Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 1 (2025)

On $\mathcal{P}$-Stable Functions

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Article History:

Received: *14-07-2024*

Revised: *29-08-2024*

Accepted: *10-09-2024*

Abstract

Let h_1, h_2 be two analytic functions defined in the open unit disc $\Delta := \{z \in \mathbb{C}: |z| < 1\}$ which are normalized by the condition $h_1(0) = 1 = h_2(0)$. Then h_1 is $\mathcal{P}$-stable with respect to h_2, whenever

$$\frac{\mathcal{P}_n(h_1, z)}{h_1(z)} \prec \frac{1}{h_2(z)} \quad (z \in \Delta),$$

holds for all $n \in \mathbb{N}$. Here '$\prec$' stands for subordination and $\mathcal{P}_n(h, z) = \mathcal{P}_n(z) * h(z)$ where $\mathcal{P}_n(z)$ denote the n-degree polynomial induced by the $(n+1)$th row entities in an admissible lower triangular matrix. The main purpose of this article is to prove that the function $((Az+1)/(Bz+1))^{\delta}$ is $\mathcal{P}$-stable with respect to $(Bz+1)^{-\delta}$, for $\delta \in (0,1]$ and $-1 \le B < A \le 0$ but not $\mathcal{P}$-stable with respect to itself, when $-1 \le B < A < 0$ and $\delta \in (0,1]$. As an application, considered different admissible lower triangular matrices to derive various results related on stability.

Keywords: Stable functions, $\mathcal{P}$-stable functions, Generalized Cesàro stable functions, Subordinations, Cesàro mean, Janowski function.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of analytic functions h in the unit disc $\Delta = \{z \in \mathbb{C}: |z| < 1\}$. Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be the subclass of $\mathcal{A}$ with the normalization $h(0) = 1$ and $h(0) = h'(0) - 1 = 0$, respectively. The class $\mathcal{S} \subset \mathcal{A}$ consists of all univalent functions in Δ. For $0 \le \gamma < 1$, a function $h \in \mathcal{A}$ belongs to the class $\mathcal{S}^*(\gamma)$ of starlike of order γ and $\mathcal{C}(\gamma)$ of convex of order γ, if h maps conformally the unit disc Δ onto the domains that are starlike and convex while the analytical characterization of these classes are given by $Re\left(\frac{zh'(z)}{h(z)}\right) > \gamma$ and $Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > \gamma$ in Δ, respectively. Also, we denote $\mathcal{S}^*(0) := \mathcal{S}^*$ and $\mathcal{C}(0) := \mathcal{C}$. These subclasses has a proper inclusion as follows: $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{S} \subset \mathcal{A}$. By Alexander transformation, we have $h \in \mathcal{C}$ if and only if $zh' \in \mathcal{S}^*$. A function $h \in \mathcal{A}_1$ is pre-starlike of order γ if $h * \mathcal{K}_\gamma \in \mathcal{S}^*(\gamma)$, where $\mathcal{K}_\gamma(z) = \frac{z}{(1-z)^{2-2\gamma}}$. If h_1, h_2 are analytic functions in Δ, we say h_1 is subordinate to h_2, written $h_1 \prec h_2$, if $h_1 = h_2 \circ \omega$ for some analytic function $\omega: \Delta \to \Delta$ with $\omega(0) = 0$. If h_2 is univalent in Δ, then $h_1 \prec h_2$ if and only if $h_1(\Delta) \subseteq h_2(\Delta)$ and $h_1(0) = h_2(0)$.

For $-1 \leq B < A \leq 1$ and $\delta \in (0,1]$, we define

$$\mathcal{J}_{A,B}^{\delta}(z) = \left(\frac{Az+1}{Bz+1}\right)^{\delta} = 1 + \sum_{j=1}^{\infty} b_j^{A,B}(\delta) z^j \quad (z \in \Delta), \tag{1.1}$$

where $b_j := b_j^{A,B}(\delta) = \sum_{i=0}^{j} \frac{[\delta]_i}{i!} \frac{(\delta)_{j-i}}{(j-i)!} A^i (-B)^{j-i}$. Also,

$$[\delta]_j = \begin{cases} 1, & j = 0 \\ \delta[\delta - 1]_{j-1}, & j \geq 1 \end{cases} \quad \text{and} \quad (\delta)_j = \begin{cases} 1, & j = 0 \\ \delta(\delta + 1)_{j-1}, & j \geq 1 \end{cases}$$

are the factorial polynomials. Moreover, we have

$$\mathcal{J}_{A,B}^{\delta}(z)' + \left(\frac{(B-A)\delta}{1+(B+A)z+BAz^2}\right) \mathcal{J}_{A,B}^{\delta}(z) = 0. \tag{1.2}$$

Note that, from (1.1) for $A = 0$ and $-1 \leq B < 0$, we have

$$\mathcal{J}_{0,B}^{\delta}(z) = (Bz+1)^{-\delta} = 1 + \sum_{j=1}^{\infty} \frac{(\delta)_j}{j!} (-B)^j z^j = 1 + \sum_{j=1}^{\infty} c_j z^j \quad (z \in \Delta). \tag{1.3}$$

Definition 1.1. [11] For $h \in \mathcal{A}_0$ and $0 < c < 1 + b$, the nth Cesàro mean of type $(b-1, c)$ of $h(z) = \sum_{j=0}^{n} a_k z^k$ is defined as

$$\sigma_n^{b-1,c}(h,z) = \frac{1}{B_n} \sum_{j=0}^{n} B_{n-j} b_j z^j = \sigma_n^{b-1,c}(z) * h(z), \quad n \in \mathbb{N} \cup \{0\} \tag{1.4}$$

where $B_j = \frac{(b)_j}{(c)_j} \frac{b-c+1}{b}$ and $B_0 = 1$.

Note that, for $b = 1 + \beta$ and $c = 1$, (1.4) represents nth Cesàro mean of order $\beta \geq 0$ which was studied by Mondal and Swaminathan in [6]. Also note that for $b = 1$ and $c = 1$ in (1.4), then we get nth partial sum of $h \in \mathcal{A}_0$.

Definition 1.2. [11] Let $\mathcal{H}_n (n \in \mathbb{N})$ be a non-empty set consisting of lower triangular matrices $H = (h_{ij})$ of order $(n+1)$ with $h_{ij} \geq 0$, for all $i, j = 0,1,2, \dots, n$ is called admissible lower triangular matrix set if each matrix satisfies the following admissible conditions:

$(i)\ h_{i0} = 1, \quad \forall\ 0 \leq i \leq n,$

$(ii)\ for\ each\ fixed\ i \geq 1, \quad h_{ij} = h_{i1} h_{i-1,j-1}, \quad \forall\ 1 \leq j \leq n,$

$(iii)\ for\ each\ fixed\ i \geq 1,\ \{h_{ij}\}\ is\ a\ decreasing\ sequence.$

Then $(n+1)$th row of each $(h_{ij}) \in \mathcal{H}_n$ induces a n-degree polynomial $\mathcal{P}_n$ defined as

$$\mathcal{P}_n(z) = 1 + \sum_{j=1}^{n} h_{nj} z^j. \tag{1.5}$$

The convolution of $h(z) = \sum_{j=0}^{n} a_k z^k \in \mathcal{A}_0$ with n-degree polynomial $\mathcal{P}_n$ is given by

$$\mathcal{P}_n(h,z) = \mathcal{P}_n(z) * h(z) = 1 + \sum_{j=1}^{n} h_{nj} b_j z^j,\ n \in \mathbb{N}. \tag{1.6}$$

Also we have,

$$\mathcal{P}_n(h,z) = h_{n1}\mathcal{P}_{n-1}(h,z) + \sum_{j=0}^{n-1}\left(h_{nj} - h_{n1}h_{n-1,j}\right)b_j z^j + h_{nn}b_n z^n. \tag{1.7}$$

Here we provide some examples of admissible lower triangular matrix which will be useful in deriving new and existing results from our main results.

Example 1.1. If we choose the admissible matrix as

$$H = (h_{ij}) = \begin{cases} 1 & 0 \le j \le i \\ 0 & j \ge i+1, \end{cases}$$

then (1.5) represents the nth partial sum of $h \in \mathcal{A}_0$.

Example 1.2. For the choice of the admissible matrix

$$H = (h_{ij}) = \begin{cases} 1 & j = 0 \\ \frac{(1+\beta)_{i-j}}{(i-j)!}\frac{i!}{(1+\beta)_i} & 1 \le j \le i \\ 0 & j \ge i+1, \end{cases}$$

then (1.5) represents the nth Cesàro mean of order $\beta \ge 0$.

Example 1.3. If the admissible matrix is

$$H = (h_{ij}) = \begin{cases} 1 & j = 0 \\ \frac{B_{i-j}}{B_i} & 1 \le j \le i \\ 0 & j \ge i+1, \end{cases}$$

then (1.5) represents nth Cesàro mean of type $(b-1,c)$.

Definition 1.3. [5] For fixed $n \in \mathbb{N}$ and $h_1, h_2 \in \mathcal{A}_0$ we say that h_1 is $\mathcal{P}_n$-stable with respect to h_2, if

$$\frac{\mathcal{P}_n(h_1,z)}{h_1(z)} \prec \frac{1}{h_2(z)}. \tag{1.8}$$

In particular, h_1 is $\mathcal{P}_n$-stable with respect to itself then it is just $\mathcal{P}_n$-stable. Suppose the above condition holds for every n, then h_1 is called as $\mathcal{P}$-stable (with respect to h_2).

Note that, for the choices of matrix $H = (h_{ij})$ given in Example 1.1, Example 1.2 and Example 1.3, the equation (1.8) represents the definition of stable[9], Cesàro stable[6] and generalized Cesàro stable[11], respectively.

The concept of stable functions was first introduced by Ruscheweyh and Salinas[9] and they proved that $\mathcal{J}_{0,-1}^{\delta}(z)$ is stable with respect to itself, for $0 < \delta \le 1$. For related works on stable functions and its application on various subclasses, we refer ([1], [2], [8], [10], [12]). Sangal and Swaminathan[11] developed nth Cesàro mean of type $(b-1,c)$, for $b+1 > c > 0$ which give rise to generalized Cesàro stable function and as an application they proved that $\mathcal{J}_{0,-1}^{\delta}(z)$, $(-1 \le \delta \le 1)$ is generalized Cesàro stable with respect to itself. Recently, Jeyaraman and Bhaskar[3] proved that $\mathcal{J}_{A,B}^{\delta}(z)$ is generalized Cesàro stable with respect to $\mathcal{J}_{0,-1}^{\delta}(z)$, for $\delta \in (0,1]$ and $-1 \le B < A \le 0$.

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Mondal $et\ al.$[5] introduced the concept of $\mathcal{P}$-stable function and proved that $\mathcal{J}^{\delta}_{1-2\alpha,-1}$ is $\mathcal{P}$-stable with respect to $\mathcal{J}^{\delta}_{0,-1}$, for $0<\delta\leq 1$ and $1/2\leq\alpha<1$. It is worth mentioning that the authors in [5] considered different admissible lower triangular matrices to derive various results on stability.

In this paper, motivated by the aforesaid works, our aim is to prove that $\mathcal{J}^{\delta}_{A,B}(z)$ is $\mathcal{P}$-stable with respect to $\mathcal{J}^{\delta}_{0,B}(z)$, for $-1\leq B<A\leq 0$ and $\delta\in(0,1]$ but not $\mathcal{P}$-stable with respect to itself, when $-1\leq B<A<0$ and $\delta\in(0,1]$. As an application, for various choices of admissible lower triangular matrices we obtain existing results on stability, Cesàro stability and generalized Cesàro stability.

We need the following Lemmas, in order to prove our main results.

Lemma 1.1. *[2, p.3]* Suppose $\mathcal{J}^{\delta}_{A,B}(z)$ is the function defined in (1.1), then for $\delta\in(0,1]$ and $-1\leq B<A\leq 0$, we have

$$(i)\ \ b_{m_1}\geq 0,$$
$$(ii)\ \ (m_1+1)(m_2+1)b_{m_1+1}+m_1m_2Bb_{m_1}\geq 0,\ \ for\ \ all\ \ m_1,m_2\in\mathbb{N}.$$

Lemma 1.2. *[7, p.54]* Let υ and ν are prestarlike function and starlike function, respectively of order $\gamma\in[0,1)$. Then for any analytic function ω in Δ, we have

$$\frac{\upsilon*(\nu\omega)}{\upsilon*\nu}(\Delta)\subset\overline{co}(\omega(\Delta)),$$

where $\overline{co}(\cdot)$ is the closed convex hull of a set.

Lemma 1.3. *[4, p.57]* Let $-1\leq B<0$ and $\delta_1,\delta_2>0$. If $H_1\prec[\mathcal{J}^{\delta_1}_{0,B}(z)]^{-1}$ and $H_2\prec[\mathcal{J}^{\delta_2}_{0,B}(z)]^{-1}$, then $H_1H_2\prec[\mathcal{J}^{\delta_1+\delta_2}_{0,B}(z)]^{-1}$, for $z\in\Delta$.

2. $\mathcal{P}$-stability of $\mathcal{J}^{\delta}_{A,B}(z)$

For a constant $a\in\mathbb{R}$ and $-1\leq B<A\leq 0$, using (1.1), (1.6) and (1.7) we have the following relations:

$$\left.\begin{aligned}
a\mathcal{P}'_n(\mathcal{J}^{\delta}_{A,B}(z),z)&=\mathcal{P}_n(a\mathcal{J}^{\delta}_{A,B}(z)',z)\quad -a(n+1)h_{nn}b_{n+1}z^n\\
&\quad+a\sum_{j=0}^{n-1}\left(h_{n,j+1}-h_{nj}\right)(j+1)b_{j+1}z^j\\
az\mathcal{P}'_n(\mathcal{J}^{\delta}_{A,B}(z),z)&=\mathcal{P}_n(az\mathcal{J}^{\delta}_{A,B}(z)',z)\\
az^2\mathcal{P}'_n(\mathcal{J}^{\delta}_{A,B}(z),z)&=\mathcal{P}_n(az^2\mathcal{J}^{\delta}_{A,B}(z)',z)\quad +anh_{nn}b_nz^{n+1}\\
&\quad+a\sum_{j=2}^{n}\left(h_{n,j-1}-h_{nj}\right)(j-1)b_{j-1}z^j.
\end{aligned}\right\}\qquad(2.1)$$

Similarly, for a constant $a\in\mathbb{R}$ and $-1\leq B<0$, using (1.3) and (1.6) the following relations hold.

$$\left.\begin{aligned} a\mathcal{P}'_n(\mathcal{J}_{0,B}^{\delta}(z),z) &= h_{n1}\mathcal{P}_{n-1}(a\mathcal{J}_{0,B}^{\delta}(z)',z)\\ az\mathcal{P}'_n(\mathcal{J}_{0,B}^{\delta}(z),z) &= \mathcal{P}_n(az\mathcal{J}_{0,B}^{\delta}(z)',z)\\ \mathcal{P}_n(az\mathcal{J}_{0,B}^{\delta}(z)',z) &= h_{n1}\mathcal{P}_{n-1}(az\mathcal{J}_{0,B}^{\delta}(z)',z) + a\sum_{j=0}^{n-1}(h_{nj}-h_{n1}h_{n-1,j})jc_jz^j + ah_{nn}nc_nz^n. \end{aligned}\right\}$$

(2.2)

Theorem 2.1. Let $H = (h_{ij})$ be admissible lower triangular matrix with $h_{i1} \le 1$, $\forall\ i \ge 1$. Then for $-1 \le B < A \le 0$ and $\delta \in (0,1]$, the function $\mathcal{J}_{A,B}^{\delta}(z)$ is $\mathcal{P}$-stable with respect to $\mathcal{J}_{0,B}^{\delta}(z)$.

Proof. To prove that $\mathcal{J}_{A,B}^{\delta}(z)$ is $\mathcal{P}$-stable with respect to $\mathcal{J}_{0,B}^{\delta}(z)$, we must show that

$$\frac{\mathcal{P}_n(\mathcal{J}_{A,B}^{\delta}(z),z)}{\mathcal{J}_{A,B}^{\delta}(z)} \prec \frac{1}{\mathcal{J}_{0,B}^{\delta}(z)} \quad (z \in \Delta).$$

Therefore, it is enough to prove that

$$\left|\frac{(Bz+1)\left[\mathcal{P}_n(\mathcal{J}_{A,B}^{\delta}(z),z)\right]^{\frac{1}{\delta}}}{(Az+1)} - 1\right| \le 1.$$

For fixed n and δ, let us consider

$$Q(z) = 1 - \frac{(Bz+1)\left[\mathcal{P}_n(\mathcal{J}_{A,B}^{\delta}(z),z)\right]^{\frac{1}{\delta}}}{(Az+1)}.$$

A simple computation yields

$$Q'(z) = \frac{(A-B)\left[\mathcal{P}_n(\mathcal{J}_{A,B}^{\delta}(z),z)\right]^{\frac{1}{\delta}-1}}{(Az+1)^2}\left[\mathcal{P}_n(\mathcal{J}_{A,B}^{\delta}(z),z) - \left(\frac{1+(A+B)z+ABz^2}{\delta(A-B)}\right)\mathcal{P}'_n(\mathcal{J}_{A,B}^{\delta}(z),z)\right]. \quad (2.3)$$

Using (2.1) and (1.2) in (2.3), we have

$$\begin{aligned} Q'(z) = {} & \frac{(A-B)\left[\mathcal{P}_n(\mathcal{J}_{A,B}^{\delta}(z),z)\right]^{\frac{1}{\delta}-1}}{(Az+1)^2}\Bigg[\mathcal{P}_n(\mathcal{J}_{A,B}^{\delta}(z) - \left(\frac{1+(A+B)z+ABz^2}{\delta(A-B)}\right)\mathcal{J}_{A,B}^{\delta}(z)',z)\\ & + \frac{1}{\delta(A-B)}\left((n+1)h_{nn}b_{n+1}z^n - \sum_{j=0}^{n-1}\left(h_{n,j+1}-h_{nj}\right)(j+1)b_{j+1}z^j\right)\\ & - \frac{AB}{\delta(A-B)}\left(nh_{nn}b_n\ z^{n+1} + \sum_{j=2}^{n}\left(h_{n,j-1}-h_{nj}\right)(j-1)b_{j-1}z^j\right)\Bigg]. \end{aligned}$$

$$= \frac{\left[\mathcal{P}_n(\mathcal{J}_{A,B}^{\delta}(z),z)\right]^{\frac{1}{\delta}-1}}{\delta}[\big((n+1)b_{n+1}+\sum_{n_1=1}^{\infty}\left((n_1+1)(n+1)b_{n+1}+n_1nBb_n\right)(-A)^{n_1}z^{n_1}\big)h_{nn}z^n$$
$$+\big((h_{n0}-h_{n1})b_1\big)\big(\sum_{n_1=0}^{\infty}(n_1+1)(-A)^{n_1}z^{n_1}\big)$$
$$+\sum_{j=1}^{n-1}\left(h_{nj}-h_{n,j+1}\right)z^j\big((j+1)b_{j+1}-ABjb_jz\big)\big(\sum_{n_1=0}^{\infty}(n_1+1)(-A)^{n_1}z^{n_1}\big)]$$
$$= \frac{\left[\mathcal{P}_n(\mathcal{J}_{A,B}^{\delta}(z),z)\right]^{\frac{1}{\delta}-1}}{\delta}[\big((n+1)b_{n+1}+\sum_{n_1=1}^{\infty}\left((n_1+1)(n+1)b_{n+1}+n_1nBb_n\right)(-A)^{n_1}z^{n_1}\big)h_{nn}z^n$$
$$+\big((h_{n0}-h_{n1})b_1\big)\big(\sum_{n_1=0}^{\infty}(n_1+1)(-A)^{n_1}z^{n_1}\big)$$
$$+\sum_{j=1}^{n-1}\left(h_{nj}-h_{n,j+1}\right)z^j\big((j+1)b_{j+1}+\sum_{n_1=1}^{\infty}\big((n_1+1)(j+1)b_{j+1}+Bn_1jb_j\big)(-A)^{n_1}z^{n_1}\big)]. \tag{2.4}$$

Using Lemma 1.1 and Definition 1.2, from (2.4) it follows that the expression

$$[\big((n+1)b_{n+1}+\sum_{n_1=1}^{\infty}\left((n_1+1)(n+1)b_{n+1}+n_1nBb_n\right)(-A)^{n_1}z^{n_1}\big)h_{nn}z^n$$
$$+\big((h_{n0}-h_{n1})b_1\big)\big(\sum_{n_1=0}^{\infty}(n_1+1)(-A)^{n_1}z^{n_1}\big)+\sum_{j=1}^{n-1}\left(h_{nj}-h_{n,j+1}\right)z^j$$
$$\big((j+1)b_{j+1}+\sum_{n_1=1}^{\infty}\big((n_1+1)(j+1)b_{j+1}+Bn_1jb_j\big)(-A)^{n_1}z^{n_1}\big)],$$

represents a series of positive Taylor's coefficients about $z=0$. Again by Lemma 1.1, we have $b_j>0$ for all $j\in\mathbb{N}$, it follows that $\mathcal{P}_n(\mathcal{J}_{A,B}^{\delta}(z),z)$ has a series representation with positive Taylor's coefficients about $z=0$ which implies that $|\mathcal{P}_n(\mathcal{J}_{A,B}^{\delta}(z),z)|\le\mathcal{P}_n(\mathcal{J}_{A,B}^{\delta}(z),|z|)$. Applying these results in (2.3), we conclude that $Q'(z)$ is a series with positive Taylor's coefficients about $z=0$. Hence, $|Q'(z)|\le Q'(|z|)$. Since $Q(0)=0$ and $Q(-B)=1$, it follows that

$$|Q(z)|=\left|\int_0^z Q'(t)\ dt\right|\le\int_0^{-B}\left|Q'\left(-\frac{tz}{B}\right)\right|\ dt\le\int_0^{-B}Q'(t)\ dt=1.$$

Therefore, $\mathcal{J}_{A,B}^{\delta}(z)$ is $\mathcal{P}_n$-stable with respect to $\mathcal{J}_{0,B}^{\delta}(z)$ for all $n\in\mathbb{N}$. Hence the proof.

Remark 2.1. (i) For $A=1-2\alpha$ and $B=-1$, Theorem 2.1 reduces to [5, Theorem 2.2].

(ii) For the choices of matrix $H=(h_{ij})$ as given in Example 1.1, Example 1.2 and Example 1.3 in Theorem 2.1, we have the result of stability[2, Theorem 3], Cesàro stability[3, Corollary 2.3] and generalized Cesàro stability[3, Theorem 2.1], respectively.

(iii) If we take $A=1-2\alpha$, $B=-1$ with the choice of matrix $H=(h_{ij})$ as per Example 1.1, Example 1.2 and Example 1.3 in Theorem 2.1, then we have the result obtained in [1, Theorem 2.1], [5, Theorem 2.3] and [5, Theorem 2.4], respectively.

Now, by extending the range of the parameter δ from (0,1] to $[-1,1]$ and letting $A=0$ in Theorem 2.1, we have the following:

Theorem 2.2. Let $H=(h_{ij})$ be admissible lower triangular matrix with $h_{i1}\le 1,\ \forall\ i\ge 1$. Then for $-1\le B<0$ and $\delta\in[-1,1]$, the function $\mathcal{J}_{0,B}^{\delta}(z)$ is $\mathcal{P}$-stable.

Proof. To prove that $\mathcal{J}_{0,B}^{\delta}(z)$ is $\mathcal{P}$-stable, we must show that

$$\frac{\mathcal{P}_n(\mathcal{J}_{0,B}^{\delta}(z),z)}{\mathcal{J}_{0,B}^{\delta}(z)}\prec\frac{1}{\mathcal{J}_{0,B}^{\delta}(z)}\quad(z\in\Delta).$$

Therefore, it is enough to prove that

$$\left|(Bz+1)\left[\mathcal{P}_n(\mathcal{J}_{0,B}^{\delta}(z),z)\right]^{\frac{1}{\delta}}-1\right| \leq 1.$$

For fixed n and δ, let us consider

$$R(z) = 1-(Bz+1)\left[\mathcal{P}_n(\mathcal{J}_{0,B}^{\delta}(z),z)\right]^{\frac{1}{\delta}}. \quad (2.5)$$

We note that for $A = 0$ in (1.2), we get

$$\mathcal{J}_{0,B}^{\delta}(z)' + \left(\frac{B\delta}{1+Bz}\right)\mathcal{J}_{0,B}^{\delta}(z) = 0. \quad (2.6)$$

Clearly for $\delta = 0$, we have $\mathcal{J}_{0,B}^{\delta}(z) = 1$ and hence $|R(z)| \leq 1$. For $\delta \neq 0$, a simple calculation using (2.5) yields

$$R'(z) = (-B)\left[\mathcal{P}_n(\mathcal{J}_{0,B}^{\delta}(z),z)\right]^{\frac{1}{\delta}-1}\left[\mathcal{P}_n(\mathcal{J}_{0,B}^{\delta}(z),z)+\left(\frac{Bz+1}{B\delta}\right)P'_n(\mathcal{J}_{0,B}^{\delta}(z),z)\right]. \quad (2.7)$$

Using (2.2) and (2.6) in (2.7) , we have

$$R'(z) = (-B)\left[\mathcal{P}_n(\mathcal{J}_{0,B}^{\delta}(z),z)\right]^{\frac{1}{\delta}-1}\left[h_{n1}P_{n-1}\left(\mathcal{J}_{0,B}^{\delta}(z)+\frac{Bz+1}{B\delta}\mathcal{J}_{0,B}^{\delta}(z)',z\right)\right.$$
$$\left.+\sum_{j=0}^{n-1}(h_{nj}-h_{n1}h_{n-1,j})\left(c_j+\frac{jc_j}{\delta}\right)z^j+h_{nn}\left(c_n+\frac{nc_n}{\delta}\right)z^n\right].$$

Therefore,

$$R'(z) = (-B)\left[\mathcal{P}_n(\mathcal{J}_{0,B}^{\delta}(z),z)\right]^{\frac{1}{\delta}-1}\left[h_{nn}\left(\frac{(-1)^nB^n(\delta+1)_n}{n!}\right)z^n\right.$$
$$\left.+\sum_{j=0}^{n-1}h_{n1}(h_{n-1,j-1}-h_{n-1,j})\left(\frac{(-1)^jB^j(\delta+1)_j}{j!}\right)z^j\right]. \quad (2.8)$$

Using the Definition 1.2 in (2.8), it follows that the expression

$$\left[h_{nn}\left(\frac{(-1)^nB^n(\delta+1)_n}{n!}\right)z^n+\sum_{j=0}^{n-1}h_{n1}(h_{n-1,j-1}-h_{n-1,j})\left(\frac{(-1)^jB^j(\delta+1)_j}{j!}\right)z^j\right],$$

represents a series of positive Taylor's coefficients about $z = 0$.

Case (i): For $\delta \in (0,1]$, since $c_j > 0$ for all $j \in \mathbb{N}$, $\mathcal{P}_n(\mathcal{J}_{0,B}^{\delta}(z),z)$ has a series representation with positive Taylor's coefficients about $z = 0$. Hence, $|\mathcal{P}_n(\mathcal{J}_{0,B}^{\delta}(z),z)| \leq \mathcal{P}_n(\mathcal{J}_{0,B}^{\delta}(z),|z|)$.

Case (ii): For $\delta \in [-1,0)$, the series (1.3) has coefficients $c_j < 0$ for all $j \in \mathbb{N}$. Thus, we can write

$$\mathcal{P}_n(\mathcal{J}_{0,B}^{\delta}(z),z) = 1+\sum_{j=1}^{n}h_{nj}c_jz^j = 1-\tau(z),$$

where $\tau(z)$ is series with positive Taylor's coefficients about $z = 0$. Therefore,

$$[\mathcal{P}_n(\mathcal{J}_{0,B}^{\delta}(z),z)]^{\frac{1}{\delta}-1} = [1-\tau(z)]^{\frac{1}{\delta}-1} = 1+\sum_{j=1}^{\infty}\frac{(1-\frac{1}{\delta})_j}{j!}(\tau(z))^j.$$

Thus, $[\mathcal{P}_n(\mathcal{J}^{\delta}_{0,B}(z),z)]^{\frac{1}{\delta}-1}$ is a series with positive Taylor's coefficients about $z=0$. Hence, $|[\mathcal{P}_n(\mathcal{J}^{\delta}_{0,B}(z),z)]^{\frac{1}{\delta}-1}| \leq [\mathcal{P}_n(\mathcal{J}^{\delta}_{0,B}(z),|z|)]^{\frac{1}{\delta}-1}$.

From the case (i) and case (ii), we observe that $R'(z)$ is a series with positive Taylor's coefficients about $z=0$. Hence, $|R'(z)| \leq R'(|z|)$. Since $R(0)=0$ and $R(-B)=1$, it follows that

$$|R(z)| = \left|\int_0^z R'(t)\ dt\right| \leq \int_0^{-B} \left|R'\left(-\frac{tz}{B}\right)\right|\ dt \leq \int_0^{-B} R'(t)\ dt = 1.$$

Therefore, $\mathcal{J}^{\delta}_{0,B}(z)$ is $\mathcal{P}_n$-stable for all $n \in \mathbb{N}$. Hence the proof.

Remark 2.2. (i) For the choice of matrix $H=(h_{ij})$ given in Example 1.3, Theorem 2.2 reduces to [3, Theorem 2.2].

(ii) For the choice $B=-1$ and matrix $H=(h_{ij})$ given in Example 1.1, Example 1.2 and Example 1.3, we have the results of [10, Theorem 1.1], [6, Theorem 2.2] and [11, Theorem 2.1], respectively.

Theorem 2.3. Let $H=(h_{ij})$ be admissible lower triangular matrix with $h_{i1} \leq 1$, $\forall\ i \geq 1$. Then for $-1 \leq B < A < 0$ and $\delta \in (0,1]$, the function $\mathcal{J}^{\delta}_{A,B}(z)$ is not $\mathcal{P}$-stable with respect to itself.

Proof. To prove the result, it is enough to show that

$$\frac{\mathcal{P}_n(\mathcal{J}^{\delta}_{A,B}(z),z)}{\mathcal{J}^{\delta}_{A,B}(z)} \nprec \frac{1}{\mathcal{J}^{\delta}_{A,B}(z)} \quad (z \in \Delta),$$

or equivalently, $S(z) = \frac{(Bz+1)\left[\mathcal{P}_n(\mathcal{J}^{\delta}_{A,B}(z),z)\right]^{\frac{1}{\delta}}}{(Az+1)}$ is not subordinate to $T(z) = \frac{Bz+1}{Az+1}$.

If $S(z) \prec T(z)$, then by definition of subordination and Schwarz lemma, we have $|S'(0)| \leq |T'(0)|$ and $S(|z| \leq r) \subseteq T(|z| \leq r)$, where $0 \leq r < 1$.

To prove $S(z) \nprec T(z)$, for $z = \frac{Bw+1}{Aw+1}$, we have to show that there exist atleast a point $z_0 \in \Delta$ with $|z_0| \leq r_0$, for which $S(z_0)$ lies outside the disc, $\left|w - \frac{r^2A-B}{B^2-r^2A^2}\right| \leq \frac{r(A-B)}{B^2-r^2A^2}$, for $-1 \leq B < A \leq 0$.

On choosing $z_0 = 0.907512 + 0.395628i,\ r_0 = 0.99,\ A_0 = -0.4,\ B_0 = -1,\ \delta_0 = 0.4$ and $n=1$, we obtain $\frac{r_0^2A_0-B_0}{B_0^2-r_0^2A_0^2} = 0.721028862$ and $\frac{r_0(A_0-B_0)}{B_0^2-r_0^2A_0^2} = 0.70447257$. To complete the proof, we need to show that $S(|z| \leq r) \nsubseteq T(|z| \leq r)$ for the values of $h_{11} \geq 0$, which is possible for various choices of admissible lower triangular matrices. Here, we provide the table for some values of h_{11}.

It is clear from the Table 1 that $S(z_0)$ lies outside the disc $|w - 0.721028862| \leq 0.70447257$ for the values of h_{11} in the interval [0,1]. Also from the Figure 1(A) and Figure 1(B), we observe that the $S(z)$ is not subordinate to $T(z)$ for the values of $h_{11} = 0.3$ and $h_{11} = 0.5$ respectively. Therefore, $\mathcal{J}^{\delta}_{A,B}(z)$ is not $\mathcal{P}_1$-stable. Hence, $\mathcal{J}^{\delta}_{A,B}(z)$ is not $\mathcal{P}$-stable, which completes the proof.

Sl. No.	Values of h_{11}	$S(z_0)$	$\left\lvert S(z_0) - \frac{r_0^2 A_0 - B_0}{B_0^2 - r_0^2 A_0^2} \right\rvert$
1	0.1	0.311155-0.5744904i	0.7057165
2	0.3	0.373144-0.6225204i	0.7131308
3	0.5	0.4403595-0.6719527i	0.7282141
4	0.8	0.5512408-0.748699i	0.7677097
5	1	0.6320364-0.8015772i	0.8065021

TABLE 1

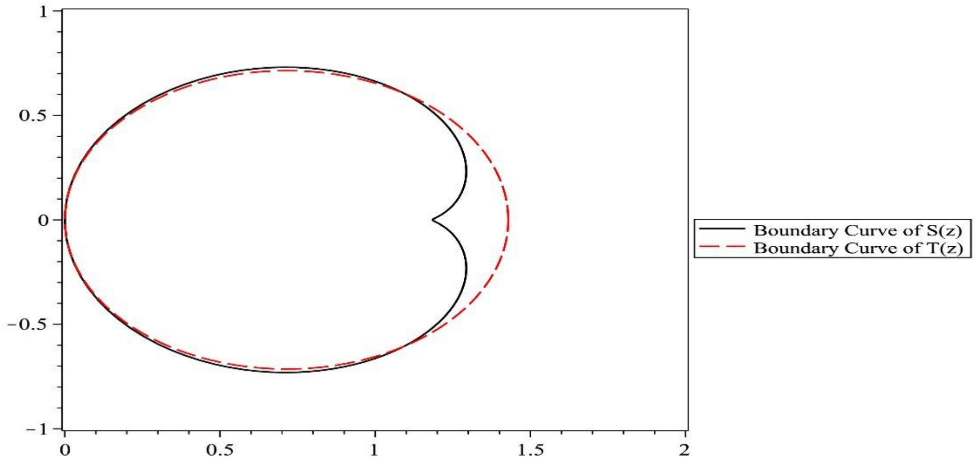

Figure 1 (A) Boundary curves for $h_{11} = 0.3$

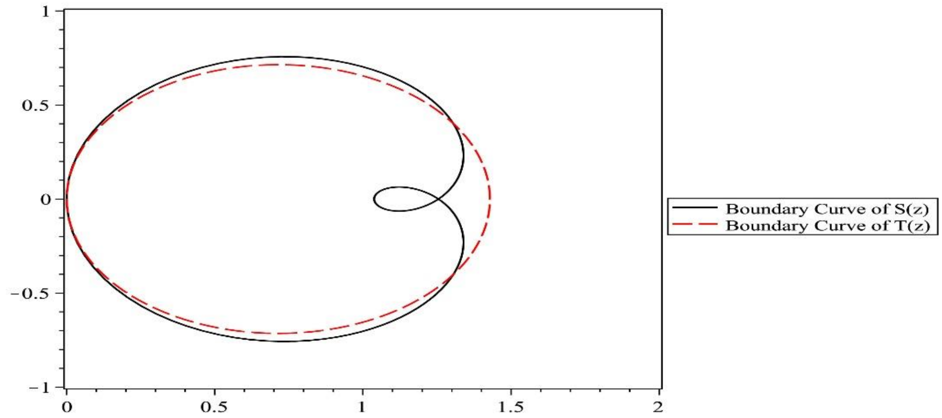

Figure 1 (B) Boundary curves for $h_{11} = 0.5$

Remark 2.3. It is clear that for $\beta \geq 0$, $\delta \in (0,1]$ and $-1 \leq B < A < 0$, $\mathcal{J}_{A,B}^{\delta}(z)$ is not a Cesàro stable with respect to itself.

3. On $\mathcal{P}$-stability of $h \in S^*(\delta)$

Also $zh_\delta(z) = \frac{z}{(1-z)^\delta} \in S^*(1-\delta/2)$, $\delta \in (0,1]$ is an extremal function which plays an important role in studying several properties like growth, distortion, etc., it is observed that for $\delta \in (0,1]$ and $B = -1$ in Theorem 2.2, we have

$$\frac{\mathcal{P}_n(h_\delta,z)}{h_\delta} \prec \frac{1}{h_\delta}. \tag{3.1}$$

Theorem 3.1. Let $h \in S^*(1-\delta/2)$, for $\delta \in (0,1]$. Then

$$\frac{\mathcal{P}_n(h/z,z)}{h(z)/z} \prec (1-z)^\delta \quad (z \in \Delta).$$

Proof. If $h \in S^*(1-\delta/2)$, then $h(z) = zh_\delta * H(z)$, where $H(z)$ is an unique prestarlike function of order $(1-\delta/2)$. Using $h(z) = zh_\delta * H(z)$, we have

$$\frac{\mathcal{P}_n(h/z,z)}{h(z)/z} = \frac{z\mathcal{P}_n(z)*h(z)}{h(z)} = \frac{H(z)*\left[(zh_\delta)\frac{\mathcal{P}_n(h_\delta,z)}{h_\delta}\right]}{H(z)*zh_\delta} \in \overline{co}\left(\frac{\mathcal{P}_n(h_\delta,z)}{h_\delta}(\Delta)\right).$$

By Lemma 1.2, we see that the range of $\frac{\mathcal{P}_n(h/z,z)}{h(z)/z}$ lies in the closed convex hull of image of $\frac{\mathcal{P}_n(h_\delta,z)}{h_\delta}$ under Δ. Now applying (3.1), for $\delta \in (0,1]$, we have

$$\frac{\mathcal{P}_n(h/z,z)}{h(z)/z} \prec (1-z)^\delta \quad (z \in \Delta).$$

Hence, the proof.

Remark 3.1. For the choice of matrix $H = (h_{ij})$ as given in Example 1.1, Example 1.2 and Example 1.3, Theorem 3.1 reduces to [9, Theorem 1.1], [6, Theorem 2.3] and [11, Theorem 2.2], respectively.

Now if we consider $0 < \delta \le \mu \le 1$ in Theorem 2.2, then we have the following result.

Theorem 3.2. Let $-1 \le B < 0$ and $0 < \delta \le \mu \le 1$. Then

$$\frac{\mathcal{P}_n\left(\mathcal{J}^\delta_{0,B}(z),z\right)}{\mathcal{J}^\mu_{0,B}(z)} \prec \frac{1}{\mathcal{J}^\mu_{0,B}(z)} \quad (z \in \Delta).$$

Proof. Using Theorem 2.2 and Lemma 1.3, we have

$$log\left[\frac{\mathcal{P}_n(\mathcal{J}^\delta_{0,B}(z),z)}{\mathcal{J}^\mu_{0,B}(z)}\right]^{\frac{1}{\mu}} = log\left[\frac{\mathcal{P}_n(\mathcal{J}^\delta_{0,B}(z),z)}{\mathcal{J}^{\mu-\delta}_{0,B}(z)\mathcal{J}^\delta_{0,B}(z)}\right]^{\frac{1}{\mu}} = log\left[\frac{1}{\mathcal{J}^{\mu-\delta}_{0,B}(z)}\right]^{\frac{1}{\mu}} + log\left[\frac{\mathcal{P}_n(\mathcal{J}^\delta_{0,B}(z),z)}{\mathcal{J}^\delta_{0,B}(z)}\right]^{\frac{1}{\mu}}$$

$$= log\left[(B\omega_1(z)+1)^{\mu-\delta}\right]^{\frac{1}{\mu}} + log\left[(B\omega_2(z)+1)^{\delta}\right]^{\frac{1}{\mu}} \prec log\left[\frac{1}{\mathcal{J}^\mu_{0,B}(z)}\right]^{\frac{1}{\mu}},$$

for some analytic functions $|\omega_1(z)| \le |z|$ and $|\omega_2(z)| \le |z|$ in Δ. Hence the proof.

Acknowledgements. The authors are thankful to the referees for their insightful suggestions.

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