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Odd-Even Congruence Labeling of Some Graphs

Rituraj[1], Shambhu Kumar Mishra[2]

[1]Research Scholar, Department of Mathematics, Patliputra University, Patna, Bihar, India. rituraj6312@gmail.com

[2]Professor, Department of Mathematics, Patliputra University, Patna, Bihar, India

Shambhumishra5@gmail.com

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Abstract:

Introduction: Graph labeling is one of the fastest growing areas in the field of graph theory. A graph labeling is basically an assignment of integers to the nodes or edges or both, subject to certain conditions. Different types of graph labeling techniques have been developed by several authors.

Objectives: The applications of labeling of graphs has diverse fields in the study of data base management, secret sharing schemes, physical cosmology, debug circuit design, X-ray, crystallography, astronomy, radar, broadcasting network, secret, coding theory and much more. The graph labeling techniques serve as useful mathematical models and solve various graph related problems.

Methods: The strength of this research paper is based on Odd-even congruence labeling of different types of graphs. Assignment of natural numbers as labels for the edges and vertices of a graph. It is based on modular arithmetic property known as congruence graph labelling of a graph.For congruence graph labeling it entails the assignment of odd integers to vertices and even integers to edges with property of congruence graph labeling.

Results: This labeling method has been identified on friendship graph, shell graph, generalized butterfly graph, fan graph, $P_2 + mK_1$ graph

Keywords: Labeling, Congruence labeling, Odd-even congruence labeling, friendship graph, shell graph, generalized butterfly graph, fan graph, $P_2 + mK_1$ graph

1. Introduction

In this paper all graphs G considered as finite, undirected, connected and without loops. Suppose $V(G)$ and $E(G)$ be the set of vertices and edges of a graph G respectively. The cardinality of vertex set is denoted by $|V(G)|$ and the edge set is denoted by $|E(G)|$ are called the order and size of the graph G. For standard terminology of Graph Theory we used [1]. For all detailed survey of graph labeling, we refer [2]. While studying graph theory, one that has gained a lot of popularity during the last 62 years is the concept of labeling of graphs due to its wide range of applications. A labeling of a graph G is one-to-one mapping that carries the set of graph elements onto a set of numbers, called labels. In 1967, Rosa [5] published a pioneering paper on graph labeling problems. Thereafter many types of graph labeling methods have been studied by several authors.

G.Thamizhendhi and K.Kanakambika [6] have introduced Odd-even congruence labeling and they proved behaviour of several graphs like bipartite graph, comb graph, star graph, graph acquired by connecting two copies of even cycle C_r by a path P_t , shadow graph of the path P_t and the tensor product of $K_{1,t}$ & P_2 as Odd-even congruence graph. They defined an odd-even congruence graph, if vertex and edge set are assigned by distinct odd and even integers respectively, further $f(u_p) \equiv f(u_q)\ (mod\ g(e))$, u_p and u_q are adjacent vertices in G. In this paper we investigate the existence of Odd-even congruence labeling for vertex switching graph, friendship graph, shell graph, generalized butterfly graph, fan graph, $P_2 + mK_1$ graph.

2. Preliminaries

Definition 2.1. [6, 7]. A graph obtained by fetching a vertex x of H, eliminating the adjacent edges of x and by adding new edges that are joining x to their non-adjacent vertices in H is called vertex switching H_x of H.

Definition 2.2. [3]. The friendship graph Fr_n is a collection of n triangles with a common vertex. It is a planar undirected graph with $2n + 1$ vertices and $3n$ edges constructed by joining n copies of the cycle graph C_3 with a common vertex.

Definition 2.3. [10]. A shell S_n is the graph obtained as a cycle C_n by taking $(n - 3)$ concurrent chords sharing a common end point called the apex. Shell graph are denoted as $C_{(n,n-3)}$. A shell S_n is also called fan f_{n-1}

Definition 2.4. [4]. The Generalized butterfly graph denoted by BF_n obtained by inserting adjoining vertices to every wing with assumption that sum of inserting vertices to every wing are same.

Definition 2.5. [9]. A fan graph $f_n,\ n \geq 2$ obtained by joining all vertices of a path P_n to a further vertex , called the center.

$f_n = K_1 + P_n$, $|V(f_n)| = n + 1$ and $|E(G)| = 2n - 1$

3. Main Results

Theorem 3.1 The graph obtained from switching of any vertex in cycle C_r is an Odd-even congruence graph.

Proof:

Suppose $x_1, x_2, \dots\dots\dots\dots\dots, x_{r-1}$ be the vertices of a cycle C_r. Let H_{x_1} is the graph obtained from switching of a vertex $x_1$in C_r. In H_{x_1} each vertex x_i other than x_2 and x_r join to x_1.

Here $\left|V\left(H_{x_1}\right)\right| = r$ and $\left|E\left(H_{x_1}\right)\right| = 2r - 5$

we have $d = \min\{2r\ ,2(2r - 5)\ \}$

$= 2(2r - 5)$

Let the edge set of H_{x_1} be

$E\left(H_{x_1}\right) = \{x_i x_{i+1}\ /\ 2 \leq i \leq r - 1\} \cup \{x_1 x_{i+1} /\ 2 \leq i \leq r - 2\}$

Where $e_i = x_i x_{i+1}\ /\ 2 \leq i \leq r - 1$ and $w_i = x_1 x_{i+1} /\ 2 \leq i \leq r - 2$

Define a labeling $f : V\left(H_{x_1}\right) \rightarrow \{1,3,7,15,31,63, \dots\dots\dots\dots, 2^{n+i} - 1\}$ is defined as follows

$f(x_1) = 1$,

$f(x_{i+1}) = 2^{i+1} - 1$ for $1 \leq i \leq r-1$

The edge labeling $k : E(H_{x_1}) \rightarrow \{4,6,8,14,16, \ldots \ldots \ldots, 4i+2\}$ is defined as

$k(e_i) = 2^i$ for $2 \leq i \leq r-1$

$k(w_i) = 2^{i+1} - 2$ for $2 \leq i \leq r-2$

Clearly $k(e_i)$ divides $|(f(x_i) - f(x_{i+1}))|$ for $2 \leq i \leq r-1$ and

$k(w_i)$ divides $|(f(x_1) - f(x_{i+1}))|$ for $2 \leq i \leq r-2$

Hence the graph obtained from switching of any vertex in cycle C_r is an Odd-even congruence graph.

Example: 3.2

Consider a graph $G = Hx_1$ obtained from C_8.

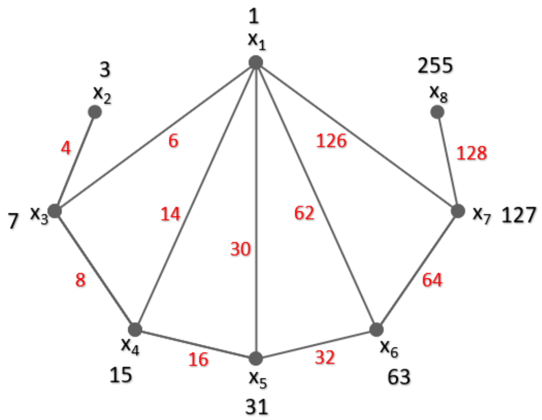

Figure-1

Fig-1 shows that graph G admits odd-even congruence labeling.

Theorem 3.3 The friendship graph Fr_n is an Odd-even congruence graph.

Proof:

Let the vertex of Fr_n be

$V(Fr_n) = \{ v_i \, / \, i = 0,1,2,3 \ldots \ldots \ldots \ldots \ldots 2n\}$ with v_0 as the central vertex
and the edge set be

$E(Fr_n) = \{v_0 v_i, \; v_i v_{i+1}, \; i = 1,2 \ldots \ldots \ldots \ldots \ldots 2n\}$

Where $e_i = \{v_0 v_i \, /1 \leq i \leq 2n\}$ and $w_i = v_i v_{i+1}$ for $i = 1,3,5, \ldots \ldots \ldots 2n-1$

With $|V| = 2n+1$ and $|E| = 3n$
For $G = Fr_n$, we have $d = min\,\{2(2n+1), 2(3n)\}$
$= 2(2n+1)$

The vertices of Fr_n are labeled as given below.

Define the bijection $f : V(Fr_n) \rightarrow \{1,3,7,\ldots\ldots\ldots\ldots\ldots,2^i-1\}$ as $f(v_{i-1}) = 2^i - 1$ for $1 \leq i \leq 2n+1$

The edge labeling $k : E(Fr_n) \rightarrow \{2,4,6,8,\ldots\ldots\ldots\ldots,4i+2\}$ is defined as

$k(e_i) = 2^{i+1} - 2$ for $1 \leq i \leq 2n$

$k(w_i) = 4^i$ for $i = 1,3,5,\ldots\ldots\ldots 2n-1$

Clearly $k(e_i)$ divides $|(f(v_i) - f(v_0))|$ for $1 \leq i \leq 2n$ and

$k(w_i)$ divides $|(f(v_i) - f(v_{i+1}))|$ for $i = 1,3,5,\ldots\ldots\ldots 2n-1$

Hence the graph friendship graph Fr_n is an odd-even congruence graph.

Example: 3.4

Consider a graph $G = Fr_n$ with $n = 7$

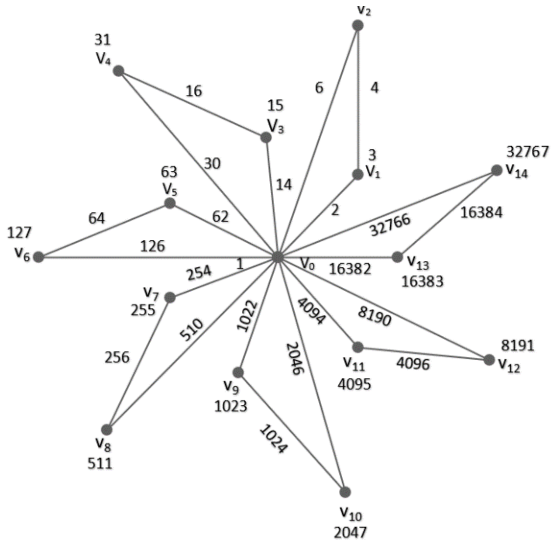

Figure-2

Fig-2 shows that friendship graph Fr_7 admits odd-even congruence labeling.

Theorem 3.5 The shell graph S_n is an Odd-even congruence graph.

Proof:

Let G be a shell graph.

Define $V(G) = \{u, v_i / 1 \leq i \leq (n-1)\}$ and

$E(G) = \{e_i = uv_i : 1 \leq i \leq n-1;\ e'_i = v_i v_{i+1} : 1 \leq i \leq n-2\}$

are the vertices and edges of a graph G

Also $|V(G)| = n$ and $|E(G)| = 2n-3$

Then $d = \min\{2n, 2(2n-3)\}$

$= 2n$

Define a bijective function $f : V(G) \rightarrow \{1,3,7, \ldots\ldots\ldots, 2^{i+1} - 1\}$ is defined as follows

$f(u) = 1$

$f(v_i) = 2^{i+1} - 1$ for $1 \leq i \leq n - 1$

The edge labeling $k : E(G) \rightarrow \{2,4,8 \ldots\ldots\ldots, 4i + 2\}$ is defined as

$k(e_i) = 2^{i+1} - 2$ for $1 \leq i \leq n - 1$

$k(e_i') = 2^i$ for $1 \leq i \leq n - 2$

Clearly $k(e_i)$ divides $|(f(u) - f(v_i))|$ for $1 \leq i \leq n - 1$ and

$k(e_i')$ divides $|(f(v_i) - f(v_{i+1}))|$ for $1 \leq i \leq n - 2$

Hence the shell graph S_n is an odd-even congruence graph.

Example: 3.6

Consider a shell graph $G = S_n$ with $n = 8$

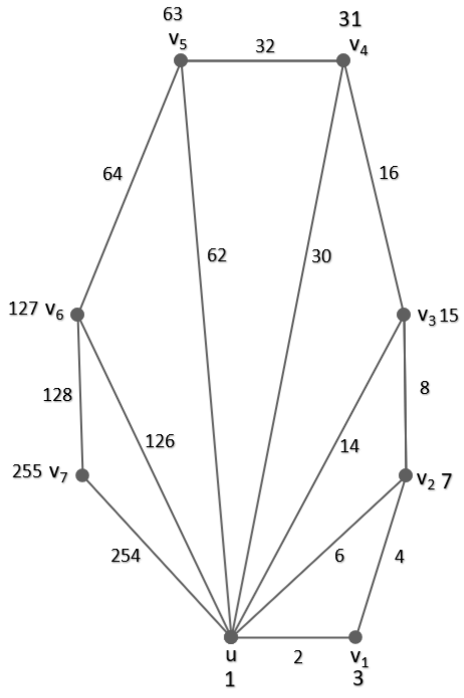

Figure-3

Fig-3 shows that shell graph S_8 admits odd-even congruence labeling.

Theorem 3.7 A generalized butterfly graph BF_n admits odd-even congruence graph for $n \geq 2$.

Proof:

Suppose the vertex set of BF_n be

Define $V(BF_n) = \{ v_i / 0 \leq i \leq 2n\}$ and the edge set of BF_n be

$E(BF_n) = \{ v_i v_{i+1} /1,2, \dots \dots , n-1, n+1, \dots \dots 2n-1\} \cup \{v_0 v_i \, / i = 1,2, \dots \dots . . , 2n\}$

Where $|V(G)| = 2n + 1$ and $|E(G)| = 4n - 2$

Then d =min $\{2(2n+1), 2(4n-2)\}$

$= 2(2n+1)$

Define a bijective function $f : V(G) \rightarrow \{1,3,7, \dots \dots \dots , 2^{i+1} - 1 \}$ is defined as follows

$f(u) = 1$

$f(v_i) = 2^{i+1} - 1$ for $1 \leq i \leq n-1$

The edge labeling $k : E(G) \rightarrow \{2,4,8 \dots \dots \dots , 4i+2\}$ is defined as

$h(v_{i-1} v_i) = 2^i$ for $2 \leq i \leq n$

$k(v_{i-1} v_i) = 2^i$ for $n + 2 \leq i \leq 2n$

$p(v_0 v_i) = 2^{i+1} - 2$ for $1 \leq i \leq 2n$

Clearly $h(v_{i-1} v_i)$ divides $|(f(v_{i-1}) - f(v_i))|$ for $2 \leq i \leq n$,

$k(v_{i-1} v_i)$ divides $|(f(v_i) - f(v_{i+1}))|$ for $n + 2 \leq i \leq 2n$ and

$p(v_0 v_i)$ divides $|(f(v_i) - f(v_0))|$ for $1 \leq i \leq 2n$

Hence the generalized butterfly graph BF_n is an odd-even congruence graph.

Example: 3.8

Consider a generalized butterfly graph BF_n with $n = 5$

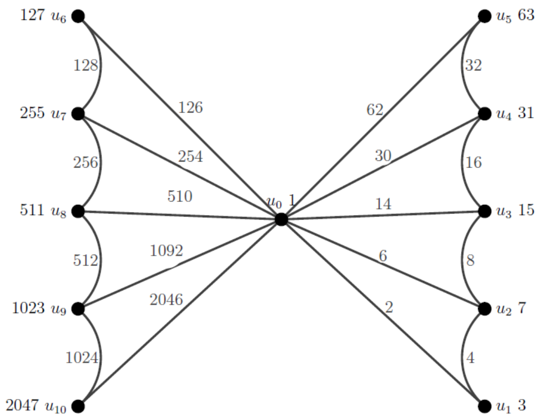

Figure-4

Fig-4 shows that friendship graph Fr_6 admits odd-even congruence labeling.

Theorem 3.9 The Fan graph F_n admits odd-even congruence graph.

Proof:

Let G be a graph of the fan graph F_n

Let $V(G) = \{v_i: 1 \leq i \leq n+1\}$ be the vertices of G and

$E(G) = \{v_1 v_{i+1}: 1 \leq i \leq n\} \cup \{v_i v_{i+1} : 2 \leq i \leq n\}$ be the edges of G.

Where $e_i = \{v_1 v_{i+1}: 1 \leq i \leq n\}$ and $w_i = \{v_i v_{i+1} / 2 \leq i \leq n\}$

Now $|V(G)| = n+1$ and $|E(G)| = 2n-1$

Then d =min $\big(2(n+1), 2(2n-1)\big)$

$= 2(n+1)$

Define the bijection $f : V(G) \to \{1,3,7, \dots \dots \dots, 2^p - 1\}$ is defined as follows

$f(v_i) = 2^i - 1$ for $1 \leq i \leq n+1$

The edge labeling $k : E(G) \to \{2,4,6,8,14, \dots \dots \dots, 2^{i+1} - 2\}$ is defined as follows

$k(e_i) = 2^{i+1} - 2$ for $1 \leq i \leq n$

$k(w_i) = 2^{i+1}$ for $2 \leq i \leq n$

Clearly $k(e_i)$ divides $\left|\left(f(v_{i+1}) - f(v_1)\right)\right|$ for $1 \leq i \leq n$ and

$k(w_i)$ divides $\left|\left(f(v_{i+1}) - f(v_i)\right)\right|$ for $2 \leq i \leq n$

Hence the fan F_n is an odd-even congruence graph.

Example: 3.10

Consider a generalized fan graph F_n with $n = 6$

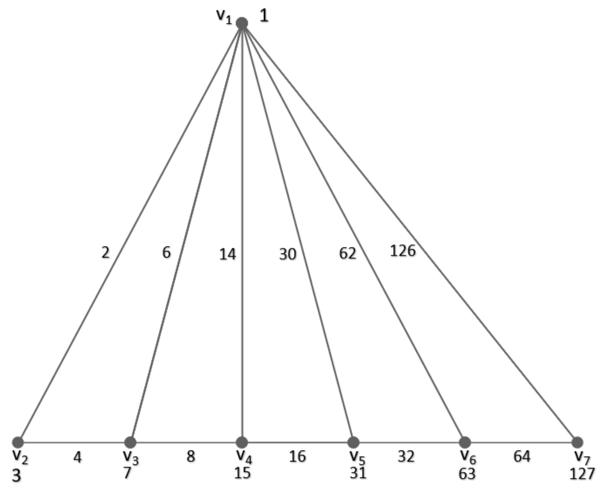

Figure-5

Fig-5 shows that friendship graph F_6 admits odd-even congruence labeling.

Theorem 3.11 The graph $P_2 + mK_1$ admits odd-even congruence graph.

Proof:

Suppose P_2 is a path having two vertices u_1, u_2.

Let $u_3, u_4, \dots\dots\dots.., u_{m+2}$ be the m isolated vertices

Connecting u_1, u_2 with $u_i,\ 3 \le i \le m+2$

We obtain $P_2 + mK_1$

Suppose $G = P_2 + mK_1$

Let the vertex set of G be

Let $V(G) = \{u_i : 1 \le i \le m+2\}$ be the vertices of G and

$E(G) = \{u_1u_2\} \cup \{u_1u_i\ /3 \le i \le m+2\} \cup \{u_2u_i\ /3 \le i \le m+2\}$ be the edges of G where $w_i = u_1u_i$ and $q_i = u_2u_i$. Also $|V(G)| = 2+m$ and $|E(G)| = 2m+1$

Then $d =$min $\big(2(2+m\ ), 2(2m+1)\big)$

$= 2(2+m)$

Define a bijective function $h : V(G) \rightarrow \{1,3,7, \dots\dots\dots.., 4i+3\}$ is defined as follows

$f(u_1) = 1\ , f(u_2) = 3$

$h(u_{i+2}) = 4i+3$ for $1 \le i \le m$

The edge labeling $k : E(G) \rightarrow \{2,4,6,8, \dots\dots\dots.., 4i+2\}$ is defined as

$f(w) = 2$

$k(w_i) = 4i+2$ for $1 \le i \le m$

$k(q_i) = 4i$ for $1 \le i \le m$

Clearly $k(w)$ divides $|f(u_1) - f(u_2)|$

$k(w_i)$ divides $\big|\big(f(u_1) - h(u_i)\big)\big|$ for $1 \le i \le m$ and

$k(q_i)$ divides $\big|\big(f(u_2) - h(u_i)\big)\big|$ for $1 \le i \le m$

Hence the graph $P_2 + mK_1$ is an odd-even congruence graph.

Example: 3.12

Consider a graph $P_2 + mK_1$ with $m = 6$

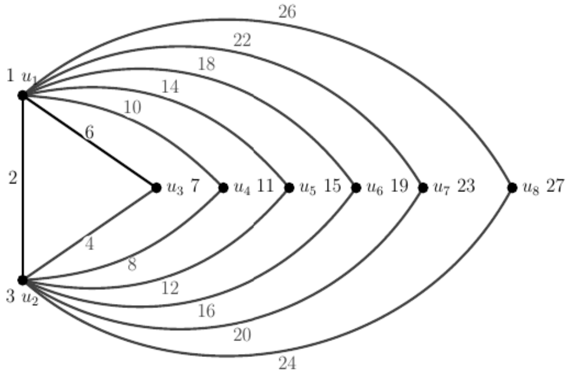

Figure-6

Fig-6 shows that a graph $P_2 + mK_1$ with $m = 6$ admits odd-even congruence labeling.

4. Conclusion

The labelling of graphs is an interesting and vast research area which is very useful and it is extended in various topics by several people. In this paper we have studied Odd-even congruence labeling behaviour of odd-even congruence labeling for friendship graph, shell graph, generalized butterfly graphs, fan graph, $P_2 + mK_1$ graph etc. To derive similar results for other graph families is an open problem.

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