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Extension and Generalization of Banach Contraction in Metric and in Menger Space

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Abstract:

The root of metric fixed point theory is Stefen Banach's contraction mapping, a research source for shrinking the distance between two points in space. As a source, many authors have introduced many contraction mappings as extensions and generalizations of Banach contraction and established fixed point theorems under the property that each such mapping in complete metric and Menger space has a unique fixed point.

This article presents updated results of Banach contraction generalization and extension forms in metric and Menger space which helps the comparative and interrelationship study in these spaces.

Keywords: Banach contraction, distribution function, Triangular norm and Menger space**.**

1. Introduction

In the nineteenth century, existence theorems emerged in analysis when basic mathematical facts were considered critically. The first mathematician to demonstrate the existence of a theorem for differential equation systems with analytic right-hand sides was A. L. Cauchy [9]. E. Picard [43] suggested the method of successive approximations to prove the existence of the theorems. In 1922, Birkhoff and Kellogg[4] gave proof of the classical existence theorem for the equation $\frac{dy}{dx} = f(x, y)$ in function spaces. However, the most elementary and by far the most fruitful method for proving theorems on the existence and uniqueness of solutions is the principle formulated and proved by S. Banach [1]1920 in his PhD thesis published in 1922. Although the idea of successive approximations in some concrete situations (solving differential and integral equations) appears in some works of E. Picard, R. Caccioppoli, et al. [10], it was Banach who placed it in the right abstract setting, making it suitable for a wide range of applications. This principle results from the geometric interpretation of Picard's method of successive approximations. Several generalizations of this principle have appeared and many authors have done their comparative study see references [[35], [43], [44], [48]] in metric space.

Menger [38] introduced the probabilistic metric space in 1942 to generalize Frechet's [25] metric space by replacing the distance function with the distribution function. It helps to solve uncertainty cases regarding the distance between two points in space. This probabilistic metric space became active for mathematicians after the great contribution in this space from Schweizer and A. Sklar. [49]

In 1966, Sehgal [50] introduced the first generalized form of Banach contraction in probabilistic metric space and established the fixed point theorem in complete Menger probabilistic metric space [51], and Hicks [32] defined another contraction in 1983. After that many authors worked in this space and introduced different variants of contraction in this space, see references [[13 – 17], [20], [26], [28], [39], [41]].

In this paper, we shall update the study of various general contractive results to the Banach contraction principle. Each mapping has a property to establish unique fixed points in complete metric and Menger space.

2. Preliminaries:

Definition 2.1 [25]: Let X be an abstract set and d be a distance function from $X \times X \rightarrow \mathbb{R}^{+}$. Then, an ordered pair (X, d) is said to be metric space if d satisfies the following conditions for all $x, y, z \in X$:

(i) $d(x, y) \geq 0$;
(ii) $d(x, y) = 0$ if and only if $x = y$;
(iii) $d(x, y) = d(y, x)$; and
(iv) $d(x, y) \leq d(x, z) + d(z, y)$

Here, (i) and (ii) guarantee that the distance between any two points of X is always positive and only zero when the points coincide. (iii) assures that the order of measurement of the distance between two points is insignificant. (iv) is a statement of the familiar triangular inequality.

Definition 2.2 [15]: A mapping F: $\mathbb{R} \rightarrow \mathbb{R}^{+}$ is said to be a ***distribution function*** if it is non-decreasing and left continuous with $inf_{x \in \mathbb{R}} F(x) = 0$, and $sup_{x \in \mathbb{R}} F(x) = 1$, where $\mathbb{R}^{+}$ denotes the set of non-negative real numbers.

Definition 2.3 [15]: Let X be a non-empty set and $F: X \times X \rightarrow L$ (set of all distribution functions) be a distribution function. Then, a pair (X, F) is said to be a ***Probabilistic metric space*** (abbreviated as ***Pm-space***) if the distribution function $F(x, y)$, also denoted by $F_{x,y}$ satisfies the following conditions:

(i) $F_{x,y}(t) = 1$ for every $t > 0$ if and only if $x = y$;

(ii) $F_{x,y}(0) = 0$ for every $x, y \in X$;

(iii) $F_{x,y}(t) = F_{y,x}(t)$ for every $x, y \in X$; and

(iv) $F_{x,z}(p + q) = 1$ if and only if $F_{x,y}(p) = 1$ and $F_{y,z}(q) = 1$.

Example 2.1: Let (X, d) be metric space where $X = [0, 5]$with usual metric $d(x, y) = |x - y|$ and distribution function F defined as:

$$F_{x,y}(t) = \begin{cases} e^{\frac{|x-y|}{t}}, & if\ t > 0, \\ 0, & if\ t = 0. \end{cases}$$ for all $x, y \in X$. Then, (X, F) be Pm space.

Definition 2.4 [28]: A function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is referred to as a ***Triangular norm***

(shortly T-norm) if it satisfies the following conditions:

T_1: $T\ (0,0) = 0$;

T_2: $T\ (a,1) = a$ for all $a \in [0,1]$;

T_3: $T\ (a,b) = t\ (b,a)$ for all $a,b \in [0,1]$;

T_4: $if\ a \le c, b \le d$ then $T\ (a,b) \le T(c,d)$ and

T_5: $T\ (t\ (a,b),c) = T\left(a,t\ (b,c)\right)$, where $a,b,c,d \in [0,1]$.

Definition 2.5. [17] A triplet (X,F,T) is said to be Menger space, where X is a non-empty set, F be a distribution function, and T is a t-norm such that the following are satisfied

for every $t,s > 0$ & $x,y,z \in X$:

(i) $F_{x,y}(t) = 1$ for every $x > 0$ if and only if $x = y$,

(ii) $F_{x,y}(0) = 0$;

(iii) $F_{x,y}(t) = F_{y,x}(t)$, and

(iv) $F_{x,z}(t+s) \ge T(F_{x,y}(t), F_{y,z}(s))$.

Definition 2.6.[36] Let $T: X \to X$ be a mapping in metric space (X,d). Given $x \in X$, $O(x) = \{f^n : n \in N\}, \bar{O}(x)$ be its closure. A point $x \in X$ is said to be regular for T if $diam\ O(x) < \infty$.

Given, $x,y \in X$, let $m(x,y) = max\{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\}$,

And $\delta(x,y) = diam\{O(x) \cup O(y)\}$.

Definition 2.7.[1] Let (X,d) be a metric space. Then, a mapping $T: X \to X$ is said to be contraction mapping if there exists a number $\lambda \in [0,1)$ such that for every $x,y \in X$, we have

$$d(Tx,Ty) \le \lambda\ d(x,y).$$ **(Contraction, in 1922)**

Example 2.2 [16]: Let function $f: [0,2] \to [0,2]$ be defined by

$$f(x) = \begin{matrix} 0 \text{ for } x \in [0,1] \\ 1 \text{ for } x \in (1,2] \end{matrix}$$

Then, f^2 is a contraction but f is not a contraction.

3. Various generalizations and extensions of Banach contraction are:

Definition 3.1. A mapping $T: X \to X$ of a metric space (X,d) is said to be contractive if

$$d(Tx,Ty) < d(x,y), \text{ for every } x \ne y \in X.$$ (**Edelstein in 1962, [24]**)

It is extended forms as:

(i) $d(Tx,Ty) < m(x,y)$ **(Rhodes in 1977, [48])**

(ii) If x and y are regular, $d(Tx,Ty) < \delta(x,y)$. **(Park in 1980, [43])**

Remark 1. A contractive map is continuous; if such a mapping has a fixed point, then this fixed point is unique. While the condition $d(Tx, Ty) < d(x, y)$ is sufficient to assure that T has a fixed point but it is too weak to guarantee the existence of one as will be seen from the following examples:

Example 3.1. Let $T: \mathbb{R} \to \mathbb{R}$ be defined by $T(x) = x + \frac{\pi}{2} - Tan^{-1}x$

Since $Tan^{-1}x < \frac{\pi}{2}$ for every x, the operator T has no fixed point although T is a contractive map for $T'(x) = 1 - \frac{1}{1+x^2} < 1$.

Example 3.2. Let $T: \mathbb{R} \to \mathbb{R}$ be defined by $T(x) = \ln(1 + e^x)$,

Differentiating, we obtain

$$T'(x) = \frac{e^x}{1+e^x} < 1.$$

i.e. T is a Contractive mapping, and it is easy to see that T has no fixed point.

Definition 3.2. Let $T: X \to X$ be a mapping of a metric space (X, d). And $\alpha(= \alpha[\, or\ (x, y) = \alpha d(x, y)]: [0, \infty) \to [0,1]$ be a monotone decreasing function. Then, T is a Rakotch contractive mapping if, for every $x \neq y \in X$, we have $d(Tx, Ty) \leq \alpha d(x, y)$. **(Rakotch in 1962, [45])**

Definition 3.3. A mapping $T: X \to X$ in metric space (X, d) is said to be Caristi contractions if there is $\emptyset: X \to [0, \infty)$ lower semi-continuous function such that $d(x, Tx) \leq \emptyset(x) - T(x)$.

(Caristi in 1966, [8])

Note that every Banach contraction T satisfies Caristi contractions if for some $\lambda \in [0,1)$,

$$\emptyset(x) = \frac{d(x, Tx)}{1 - \lambda}.$$

Definition 3.4. Let $T: X \to X$ be a contractive mapping of a metric space (X, d). Then, T is said to be Kanan contraction if there exists $\alpha \in (0, \frac{1}{2})$ such that, we have

$d(Tx, Ty) \leq \alpha[d(x, Tx) + (y, Ty)$, for every $x, y \in X$ **(Kanan in 1968, [37])**

The following example shows that T is not continuous but T is a Kanan contraction at $\alpha = \frac{1}{5}$:

Example 3.3: Let $X = \mathbb{R}$ be a usual metric and mapping $T: X \to X$ be defined by

$$T(x) = \begin{cases} 0, & for\ x \in (-\infty,\ 2] \\ \frac{1}{2}, & for\ x \in (2,\ +\infty). \end{cases}$$

Bianchini extends the Kanan contractions as:

Definition 3.5. A mapping $T: X \to X$ in metric space (X, d) is said to be Bianchini contractions if $d(Tx, Ty) \leq r\ max\{(d(x,\ Tx), d(y,\ Ty)\}$ for all $x, y \in X$, and $r \in [0,1)$.

(Bianchini in 1972, [6])

Definition 3.6. A mapping $T: X \to X$ in metric space (X, d) is said to be Reich contractions if $d(Tx, Ty) \leq a\, d(x, Tx) + b\, d(y, Ty) + c\, d(x, y)$, where a, b, c are nonnegative and satisfy $a + b + c < 1$ and for all $x, y \in X$. **(Reich in 1971, [46])**

Note that $a = b = 0$ yields Banach's fixed point theorem, while $a = b, c = 0$ yields Kannan's theorem.

Definition 3.7. Let $T: X \to X$ be a contractive mapping of a metric space (X, d). Then, T is said to be Chatterjee contraction if there exists $\alpha \in (0, \frac{1}{2})$ such that, we have

$d(Tx, Ty) \leq \alpha[d(x, Ty) + (y, Tx)$, for every $x, y \in X$ **(Chatterjee in 1972, [12])**

Definition 3.8. Let $T: X \to X$ be a contractive mapping of a metric space (X, d). Then, T is said to be T. Zamfirescu contraction if there exist real numbers a, b, c satisfying $0 \leq a < 1, 0 \leq b, c < \frac{1}{2}$, such that for every $x, y \in X$, at least one of the following is true: **(Zamfirescu in 1972, [57])**

(i) $d(Tx, Ty) \leq a[d(x, y)$;
(ii) $d(Tx, Ty) \leq b[d(x, Tx) + (y, Ty)]$;
(iii) $d(Tx, Ty) \leq c[d(x, Ty) + (y, Tx)$.

Definition 3.9. A mapping $T: X \to X$ in metric space (X, d) is said to be Sehgal contractions if, for every $x, y \in X, x \neq y$, we have

$d(Tx, Ty) < \max\{d(x, Tx), d(y, Ty), d(x, y)\}$ **(Sehgal in 1972, [52])**

Definition 3.10. A mapping $T: X \to X$ in metric space (X, d) is said to be Rhodes contractions if, for each $x, y \in X, x \neq y$,

(i) $d(Tx, Ty) < max\{(d(x, Ty), d(y, Tx)\}$. **(Rhodes in 1977, [48])**
(ii) $d(Tx, Ty) \leq max\{(d(x, Ty), d(y, Tx), d(x, y)\}$. **(Rhodes in 1977, [48])**
(iii) $d(Tx, Ty) \leq max\{(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$

(Rhodes in 1977, [48])

Definition 3.11. A mapping $T: X \to X$ in metric space (X, d) is said to be Ciric contractions if there exists a non-negative number $q(x, y), r(x, y), s(x, y), and\ t(x, y)$ such that

$$\sup_{x,y \in X}\{q(x, y) + r(x, y), +s(x, y) + 2\, t(x, y)\} < 1$$

And $d(Tx, Ty) \leq \mathrm{q(x, y)\ d(x, y) + r(x, y)\ d(x, Tx) + s(x, y)\ d(y, Ty) + t(x, y)\ d(x, Ty)} + d(y, Tx)$

for every $x, y \in X$. **(Ciric in 1971, [19])**

Definition 3.12. A mapping $T: X \to X$ in metric space (X, d) is said to be Hardy and Roger contractions if there exists a, b, c, e, f monotonically decreasing function from $[0, \infty)\ to\ (0,1)$, and

$a + b + c + e + f < 1$ such that

$d(Tx, Ty) \leq$a d(x, Tx)+b d(y, Ty)+c d(x, Ty)+$ed(y, Tx) + fd(x, y)$, for every $x, y \in X$,with

$a = a\big(d(x, y)\big), b = b\big(d(x, y)\big), c = c\big(d(x, y)\big), e = e\big(d(x, y)\big), f = f\big(d(x, y)\big)$

(Hardy and Roger in 1973, [31])

The Browder contraction [7] was extended as follows:

Definition 3.13. Let (X, d) be a metric space and $T: X \to X$ be a mapping.

There exists a non-decreasing right continuous function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) < t \; for \; t > 0$ and, for any $x, y \in X$,

(i) $d(Tx, Ty) \leq \phi(d(x, y))$ (**Browder [7]**)
(ii) $d(Tx, Ty) \leq \phi(m(x, y))$ (**Danes [21]**)
(iii) $d(Tx, Ty) \leq \phi(\delta(x, y))$ if x, y are regular (**Kasahara [36]**)

Definition 3.14. Let (X, d) be a metric space, and let $T: X \to X$ be a mapping. Then, T is said to be Meir Keeler's contraction if for any $\varepsilon > 0, \exists \delta > 0$, such that

$\varepsilon \leq d(x, y) < \varepsilon + \delta$ implies $d(Tx, Ty) \leq \varepsilon$ for all $x, y \in X \; with \; x \neq y$.

(Meir Keeler in 1969, [40])

Its extended forms are as:

(i) $\varepsilon \leq m(x, y) < \varepsilon + \delta \; implies \; d(Tx, Ty) < \varepsilon$;
(ii) $\varepsilon \leq \delta(x, y) < \varepsilon + \delta \; implies \; d(Tx, Ty) < \varepsilon$. **(Park in 1980, [43])**

Definition 3.15. Let (X, d) be a metric space, and let $T: X \to X$ be a mapping. Then, T is said to be Park contraction if for any $\epsilon > 0, \exists \epsilon_0 < \epsilon$, and $\delta_0 > 0$ such that for any $x, y \in X$,

(i) $\epsilon \leq d(x, y) < \epsilon + \delta_0$ implies $d(Tx, Ty) \leq \epsilon_0$. **(Park in 1980, [43])**
(ii) $\epsilon \leq m(x, y) < \epsilon + \delta_0$ implies $d(Tx, Ty) \leq \epsilon_0$. **(Park in 1980, [43])**
(iii) $\epsilon \leq \delta(x, y) < \epsilon + \delta_0$ implies $d(Tx, Ty) \leq \epsilon_0$. **(Hegedlis-Szilagyi [29]).**

Definition 3.16. Let (X, d) be a metric space. Then a mapping $T: X \to X$ is said to be contractive if there exists $\alpha, \beta > 0$ with $\alpha + \beta < 1$ satisfying

$d(Tx, Ty) \leq \alpha \frac{d(y,Ty)(1 + d(x,Tx))}{1 + d(x,y)} + \beta d(x, y), \forall x, y \in X$. **(Das and Gupta in1975, [22])**

Definition 3.17. A mapping $T: X \to X$ is said to be contractive in metric space (X, d) if for all $x, y \in X, x \neq y$ and some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ satisfying

$d(Tx, Ty) \leq \alpha \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta d(x, y)$ **(Jaggi in 1977, [33])**

Definition 3.18. Consider a right upper semi-continuous function $\psi: \bar{P} \to [0, \infty)$ satisfying $\psi(t) < t \; for \; all \; t \in \bar{P} \setminus \{0\}$, where P is the range of d. A mapping $T: X \to X$ in metric space (X, d) is said to be Boyd-Wong contractions if $d(Tx, Ty) \leq \psi(d(x, y))$ for every $x, y \in X$. **(Boyd-Wong in 1969, [5])**

Definition 3.19. A mapping $T: X \to X$ in metric space (X, d) is said to be Singh contractions if there exists a positive number m and a number $\alpha \in (0, \frac{1}{2})$, for each $x, y \in X$, we have

$d(f^m(x), f^m(y)) \leq \alpha[d(x, f^m(x)) + d(y, f^m(y))$. **(Singh in 1969, [53])**

Definition 3.20. A mapping $T: X \to X$ in metric space (X, d) is said to be Yen contractions if there exists a positive integer m and n and a number $\alpha \in (0,1)$, such that for each $x, y \in X$, we have

$d(f^m(x), f^n(y)) \leq \alpha d(x, y)$. **(Yen in 1972, [56])**

Definition 3.21. A mapping $T: X \to X$ in metric space (X, d) is said to be Guseman contractions if there exists a positive integer m and a number $\alpha \in (0,1)$, such that for each $x, y \in X$, we have

$d(f^m(x), f^m(y)) \leq \alpha d(x, y)$. **(Guseman in 1970, [27])**

Definition 3.22. A mapping $T: X \to X$ in metric space (X, d) is said to be Bailey contractions if there exists a positive integer m and $0 < d(x, y)$, such that for each $x, y \in X$, we have

$d(f^m(x), f^m(y)) \leq d(x, y)$. **(Bailey in 1966, [2])**

Definition 3.23. A mapping $T: X \to X$ in metric space (X, d) is said to be Berinde contractions if there exists $\alpha \in [0,1)$, and $\lambda \geq 0$ such that for each $x, y \in X$, we have

$d(Tx, Ty) \leq \alpha d(x, y) + \lambda d(y, Tx)$ **(Berinde weak contractions in 2004, [3])**

Definition 3.24. A mapping $T: X \to X$ in metric space (X, d) is said to be a contractive mapping for every $x, y \in X$, $\varphi\big(d(Tx, Ty)\big) \leq \varphi\big(d(x, y)\big) - \emptyset\big(d(x, y)\big)$, where $\varphi, \emptyset: [0, \infty) \to [0, \infty)$ are continuous, non-decreasing, and $\varphi^{-1}(\{0\}) = \emptyset^{-1}(\{0\}) = \{0\}$. **(Dutta and Choudhury in 2008, [23])**

Definition 3.35. Let (X, d) be a metric space and $T: X \to X$ be a mapping. Define a non-increasing function $\theta: [0, 1) \to (\frac{1}{2}, 1]$ by

$$\theta(r) = \begin{cases} 1, & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ (1-r)r^{-2} & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq 2^{-\frac{1}{2}} \\ (1+r)^{-1} & \text{if } 2^{-\frac{1}{2}} \leq r < 1. \end{cases}$$

Assume that there exists $r \in [0, 1)$ such that $\theta(r)\, d(x, T x) \leq d(x, y)$ implies

$d(T x, T y) \leq r\, d(x, y)$ for all $x, y \in X$. **(Suzuki contraction in 2008, [54])**

Definition 3.36. A mapping $T: X \to X$ in metric space (X, d) is said to be F contractions if there exist $\tau > 0$ such that for every $x, y \in X$, $d(Tx, Ty) > 0$ *implies* $\tau + F\big(d(Tx, Ty)\big) \leq Fd(x, y)$,

Where $F: \mathbb{R}_+ \to \mathbb{R}$ be a mapping satisfying:

(i) F is strictly increasing i.e. for all $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta, F(\alpha) < F(\beta)$;

(ii) For each sequence $\{x_n\}, n \in N$ of positive numbers $\lim_{n\to\infty} x_n = 0$ if and only if $\lim_{n\to\infty} F(x_n) = -\infty$;

(iii) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

(F Contractions Wardowaski in 2012, [55])

Definition 3.37. A mapping $T: X \to X$ in metric space (X, d) is said to be θ-contraction if there exist $k \in (0,1)$ such that for every $x, y \in X$, $d(Tx, Ty) \neq 0 \Rightarrow \theta(d(Tx, Ty)) \leq [\theta(d(x,y))]^k$,

Where $\theta: (0, \infty) \to (1, \infty)$ is a function satisfying the following conditions:

(i) θ is non-decreasing;
(ii) for each sequence,$\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$if and only if $\lim_{n\to\infty} t_n = 0^+$;
(iii) θ is continuous on $(0, \infty)$;
(iv) there exist $r \in (0,1)$, and $l \in (0, \infty]$such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.

(Jleli and Samet in 2014, [34])

Definition 3.38. Let (X, d) be a metric space, and $T: X \to X$ be a mapping. Then for all x, y ∈ X, we denote

$m(Tx, Ty) = ad(x, y) + b\, max\{d(x, Tx), d(y, Ty)\} + c[d(x, Ty) + d(y, Tx)]$,

where a, b, and c are non-negative reals such that $a + b + 2c = r\; with\; r \in [0, 1)$.

Now, we consider the following generalized contractive condition:

$\theta(r)\, min\{d(x, Tx), d(x, Ty)\} \leq d(x, y)\; implies\; d(Tx, Ty) \leq m(Tx, Ty)$.

(Chandra, Joshi, and Joshi in 2022,]11])

4. Generalization of contraction mapping in Menger Space:

Sehgal [50] first defined probabilistic contraction in his PhD dissertation in 1966 as:

Definition 4.1: Let (X, F) be a probabilistic metric space. A mapping $T: X \to X$ is a **probabilistic contraction or Sehgal contraction** if there exists $k \in (0,1)$ such that $F_{Tp,Tq}(kt) \geq F_{p,q}(t)$

for all $p, q \in X$, and $t > 0$.

Hicks [32] defined another contraction mapping in probabilistic metric space in 1983:

Definition 4.2: A mapping $T: X \to X$ in probabilistic metric space (X, F) is said to be Hicks contraction or **C-contraction** if there exists $k \in (0,1)$ such that for every $p, q \in X$, and every $t > 0$:

$$F_{p,q}(t) > 1 - t \Rightarrow F_{Tp,\ Tq}(kt) > 1 - kt.$$

A weaker form of Hicks contraction was introduced by D. Mihet [39] in 2005 as:

Definition 4.3: A mapping $T: X \to X$ is said to be **weak - Hicks contraction (w-H contraction)** if there exists $k \in (0,1)$ such that, for all $p, q \in S$.

$$(w - H): t \in (0,1),\ F_{p,q}(t) > 1 - t \Rightarrow F_{Tp,\ Tq}(kt) > 1 - kt.$$

Example 4.1: Let $X = [0, \infty)$, and $F_{p,q}(t) = \frac{\min(p,q)}{\max(p,q)}, \forall\, p, q \in X,\ p \neq q$. Then, (X, F, T) be a complete Menger space under triangular norm $T = T_P > T_L$. It can be seen that the mapping $g: X \to X$, $g(x) = \{ \begin{smallmatrix} 0, if\ x=0 \\ 1, if\ x>0 \end{smallmatrix}$

is a $w - H$ contraction for $k \in (0,1)$.

4.1 **Generalized form of probabilistic contraction:**

A probabilistic (m, k) contraction is a generalization of Sehgal contraction, where $m \geq 1$ and $k \in (0,1)$ and is defined as:

Definition 4.1.1: [16] If (X, F) is a PM - space, $m \geq 1$ and $k \in (0,1)$, a function $f: X \to X$ is called probabilistic **(m, k)-contraction** if for any $p, q \in X$ there is an i with $1 < i < m$ such that for every $t > 0$,

$$F_{f^i p, f^i q}(k^i t) \geq F_{p,q}(t).$$

If $m = 1$ and $k \in (0,1)$ then a probabilistic $(1, k)$-Sehgal contraction, f is a probabilistic

Sehgal contraction.

Following is a generalization of Hicks C-contraction:

Definition 4.1.2: [16] If (X, φ) is a PM - space, $m \geq 1$ and $k \in (0,1)$, a function $f: X \to X$ is called a **(m, k)-C-contraction** if for any $p, q \in X$ there is an i with $l < i < m$ such that for every $t > 0$.

$$F_{p,q}(t) > 1 - t \Rightarrow F_{f^i p, f^i q}(k^i t) > 1 - k^i t.$$

If $m = 1$ and $k \in (0,1)$ then a probabilistic $(1, k)$-C-contraction f is a probabilistic C-contraction.

Definition 4.1.3: [26] Let f, g be two mappings defined on a Menger space (X, F, T) with values into itself, and let us suppose that g is bijective. The mapping f is called a **probabilistic g-contraction** with a constant $k \in (0,1)$ if

$t > 0$ and $F_{g(x),g(y)}(t) > 1 - t$ impies $F_{f(x),f(y)}(kt) > 1 - kt$.

Conclusion: This study discusses only Banach contraction's extended and generalized form in complete metric space and Menger space so that each mapping establishes unique fixed point theorems. It helps in comparative studies and may solve many related open problems.

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