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Bound Inequalities on Degree Sum Energy of Graph

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Abstract

$E_{ds}(G)$,the Degree sum energy of a graph G is the total of all the absolute values of its Degree Sum eigenvalues. In this investigation one upper and lower constraints on the degree Sum energy are obtained in this study.

Keywords: Degree sum matrix, Degree sum eigenvalues, Degree Sum energy.

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1. Introduction

Let us assume that G is a simple graph, and that $V\ (G) = \{v_1, v_2, \dots, v_n\}$ is its vertex set. When the vertices v_i and v_j are adjacent, the adjacency matrix $A(G)$ of the graph G is a square matrix of rank n with the $(i:j) -$ entry equal to unity, otherwise, it is equal to zero. The eigenvalues of the graph G are are $\boldsymbol{\delta_1}, \boldsymbol{\delta_2}, \dots., \boldsymbol{\delta_n}$, of $A(G)$, which are considered to be non-increasing in order.

I. Gutman [6] originally defined the energy of G in 1978 as the total of its eigenvalues absolute values: $\boldsymbol{E(G) = \sum_{k=1}^{n} |\delta_k|}$. There has been a steady flow of articles on this subject since I. Gutman first established the graph energy $E(G)$ of a simple graph G. For basic mathematical properties of the theory of graph energy including its upper and lower bounds one can see [4, 11]. Erich Huckle[8], employed the energy of graphs technique in the early 1930s to develop approximations solutions for a family of organic molecules known as conjugated hydro carbons.

Numerous matrix types, including Incidence [10], Distance [9], Lapalcian [7], Maximum Degree Matrix [1] and others are established and researched for graphs, with inspiration drawn from the adjacency matrix of a graph. In their publication [12], Ramane et al. introduced and investigated sum-degree energy of G, defined as follows:

Let G be a simple graph with connections. The matrix $DSM(G) = [d_{kj}]$ needs to be defined as,

$$d_{kj} = \begin{cases} d_k + d_j, & when\ v_k\ and\ v_j\ are\ adjacent \\ 0 & othewise, \end{cases}$$

This is referred to as G's degree sum matrix.

The degree sum energy DSE of G is then written as $E_{ds}(G) = \sum_{k=1}^{n} |\xi_k|$, where, ξ_k are the eigenvalues of DSM(G), Furthermore, these eigenvalues are real numbers and are sorted in ascending order.

Note that DSM(G) has $trace = 0$, and $\sum_{k=1}^{n} {\xi_k}^2 = 2\mathfrak{E}, where\ \mathfrak{E} = \sum_{1 \le k < j \le n} (d_k + d_j)^2$.

2. Bounds for Degree sum Energy

Throughout this section G denotes a simple graph. This section is aimed to discuss upper and lower bounds for Degree sum Energy (DSE) of G.

Theorem 2.1. Let G be a connected graph with n vertices and m edges and $2\mathfrak{E} \geq n$ then

$$E_{ds}(G) \leq \frac{2\mathfrak{E}}{\boldsymbol{n}} + \frac{\mathbf{1}}{\boldsymbol{n}}\sqrt{2\mathfrak{E}(n-1)(\boldsymbol{n}^{\mathbf{2}} - 2\mathfrak{E})}.$$

Proof: Cauchy-Schwarz inequality states that if $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are $n-\ vectors$ then:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right).$$

For $a_k = 1,\ b_k = |\xi_{\boldsymbol{k}}|$ and $2 \leq k \leq n$, in the above inequality, we obtain

$$\left(\sum_{k=1}^{n} |\xi_{\boldsymbol{k}}|\right)^2 \leq \left(\sum_{k=1}^{n} 1^2\right)\left(\sum_{k=1}^{n} |\xi_{\boldsymbol{k}}|^2\right).$$

Therefore,

$$(E_{ds}(G) - \xi_{\mathbf{1}})^{\mathbf{2}} \leq (\boldsymbol{n} - \mathbf{1}) \sum_{k=1}^{n} \xi_{\boldsymbol{k}}{}^2 = (n-1)(2\mathfrak{E} - \xi_{\mathbf{1}}{}^{\mathbf{2}}),$$

$$\boldsymbol{E_{ds}(G)} = \xi_{\mathbf{1}} + \sqrt{(n-1)(2\mathfrak{E} - \xi_{\mathbf{1}}{}^{\mathbf{2}})}.$$

Now consider the function,

$$\boldsymbol{f(x)} = \boldsymbol{x} + \sqrt{(n-1)(2\mathfrak{E} - \boldsymbol{x}^{\mathbf{2}})}$$

Note that f is decreasing for $x \geq \sqrt{\frac{2\mathfrak{E}}{n}}$, for

$$f'(x) = 1 - \frac{(n-1)x}{\sqrt{(n-1)(2\mathfrak{E} - x^2)}} \leq 0,$$

If and only if, $x \geq \sqrt{\frac{2\mathfrak{E}}{n}}$.

Since, $1 \leq \sqrt{\frac{2\mathfrak{E}}{n}} \leq \frac{2\mathfrak{E}}{n} \leq \xi_{\mathbf{1}}$, we have,

$$f(\xi_{\mathbf{1}}) \leq \boldsymbol{f}\left(\frac{2\mathfrak{E}}{n}\right).$$

Therefore,

$$\boldsymbol{E_{ds}(G)} \leq f(\xi_{\mathbf{1}}) \leq \boldsymbol{f}\left(\frac{2\mathfrak{E}}{n}\right).$$

Hence,

$$\boldsymbol{E_{ds}(G)} \leq \frac{2\mathfrak{E}}{\boldsymbol{n}} + \sqrt{(\boldsymbol{n} - \mathbf{1})\left(2\mathfrak{E} - \left(\frac{2\mathfrak{E}}{n}\right)^2\right)}$$

or equivalently,

$$\boldsymbol{E_{ds}}(\boldsymbol{G}) \leq \frac{2\mathfrak{E}}{\boldsymbol{n}} + \frac{\mathbf{1}}{\boldsymbol{n}}\sqrt{2\mathfrak{E}(n-1)(\boldsymbol{n^2} - 2\mathfrak{E}).}$$

Theorem 2.2. Let G be simple graph connected having order n and size m, then

$$E_{ds}(G) \leq \frac{4\mathfrak{E}}{(\xi_1 - \xi_n)}. \qquad \mathbf{2.1}$$

Proof: Considering**,** $x = x_k \; and \; y = y_k \,, 1 \leq k \leq n$ as real sequence such that $\sum_{k=1}^{n}|x_k| = 1 \; and \; \sum_{k=1}^{n}|x_k| = 0$, the inequality stated below has been proved in[11]:

$$\left|\sum_{k=1}^{n} x_k y_k \right| \leq \frac{1}{2}\left(\max_{1\leq k\leq n}(y_k) - \min_{1\leq k\leq n}(y_k)\right) \qquad 2.2$$

Since, $\sum_{k=1}^{n}|\boldsymbol{\xi_k}| = 0$, for $y_{k} = \xi_k$ and $x_k = \frac{\xi_k}{\sum_{k=1}^{n}|\xi_k|,}$ for each $k \in \{1,2,\dots,3\}$ we have,

$$\sum_{k=1}^{n} x_k = \frac{\sum_{k=1}^{n} \boldsymbol{\xi_k}}{\sum_{k=1}^{n}|\boldsymbol{\xi_k}|} = 0$$

and

$$\sum_{k=1}^{n}|x_k| = \frac{\sum_{k=1}^{n}|\boldsymbol{\xi_k}|}{\sum_{k=1}^{n}|\boldsymbol{\xi_k}|} = \mathbf{1}$$

Thus, the inequality (2.2) holds.

Since, $\sum_{k=1}^{n} {\xi_k}^2 = 2\mathfrak{E}$, we have

$$\left|\sum_{k=1}^{n} x_k y_k \right| = \left|\sum_{k=1}^{n}|\boldsymbol{\xi_k}| \cdot \frac{\boldsymbol{\xi_k}}{\sum_{k=1}^{n}|\boldsymbol{\xi_k}|}\right| = \left|\frac{\sum_{k=1}^{n}(\boldsymbol{\xi_k})^2}{\sum_{k=1}^{n}|\boldsymbol{\xi_k}|}\right| = \frac{2\mathfrak{E}}{E_{DS}(G)}.$$

Applying this in (2.2), we get,

$$\frac{2\mathfrak{E}}{E_{ds}(G)} \leq \frac{\mathbf{1}}{\mathbf{2}}\left(\boldsymbol{max}(\boldsymbol{\xi_k}) - \boldsymbol{min}(\boldsymbol{\xi_k})\right),$$

From which , we have

$$\frac{2\mathfrak{E}}{E_{ds}(G)} \leq \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{\xi_1} - \boldsymbol{\xi_n}).$$

If $G \cong K_n$, then we see that,

$$\boldsymbol{\xi_k} = (\boldsymbol{n} - \mathbf{1})^2, \boldsymbol{\xi_2} = -(\boldsymbol{n} - \mathbf{1}), \dots, \boldsymbol{\xi_n} = -(\boldsymbol{n} - \mathbf{1})$$

and,

$$\boldsymbol{\xi_1} - \boldsymbol{\xi_n} = \boldsymbol{n}(\boldsymbol{n} - \mathbf{1}).$$

So the equality holds in (2.1).

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