal{D}}(\mathfrak{A}) = [d_{k\,j}]$, is the MDM of $\mathfrak{A}$, whose $kj - th$ element is given by

$$d_{kj} = \begin{cases} 1, & if\ k = j\ and\ v_j\ \ in\, \mathcal{D}; \\ 1, & if\ v_k\ and\ v_j\ \ are\ adjacent; \\ 0, & otherwise. \end{cases}$$

$\Gamma(\mathfrak{A} : \chi) = det(\chi I - M_{\mathcal{D}}(\mathfrak{A}))$ is the characteristic polynomial of $M_{\mathcal{D}}(\mathfrak{A})$. The minimum dominating eigenvalues of $\mathfrak{A}$ are the eigenvalues $\chi_1, \chi_2, \ldots, \chi_n$ of $M_{\mathcal{D}}(\mathfrak{A})$. The matrix $M_{\mathcal{D}}(\mathfrak{A})$ is real as well as symmetric. The real numbers that make up the eigenvalues of $M_{\mathcal{D}}(\mathfrak{A})$ are arranged to be: $\chi_1 \geq \chi_2 \geq \ldots \geq \chi_n$. The formula

$$\mathrm{E}_{M\mathcal{D}}(\mathfrak{A}) = \sum_{k=1}^{n} |\chi_k|$$

defines $\mathfrak{A}'s$ minimum dominating energy (MDE).

Note that $M_{\mathcal{D}}(\mathfrak{A})$ has trace =Domination number = d, and $\sum_{k=1}^{n} \chi_k^2 = 2|\mathfrak{E}| + |\mathcal{D}| = 2m + |d|$

We derive some upper and lower bounds for the MDE, $\mathrm{E}_{M\mathcal{D}}(\mathfrak{A})$, in this study.

## 2. Upper Bounds for MDE

Throughout this series $\mathfrak{A}$ denotes a simple graph. This section is aimed to discuss upper bounds for MDE of $\mathfrak{A}$.

**Theorem 2.1** Let $\mathfrak{A}$ be graph of order $n$ and size $m$. Then

$$\mathrm{E}_{M\mathcal{D}}(\mathfrak{A}) \leq \sqrt{\left(\frac{1}{2}(n^2 + |d|^2) + 2m(m + |d|)\right)}$$

**Proof:** We recall the following well-known inequality from [11]:

$$\left(\sum_{k=1}^{n} p_k e_k^2\right)\left(\sum_{k=1}^{n} q_k f_k^2\right) + \left(\sum_{k=1}^{n} p_k g_k^2\right)\left(\sum_{k=1}^{n} q_k h_k^2\right) \geq 2\left(\sum_{k=1}^{n} p_k e_k g_k\right)\left(\sum_{k=1}^{n} q_k f_k h_k\right) \qquad (2.1)$$

where $e_k, f_k, g_k$ *and* $h_k$ denote sequence of real numbers; $p_k$ *and* $q_k$ denote non-negative numbers for $1 \leq k \leq n$.

For $p_k = q_k = e_k = f_k = 1$ and $g_k = h_k = |\chi_k|$, $1 \leq k \leq n$, the inequality (2.1) reduces to

$$\left(\sum_{k=1}^{n} 1\right)\left(\sum_{k=1}^{n} 1\right) + \left(\sum_{k=1}^{n} |\chi_k|^2\right)\left(\sum_{k=1}^{n} |\chi_k|^2\right) \geq 2\left(\sum_{k=1}^{n} |\chi_k|\right)\left(\sum_{k=1}^{n} |\chi_k|\right).$$

Using,

$$\sum_{k=1}^{n} |\chi_k|^2 = \sum_{k=1}^{n} \chi_k^2 = 2m + |d|$$

in the above inequality, we deduce that

$$n.n + (|d| + 2m)(|d| + 2m) \geq 2.\mathrm{E}_{M\mathcal{D}}(\mathfrak{A}).\mathrm{E}_{M\mathcal{D}}(\mathfrak{A})$$

which gives

$$2.\mathrm{E}_{M\mathcal{D}}(\mathfrak{A})^2 \leq n^2 + (|d| + 2m)^2.$$

Hence,

$$\mathrm{E}_{M\mathcal{D}}(\mathfrak{A}) \leq \sqrt{\left(\frac{1}{2}(n^2 + |d|^2) + 2m(m + |d|)\right)}.$$

**Theorem 2.2** Let $\mathfrak{A}$ be graph of order $n$ and size $m$. Then

$$\mathrm{E}_{M\mathcal{D}}(\mathfrak{A}) \leq m + \frac{1}{2}(n + |d|).$$

**Proof:** We recall the following well-known inequality from [11]:

$$\left(\sum_{k=1}^{n} p_k e_k^2\right)\left(\sum_{k=1}^{n} q_k f_k^2\right) + \left(\sum_{k=1}^{n} p_k g_k^2\right)\left(\sum_{k=1}^{n} q_k h_k^2\right) \geq 2\left(\sum_{k=1}^{n} p_k e_k g_k\right)\left(\sum_{k=1}^{n} q_k f_k h_k\right) \qquad (2.2)$$

where $e_k,\ f_k,\ g_k\ and\ h_k$denote sequence of real numbers; $p_k\ \ and\ q_k$ denote non-negative numbers for $1 \leq k \leq n$.

For $p_k\ =\ q_k = e_k = f_k = h_k = 1$ and $g_k = |\chi_k|$, $1 \leq k \leq n$, the inequality (2.2) yields

$$\left(\sum_{k=1}^{n} 1\right)\left(\sum_{k=1}^{n} 1\right) + \left(\sum_{k=1}^{n} |\chi_k|^2\right)\left(\sum_{k=1}^{n} 1\right) \geq 2\left(\sum_{k=1}^{n} |\chi_k|\right)\left(\sum_{k=1}^{n} 1\right).$$

That is,

$$n^2 + \left(\sum_{k=1}^{n} |\chi_k|^2\right) n \geq 2n.\left(\sum_{k=1}^{n} |\chi_k|\right)$$

Which gives,

$$n + 2m + |d| \geq 2\mathrm{E}_{M\mathcal{D}}(\mathfrak{A}).$$

Hence,

$$\mathrm{E}_{M\mathcal{D}}(\mathfrak{A}) \leq m + \frac{1}{2}(n + |d|).$$

## 3. Lower Bounds for MDE

Throughout this section $\mathfrak{A}$ denotes a simple graph. This section is aimed to discuss lower bounds for MDE of $\mathfrak{A}$.

**Theorem 3.1** Let $\mathfrak{A}$ be a bipartite graph of order $n \geq 2$ and size $m$ with spectral radius $\chi_1$. Then

$$\frac{|d| + 2m}{\chi_1} \leq \mathrm{E}_{M\mathcal{D}}(\mathfrak{A}).$$

**Proof:**Let $e_k,\ f_k$ be non-negative decreasing sequences where $e_k,\ f_k \neq 0$, and $j_k$ be a non-negative sequence for $1 \leq k \leq n$. Then we have the following inequality [11]:

$$\left(\sum_{k=1}^{n} j_k e_k^2\right)\left(\sum_{k=1}^{n} j_k f_k^2\right) \leq \max\left\{f_1 \sum_{k=1}^{n} j_k e_k^2,\ \ e_1 \sum_{k=1}^{n} j_k f_k^2\right\}\left(\sum_{k=1}^{n} j_k e_k f_k\right) \qquad (3.1)$$

For $e_k = f_k = |\chi_k|$ and $j_k = 1$, $1 \leq k \leq n$, the inequality (3.1) gives,

$$\left(\sum_{k=1}^{n} 1.|\chi_k|^2\right)\left(\sum_{k=1}^{n} 1.|\chi_k|^2\right) \leq \max\left\{\chi_1 \sum_{k=1}^{n}|\chi_k|, \chi_1 \sum_{k=1}^{n}|\chi_k|\right\}\left(\sum_{k=1}^{n}|\chi_k|^2\right)$$

That is,

$$\sum_{k=1}^{n}|\chi_k|^2 \leq \chi_1 \sum_{k=1}^{n}|\chi_k|$$

Which implies

$$\chi_1 \mathrm{E}_{M\mathcal{D}}(\mathfrak{A}) \geq \sum_{k=1}^{n}|\chi_k|^2$$

Hence,

$$\frac{|d| + 2m}{\chi_1} \leq \mathrm{E}_{M\mathcal{D}}(\mathfrak{A})$$

**Lemma 1** [22] Let $n$ be a positive integer. If $l_1, l_2, \dots, l_n$ are non-negative numbers with $l_1 \geq l_2 \geq \cdots \geq l_n$, then

$$(l_1 + l_2 + \dots + l_n)(l_1 + l_n) \geq l_1^2 + \cdots + l_n^2 + n l_1 l_n \qquad (3.2)$$

Further, equality holds in (3.2) if and only if for some $r$, $1 \leq r \leq n$, $l_1 = \cdots = l_r$ and $l_{r+1} = \cdots = l_n$.

**Theorem 3.2** Let $\mathfrak{A}$ be a graph with order $n \geq 2$ and size $m \geq 1$. Assume that $\chi_1, \dots, \chi_n$ are all eigenvalues of $\mathfrak{A}$, such that $|\chi_n| \geq \cdots \geq |\chi_1| \geq 0$, then

$$\mathrm{E}_{M\mathcal{D}}(\mathfrak{A}) \geq \frac{2\sqrt{(2m + |d|)n}\sqrt{|\chi_1 \chi_n|}}{|\chi_1| + |\chi_n|}$$

**Proof:** Since there is at least one edge in the graph $\mathfrak{A}$, it follows that $\mathfrak{A}$ has at least one eigenvalue different from zero. Applying Lemma 1, we get

$$(|\chi_1| + \cdots + |\chi_n|)(|\chi_1| + |\chi_n|) \geq |\chi_1|^2 + \cdots + |\chi_n|^2 + n|\chi_1||\chi_n| \qquad (3.3)$$

and equality holds in (3.3) if and only if $|\chi_1| = \cdots = |\chi_r|$ and $|\chi_{r+1}| = \cdots = |\chi_n|$ for some$r \in 1, \cdots, n$ since $|\chi_1|^2 + \cdots + |\chi_n|^2 = 2m + |d|$. By equation (3.3) we get,

$$\mathrm{E}_{M\mathcal{D}}(\mathfrak{A})(|\chi_1| + |\chi_n|) \geq 2m + |d| + n|\chi_1||\chi_n|$$

$$\mathrm{E}_{M\mathcal{D}}(\mathfrak{A}) \geq \frac{2m + |d| + n|\chi_1||\chi_n|}{|\chi_1| + |\chi_n|} \qquad (3.4)$$

and the equality holds if and only if $|\chi_1| = \cdots = |\chi_r|$ and $|\chi_{r+1}| = \cdots = |\chi_n|$ for some $r \in 1, \cdots, n$. We all know that for every real number $a \geq 0$ and $b \geq 0$,

$$a + b \geq 2\sqrt{ab}$$

and equality holds if and only if $a = b$. Using this fact in (3.4), we get

$$\mathrm{E}_{M\mathcal{D}}(\mathfrak{A}) \geq \frac{(2m+|d|)+n|\chi_1||\chi_n|}{|\chi_1|+|\chi_n|}$$

$$\geq \frac{2\sqrt{(2m+|d|)n|\chi_1\chi_n|}}{|\chi_1|+|\chi_n|}$$

$$= \frac{2\sqrt{(2m+|d|)n}\sqrt{|\chi_1\chi_n|}}{|\chi_1|+|\chi_n|}$$

This proves the result.

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