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L^1-Convergence of Double Fourier Transform in $L^p(R)$ Spaces, $p \geq 1$

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Abstract

In this research article, we have given a method which restrict the double fourier transform of fϵL^p (R) spaces, 1≤p≤∞ . Further, we have discussed the convergence by using the approximate identities. The aim of this paper is to extend the results of K. Devendra and S. Dimple[3] from one dimensional to two-dimensional trigonometric series.

Keywords: Schwartz space, convolution operator, approximate identities, L^p-Convergence.

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1. Introduction:

Let $f \epsilon L^1(R)$. The fourier transform of $f(x, y)$ is denoted by $g(\zeta, \phi)$ and is defined by:

$$g(\zeta, \phi) = \frac{1}{\sqrt{2\pi}} \int_R f(x, y) exp^{-\iota(\zeta x + \phi y)} dxdy, \zeta \epsilon R$$

If $f, g \epsilon L^1(R)$, then the inverse fourier transform of g is defined as::

$$f(x, y) = \frac{1}{\sqrt{2\pi}} \int_R g(\zeta, \phi) exp^{\iota(\zeta x + \phi y)} d\zeta d\phi$$

for $x \epsilon R$.

“As we know that several functions such as elementary constant functions $sinwt, coswt$ do not converge in $L^1(R)$ and thus they do not have fourier transforms. But when these functions are multiplied by characteristic functions , then the resultiong functions converge in $L^1(R)$ and have fourier transforms".
As we know, Lebesgue lemma states that if $f \epsilon L^1(R)$ then $\lim_{|\zeta| \to \infty} |g(\zeta)| = 0$. From which it follows that “Fourier transform is a continuous linear operator from $L^1(R)$ into $C_0(R)$, the space of all continuous functions on R which decay at infinity , i.e. $f(x) \to 0$ as $|x| \to \infty$. We say that if $f \epsilon L^1(R)$, it is not necessary that g also belongs to $L^1(R)$.

In the present article, we provide a method for restricting Fourier transform of $f \epsilon L^p(R)$ spaces using the pointwise convergence of convolution operators for approximate identities." **Definition 1.1.** Let $\psi \varepsilon L^1(R)$ suct that $\xi(0) = 1$. Then $\psi_\epsilon(x, y) = \epsilon^{-1} \psi\left(\frac{x}{\epsilon}, \frac{y}{\epsilon}\right)$ is called an approximate identity if

(i) $\int_R \psi_\epsilon(x,y)dxdy = 1$
(ii) $sup_{\epsilon>0} \int_R |\psi_\epsilon(x,y)|dxdy < \infty$,
(iii) $\lim_{\epsilon\to 0} \int_{|x|>\delta,|y|>\delta} |\psi_\epsilon(x,y)|dxdy = 0$, for all $\delta > 0$ "

Proof. We can prove properties (i) and (ii) by following:

$$\int_R \psi_\epsilon(x,y)dxdy = \int_R \epsilon^{-1}\psi\left(\frac{x}{\epsilon},\frac{y}{\epsilon}\right)dxdy = \int_R \psi\left(\frac{x}{\epsilon},\frac{y}{\epsilon}\right)d\left(\frac{x}{\epsilon},\frac{y}{\epsilon}\right) = 1$$

For (iii), it follows that:

$$\int_{|x|>\delta,|y|>\delta} \psi_\epsilon(x,y)dxdy = \int_{|x|>\delta,|y|>\delta} \frac{1}{\epsilon}\psi\left(\frac{x}{\epsilon},\frac{y}{\epsilon}\right)dxdy$$

$$= \int_\delta^\infty \frac{1}{\epsilon}\psi\left(\frac{x}{\epsilon},\frac{y}{\epsilon}\right)dxdy + \int_{-\infty}^{-\delta} \frac{1}{\epsilon}\psi\left(\frac{x}{\epsilon},\frac{y}{\epsilon}\right)dxdy Substituting\ z = \frac{x}{\epsilon}, t = \frac{y}{\epsilon}$$

, we get

$$\lim_{\epsilon\to 0} \int_{\frac{\delta}{\epsilon}}^\infty \psi(z,t)dzdt + \int_{-\infty}^{-\frac{\delta}{\epsilon}} \psi(z,t)dzdt = 0.$$

Definition 1.2. “A sequence of functions $h_{n_{n\varepsilon N}}$ such that $h_n(x,y) = nh(nx,ny)$ where

$$n = \frac{1}{\varepsilon}, n \to \infty, \varepsilon \to 0$$

is called an approximate identity if

(i) $\int_R h_n(x,y)dxdy = 1$ for all n,

(ii) $sup_n \int_R h_n(x,y)dxdy < +\infty$,

(iii) $\lim_{n\to\infty} \int_{|x|>\delta} h_n(x,y)dxdy = 0$ for every $\delta > 0$."

By following the above definition , the following proposition can easily prove:

Proposition 1.1. “A sequence of functions $h_{n_{n\varepsilon N}}$ with $h_n \geq 0, h_n(0,0) = 1$ is an approximate identity if for every $\varepsilon > 0$ there exists $n_0 \varepsilon N$ so that for all $n \geq n_0$ we have $\int_{-\varepsilon}^{\varepsilon} h_n > 1 - \varepsilon$. Let us consider the class $S^*(R)$ of C^∞-functions on R which are rapidly decreasing i.e. Schwartz class such that

$S^*(R) = f: R \to R, sup_{x\to R}(x,y)\frac{d^m}{dx^m}\frac{d^m}{dy^m}f(x,y) < \infty; n, m\varepsilon N \cup (0)$".

We know that if $f\varepsilon S^*(R)$, then $g\varepsilon S^*$ and “$S^*(R) \subset L^p(R)$". To prove the denseness of $S^*(R) \subset L^p(R)$, we have $\eta\varepsilon S^*(R) \Rightarrow |\eta(x,y)| \leq \frac{c}{1+|xy|^n}$.

For $1 \leq p < \infty$,

$$\int_R |\eta(x,y)|^p dxdy \leq \int_R \frac{c^p}{(1+|xy|^n)^p} < \infty which\ gives\ \eta\varepsilon L^p(R)$$

. Define a sequence η_N such that
$\eta_N(x,y) = f(x,y)$, if $-N \leq x, y \leq N$; and otherwise it will become 0.
$\Rightarrow \exists \eta_N \varepsilon S(R), f \varepsilon L^p(R)$ such that
$\int_R \ |\eta_N - f|^p dx \to 0.$

as $N \to \infty$.
"Hence $S^*(R)$ is dense in $L^p(R)$."

Proposition 1.2. Let $h_n = \alpha_n \psi_n + (1-\alpha_n)\sigma_n$, where $\{\psi_n\}_{n\varepsilon N}, \{\sigma_n\}_{n\varepsilon N}$ are approximate identities and $0 \leq \alpha_n \leq 1$.
"(a) For $1 \leq p \leq +\infty$ and every $f\varepsilon L^p(R)$, $\lim_{n\to\infty}(h_n - \psi_n) \star f \to 0$ and $\lim_{n\to\infty}(h_n - \sigma_n) \star f \to 0$.
(b) For every $f\varepsilon L^\infty(R)$, $\lim_{n\to\infty}(h_n - \psi_n) \star f \to 0$ a.e..
(c) For $1 \leq p \leq +\infty$, if $\sum_n \ (1-\alpha_n)^p < +\infty$, then for every $f\varepsilon L^p(R)$, $\lim_{n\to\infty}(h_n - \psi_n) \star f \to 0$" a.e.

Proof. (a) "If $1 \leq p \leq +\infty$ and every $f\varepsilon L^p(R)$.
Using Minkowski's inequality,

$$\left|\left|(h_n - \psi_n) \star f\right|\right|_p \leq (1-\alpha_n)\left(\left|\left|\sigma_n \star f - f\right|\right|_p + \left|\left|\psi_n \star f - f\right|\right|_p\right)$$

Now, As proved by Singh D. and Singh D. [2] , we have
If $h_n(x)$ is an approximate identity and $f\varepsilon L^p(R)$, then $h_n \star f \to f\varepsilon L^p(R)$.
So, by using the above, we obtain $\left|\left|(h_n - \sigma_n) \star f\right|\right|_p \to 0$."
(b) "For $f\varepsilon L^\infty(R), |(h_n - \psi_n) \star f| \leq ||(h_n - \psi_n) \star f|| \to 0$ by part (a)."
(c) "For $f\varepsilon L^p(R)$,
$\int_R \ \sum_n \ (1-\alpha_n)^p |\sigma_n \star f(x,y)|^p dxdy = \sum_n \ ||\ (1-\alpha_n)\sigma_n \star f(x,y)||_p^p \leq \sum_n \ (1-\sigma_n)^p ||f||_p^p < +\infty$."
Then $(1-\alpha_n)\sigma_n \star f \to 0$ a.e. . Similarly $(\alpha_n - 1)\psi_n \star f \to 0$ a.e.

Definition 1.3. "An approximate identity $\{h_n\}$ is called L^p-good if $h_n \star f \to f$ a.e. for all $f\varepsilon L^p(R)$, and it is called good if it is L^p-good for every $1 \leq p \leq +\infty$. An approximate identity $\{h_n\}$ is called L^p -bad if therte exists $f\varepsilon L^p(R)$ such that $h_n \star f$ not approachable to f on a set of positive measure.
Definition 1.4. Let $\{\psi_n\}_{n\varepsilon N}$ and $\{\sigma\}_{n\varepsilon N}$ be approximate identities , α_n be a sequence of real numbers with $0 \leq \alpha_n \leq 1$ and $\alpha_n \to 1$. We call preturbed approximate identities any approximate identity $\{h_n\}_{n\varepsilon N}$ of the form $h_n\psi_n + (1-\alpha_n)\sigma_n$."

2. **Main Results.**

Theorem 2.1.

(i)" Given any good approximate identity $\{\psi_n\}_{n\varepsilon N}$ there exists a perturbed approximate identity $\{h_n\}_n \varepsilon N$ such that $f\varepsilon L^q(R)$

$$\left(h_n \star \hat{f}\right)(\zeta,\phi) = \widehat{h_n}(\zeta,\phi)\hat{f}(\zeta,\phi)\left(\widehat{h_n}(\zeta,\phi)\hat{f}(\zeta,\phi)\right) \to f(x,y) 1 \leq q < p$$

(ii) $\left(\widehat{h_n}(\zeta,\phi)\hat{f}(\zeta,\phi)\right) \to f(x,y)$ for $q > p$ and $\left(\widehat{h_n}(\zeta,\phi)\hat{f}(\zeta,\phi)\right)$ not approaches to$f(x,y)$ for $1 \leq q \leq p$.

(iii) $\left(\widehat{h_n}(\zeta,\phi)\hat{f}(\zeta,\phi)\right) \to f(x,y)$ for $q = \infty$

$(\widehat{h_n}(\zeta,\phi)\hat{f}(\zeta,\phi)$ not approachable to $f(x,y)$ for $1 \leq q < \infty$."

Proof. (i) Let

$$g_n(x) = \frac{1}{\sqrt{2\pi}}\int_R e^{\iota(x\zeta+y\phi)}\widehat{h_n}(\zeta,\phi)\hat{f}(\zeta,\phi)d\zeta d\phi = \frac{1}{\sqrt{2\pi}}\int_R e^{\iota(x\zeta+y\phi)}\widehat{h_n}(\zeta,\phi)\int_R f(x,y)dxdy$$
$$= \frac{1}{\sqrt{2\pi}}\int_R h_n(x+y)f(x,y)dxdyor \ = (h_n \star f)(x,y)\widehat{h_n}(\zeta,\phi)\hat{f}(\zeta,\phi)$$
$$= \frac{1}{\sqrt{2\pi}}\int_R e^{\iota(x\zeta+y\phi)}\widehat{h_n}(\zeta,\phi)\hat{f}(\zeta,\phi)d\zeta d\phi = (h_n \star f)(x,y)\text{“}Fix\ q \geq p$$

and taking $1-\alpha_n = \frac{1}{(nlog^2n)^{1/p}}$. Since $\sum_n \ (1-\alpha_n)^q < +\infty$ and ψ_n is an L^q-good approximate identity , using Proposition 1.4. we obtain that h_n is also an L^q-good approximate identity." Hence for $q \geq p$,$(h_n \star f)(x,y) \to f(x,y)$

Now, we have to prove that for each $1 \leq q < p$. there exists $f_q \varepsilon L^q(R)$ so that “$limsup_k|x|^k \frac{d^k}{dx^k}\left(h_k \star f_q \to \infty\right)$" on a set of positive measure.

Set,

$f_q(x,y) = \frac{1}{\left(xylog^2(x/2,y/2)\right)^{1/q}}\chi_{[0,1]}(x)\varepsilon L_q(R).$

Take

$r_n = \frac{1}{n^{1+1/p}(logn)^{2/p}}, a_n = r_n^{\frac{1}{p+1}} = \frac{1}{n^{1/p}}(logn)^{\frac{2}{p(p+1)}},$

“$J_n = [a_n - r_n, a_n + r_n]$ and

$U_n = [-a_n + r_n, -a_{n+1} + r_{n+1}],$"

for sufficiently large n and for all $k \geq n, x\varepsilon U_k$.

$$h_k \star f_q(x,y) \geq (1-\alpha_k)\sigma_k \star f_q(x) \geq \frac{1}{\left(klog^2(k)\right)^{1/p}}\int_{-J_k}\sigma_k(x,y)f_q(x-y)dxdyNow, we\ have,$$

$$h_k \star f_q(x,y) \geq \frac{f_q\left(C_{r_k}(logk)^{2/p+1}\right)}{(klog^2k)^{1/p}}\int_{-J_k}\sigma_k(x,y)dxdy$$

or , $f_q\left(C_{r_k}(logk)^{2/p+1}\right) = \frac{k^{1/q+1/pq}(logk)^{\frac{2}{pq(p+1)}}}{C^{1/q}\left(log\left(C/2k^{(p+1)/p}(logk)^{2/p(p+1)}\right)\right)^{2/q}}$

Then, $h_k \star f_q(x,y) \geq C_k^{\frac{1}{q}-\frac{1}{p}+\frac{1}{pq}}H_q(k) > k^\delta \geq n^\delta),$

where, $H_q(k) = \frac{(logk)^{\frac{2}{pq(p+1)}-\frac{2}{p}}}{C^{1/q}(log(C/2k^{(p+1)/p}(logk)^{2/p(p+1)})^{2/q}}$

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and $0 < \delta < 1/q - 1/p + 1/pq$.
So, $\frac{d^k}{dx^k}\frac{d^k}{dy^k}\left(h_k \star f_q(x,y)\right) \geq C\frac{d^k}{dx^k}\frac{d^k}{dy^k}\left(k^{1/q-1/p+1/pq}H_q(k)\right)$
or, $|xy|^n\frac{d^n}{dx^n}\frac{d^n}{dy^n}\left(h_k \star f_q(x,y)\right) \geq |xy|^n \int_{-J_k} f_q(x-y)\frac{d^n}{dx^n}\frac{d^n}{dy^n}\sigma_k(x,y)dxdy$
for $k \geq n$, $|xy|^k\frac{d^k}{dx^k}\frac{d^k}{dy^k}\left(h_k \star f_q(x,y)\right) \geq |xy|^n\frac{d^n}{dx^n}n^\delta \geq |xy|^n\frac{d^n}{dx^n}\frac{d^n}{dy^n}\left(\frac{1}{(x-y)^{p\delta}}\right)$

$$= \frac{|xy|^n(-1)^n(p\delta+n-1)!}{(p\delta)!(x-y)^{p\delta+n}}$$

$$\geq |xy|^n\frac{(-1)^n(p\delta+n-1)!}{(p\delta)!Cr_n(logn)^{2/p+1}(logn)^{2\delta/p+1}} \to \infty \text{ as } n \to \infty.$$

"In view of Sawyer's Principle [4] , there exists a functions $f\varepsilon L^q([0,1)) \subseteq L^q(R)$ such that $limsup_n|xy|^n\frac{d^n}{dx^n}\frac{d^n}{dy^n}(h_n \star f) \to \infty$ a.e. on a set of positive measure in R. It follows that $(h_n \star f)$ not belongs to $S(R)$ or $h_n \star f$not approachable to f or $\widehat{h_n}(\zeta,\phi)\hat{f}(\zeta,\phi)$ not approachable to $f(x,y)$ for $1 \leq q < p$.
Let p_n be a decreasing sequence of real numbers such that $p_1 > p_2 > \ldots. p_n > \ldots\ldots p$. for each p_i we can construct a perturbation $\{h_n^i\}_n$ of $\{\psi_n\}$ that is L^q-good for $q \geq p_i$, and L^q-bad for $1 \geq q < p_i$. Consider a sequence of blocks $\{T_k\}_{k\varepsilon N}$, where $T_k = \{h_{n_{k-1}+1}, \ldots\ldots, h_{n_k}^k\}$ and $\{n_k\}$ is a sequence of positive integers increasing to infinity. Let $S_k = \{n_{k-1}+1, \ldots., n_k\}$. and let $\{h_n\}_n = U_k T_k$. Now, fix $q > p$. There exists $n_0 \varepsilon N$ so that for all $n > n_0$ we have $p_n < q$".

$$\sum_{k=n_0}^{\infty} \sum_{n\varepsilon S_k} (1-\alpha_n^k)^q \leq \sum_{k=n_0}^{\infty} \sum_{n\varepsilon S_k} \frac{1}{(nlog^2n)}^{q/p_{n_0}} \leq \sum_n \frac{1}{(nlog^2n)^{q/p_{n_0}}} < \infty.$$

Using Proposition 1.2(c), we get $h_n \star f \to f$ for $f\varepsilon L^q(R), q > p$, or $\widehat{h_n}(\zeta,\psi)\hat{f}(\zeta,\psi) \to f(x,y)$ for $q > p$.
Now consider a sequence "$C_i^N \to \infty$ as $i \to \infty$. Since $\{h_n^i\}_n$ is L^q-bad for all $q < p_i$, it is also L^p-bad. These exists $f_i \varepsilon L^p([0,1))$ and $\lambda_i^N > 0$ such that

$$||sup_{n>n_{i-1}} h_n^i \star f_i(x,y) \,\| > \int ||h_n^i \star f_i(x,y) \,\|^p \, dxdy > C^N||f_i(xy-\lambda_i^N)||_p^p$$

$$= 2C_i^N, \left[||f_i(xy-\lambda_i^N)||_p = 2^{1-i}, C^N = 2^{(i-1)p+1}C_i^N\right]."$$

It follows that there exists $n_i > n_{i-1}$, so that $||sup_{n_{i-1}<n\leq n_i} h_n^i \star f_i \,\| > C_i^N$.

Set $\quad \tilde{f} = \sum_i f_i$, then $\|\tilde{f}||_p \leq \sum_i ||f_i||_p \leq 2$.

"Suppose that $\{h_n\}$ satisfies a weak (p,p) inequality in $L^p([0,1))$, We know that if μ be a finite positive Borel measure, then there exists a sequence μ_n of atomic measure that converges to μ weakly or if f has compact support then

$$\int_R d\mu_n f(x,y) \to \int_R f(x,y)d\mu d\gamma$$

where, $\mu_n \to \mu$ and $\gamma_n \to \gamma$, weakly.
If $f\varepsilon L^1(R), d\mu d\gamma = |f(x,y)|dxdy$ is a finite Borel measure, so we can find
$\gamma_n\mu_n = \sum_{i=1}^N C_i^N \delta_{\lambda_i^N} \to \mu\gamma$ weakly.

Consider

$$\left|\{sup_n\left(h_n^i \star f\right)\}\right| = \int_{-J_k} \left|h_n^i(x,y)f(x-y)\right|^p dxdy \leq \int_{-J_k} \left|h_n^i(x,y)d\mu_n(x-y)\right|^p dxdy$$

$$\leq ||\sum_{i=1}^{N} f(xy-\lambda_i^N)C_i^N||_p^p \sum_{i=1}^{N} C_i^N||f(xy-\lambda_i^N)||_p^p \leq C_0^N||f||_p^p$$

$$= 2^p C_0^N.\text{"}\ (1)$$

On the other hand

$$|\{sup_n(h_n \star f)\}| \leq \left|\{sup_{n_{i-1}<n\leq n_i}\left(h_n^i \star f(i)\right)\}\right| > C_i^N\ (2)$$

Combining equations, we get

$$C_0^N > C_i^N But\ C_i^N \to \infty$$

as $i \to +\infty$. Hence $h_n \star f$ not approachable to f in $L^p([0,1))$. Since the spaces $L^q([0,1))$ are nested , $\{h_n\}$ is $L^q([0,1))$-bad for all $1 \leq q \leq p$. Therefore, such a choice of $\{n_k\}$ makes $\{h_n\}$ $L^q(R)$-bad for all $1 \leq q \leq p$. This implies that $\widehat{h_n}(\zeta,\psi)\hat{f}(\zeta,\psi)$ not approachable to $f(x,y)$ for $1 \leq q \leq p$. (iii) "Let $\{\psi_n\}_{n\varepsilon N}$ be a good approximate identity and let $\{\zeta_n\}_{n\varepsilon N}$ be any approximate identity. Let $\{p_n\}$ be a sequence of real numbers satisfying

$$1 \leq p_1 < p_2 <\ldots\ldots< p_n \to \infty$$

Consider the blocks $\{T_k\}$, where each block T_k is related to p_i , for $i\varepsilon S_n$, let

$$h_i = \alpha_i^k\psi_i^k + \left(1-\alpha_i^k\right)\sigma_i^k.$$

Choose n_k such that $\alpha_i^k \to 1$. Then since $\{\psi_n\}$ is L^∞ good,

$\psi_n \star f \to f$a.e. for all $f\varepsilon L^\infty(R)$,

and, $\alpha_i^k\psi_i^k \star f \to f$ a.e. for all $f\varepsilon L^\infty(R)$
Since, $\sigma_i^k \star f(x) \leq ||f||_\infty$.
$\left(1-\alpha_i^k\right)\sigma_i^k \star f \to 0$ a.e. for all $f\varepsilon L^\infty(R)$.
it follows that $h_n \star f \to 0$ a.e. for all $f\varepsilon L^\infty(R)$.
This implies that $\left(\widehat{h_n}(\zeta,\psi)\hat{f}(\zeta,\psi)\right) \to f(x,y)$ for $q = \infty$.

The approximate identity $\{h_n^k\}_n$ is L^{pm}-bad for every $m\varepsilon\{1,\ldots.,k\}$, since it is L^q-bad for every $1 \leq q \leq p_k$. There exists $f_m^k\varepsilon L^{p_m}([0,1))$ with $\|f_m^k(xy-\lambda_m^{k(N)}|| = 2^{-k}, \lambda_m^{k(N)} > 0$ and $n_m^k > m_{k-1}$ so that

$\left|\{sup_{n_{k-1}} < n < n_m^k(h_n^k \star f_m^k)\}\right| > C^N||f_m^k(xy-\lambda_m^{k(N)}||_{p_m}^{p_m} = \frac{C_k^N}{2^{kp_m}}$
Let $\tilde{f} = \sum_{k\geq k_0} f_{k_0}^k$, then $\|\tilde{f}||_{pk_0} < 2$.
So, $\left|\{sup_n\left(h_n \star \tilde{f}\right)\}\right| \leq C_0||\tilde{f}||_{pk_0}^{pk_0} \leq 2^{p_{k_0}}C_0^N$. (3)
Hence, $\left|\{sup_n\left(h_n \star \tilde{f}\right)\}\right| \geq \left|\{sup_{n_{k-1}} < n < n_m^k\left(h_n^k \star f_{k_0}^k\right)\}\right| > \frac{C_k^N}{2^{kp_{k_0}}}$" (4)

Using the equations, we get

$$C_0^N > \frac{C_k^N}{2^{kp_{k_0}(k+1)}} \to +\infty$$

Thus we conclude that $\widehat{h_n}(\zeta,\psi)\hat{f}(\zeta,\psi)$ not approachable to $f(x,y)$ for $1 \leq q \leq \infty$.

Hence, the proof is completed.

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