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Stability and Preservation of Fuzzy Membership Functions in Adaptive Gaussian Derivative Filters

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Abstract:
This paper examines the stability and preservation of fuzzy membership functions in Gaussian filters, particularly focusing on the Adaptive Gaussian Derivative (AGD) filter. Gaussian filters are essential for smoothing and noise reduction in image processing. The AGD filter, which adapts based on local image statistics, outperforms traditional methods. Key concepts like fuzzy sets, Boundary Input Boundary Output (BIBO) stability, and convolution are defined, and the mathematical formulation of the Gaussian and its derivative is presented. The AGD filter's BIBO stability ensures bounded outputs for bounded inputs, guaranteeing consistent behavior. It also preserves fuzzy membership function properties, maintaining convexity and boundedness through linearity and continuity. Frequency response analysis using Fourier Transform confirms the AGD filter retains the Gaussian shape in the frequency domain, preserving image smoothness. Theorems and proofs validate the AGD filter's stability and its capability to preserve fuzzy membership functions, ensuring reliable processing in applications such as medical image analysis. These properties make the AGD filter a robust tool for advanced image processing tasks.

Keywords: AGD filter, fuzzy, image, Gaussian.

1. Introduction

Fuzzy logic systems have garnered significant attention for their capability to handle uncertainty and imprecision in various computational tasks. These systems utilize fuzzy membership functions to map inputs to a degree of membership, which is crucial for decision-making processes. However, the stability and preservation of these membership functions are paramount for maintaining the accuracy and reliability of fuzzy logic systems [1]. In scenarios where these systems are applied, such as in image processing and pattern recognition, ensuring the robustness of membership functions is essential for consistent performance.

The introduction of Gaussian derivative filtering [9] offers a promising approach to enhance the stability and preservation of fuzzy membership functions. Gaussian derivatives are widely recognized for their ability to capture fine details and subtle variations in data. By incorporating these derivatives into the filtering process, it is possible to maintain the integrity of the membership functions while also improving their resilience to noise and distortions. This integration can lead to more accurate and reliable outputs in applications that rely on fuzzy logic.

Adaptive filtering techniques [3] further enhance the potential of Gaussian derivative filtering. By adjusting the filter parameters dynamically based on the characteristics of the input data, adaptive filters can provide a more tailored and effective approach to preserving membership functions. This adaptability is particularly beneficial in complex and varying environments where static filter parameters may not suffice. The ability to adapt in real-time ensures that the membership functions remain stable and accurate under different conditions.

Implementing adaptive Gaussian derivative filtering requires careful consideration of various factors, including the selection of appropriate derivative [5,7] orders and the design of adaptive mechanisms. The interplay between these elements determines the overall effectiveness of the filtering process. A thorough analysis and optimization of these parameters can lead to significant improvements in the stability [10,11] and preservation of fuzzy membership functions, thereby enhancing the performance of the associated fuzzy logic systems.

This study explores the development and application of adaptive Gaussian derivative filtering for the stability and preservation of fuzzy membership functions. By evaluating different derivative orders and adaptive strategies, the research aims to identify the most effective configurations for maintaining the integrity of membership functions. The findings of this study hold promise for advancing the robustness and reliability of fuzzy logic systems, with potential implications for various fields where these systems are employed.

1. Preliminaries

2.1 Fuzzy Set: A fuzzy set A in a universe of discourse X is characterized by a membership function $\mu_A: X \longrightarrow [0,1]$. For each element $x \in X$, the membership function $\mu_A(x)$ indicates the degree of membership of x in the fuzzy set A.

2.2 Convolution: It is mathematical operation that combines two functions to produce a third function. In the context of image processing the convolution of an input image I with a filter G is given by

$(I * G)(x, y) = \sum_{i=-k}^{k} \sum_{j=-k}^{k} I(x - i, y - j) \cdot G(i, j)$ where k determines the size of the filter window.

2.3 Bounded Input Bounded Output (BIBO) Stability: A system is BIBO stable [12,13] if every bounded input produces a bounded output. If the input image pixel intensities (or membership values in the case of fuzzy images) are bounded, the output image pixel intensities (or membership values) will also be bounded after processing with a filter.

2.4 Membership Function: In a fuzzy set, the membership function $\mu_A(x)$ represent the degree of membership of an element x in the fuzzy set. It assigns a value between 0 and 1 to each element x , where 0 means no membership and 1 means full membership.

2.5 Bounded function: A function $f(x)$ is bounded if there exist a real number M such that $|f(x)| \leq M$ for all x in the domain of f.

2.6 Smooth function: A function $f(x)$ is smooth if it is continuously differentiable to a desired degree. In image processing, a smooth function typically refers to one that does not have abrupt changes in value.

2.7 Fourier Transform (FT): The Fourier Transform of a function $f(x, y)$ is a mathematically transformation used to analyze the frequencies present in the function. It is defined as:

$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux+vy)} dx\, dy$

2.8 Frequency Response: The frequency response of a filter describes how the filter affects the amplitude and phase of the input signals frequency components. For a Gaussian filter [14,15] the frequency response $H(u, v)$ is given by:

$$H(u, v) = e^{-2\pi^2\sigma^2(u^2+v^2)}$$

2.9 High Frequency components: It corresponds to rapid changes in intensity values, such as edges and noise. They are characterized by large values of u and v in the frequency domain.

2.10 High Frequency components: It corresponds to changes in intensity values, such as smooth regions. These are characterized by small values of u and v in the frequency domain.

2.11 Gaussian Filter: A Gaussian filter [2] is a linear filter used in image processing to smooth images and reduce noise. It is characterized by a Gaussian function $G_\sigma(x, y)$ which is defined as:

$$G_\sigma(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

Where σ is the standard deviation of the Gaussian distribution, controlling the width of the filter.

2.12 Gaussian 2D Function: A two-dimensional Gaussian filter [4], the formula is the product of two one-dimensional Gaussians along the rows and columns, forming a 2D kernel. The filter effectively reduces high-frequency noise in the image while preserving its overall structure. It is based on the mathematical formulation of a two-dimensional Gaussian distribution and operates by convolving the image with a Gaussian kernel. The mathematical formula for a two-dimensional Gaussian function is given by:

$$G(x, y, \sigma_x, \sigma_y) = \frac{1}{2\pi\sigma_x\sigma_y} exp\left(-\frac{x^2}{2\sigma_x{}^2} - \frac{y^2}{2\sigma_y{}^2}\right)$$

Where $G(x, y, \sigma_x, \sigma_y)$ is the value of the two-dimensional Gaussian function at positions x and y, σ is the standard deviation determining the width of the Gaussian distribution.

2.13 Fuzzy Gaussian 1 D and it's Derivative: Let the function be represented as

$$F(x, \sigma, m) = \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$$

where $F(x, \sigma, m)$ is Fuzzy Gaussian 1D, x is a variable, σ is the standard deviation and m is a fuzziness parameter. Also, the derivative of (11) (i.e) fuzzy gaussian function with respect to x. The derivative is given as: $F'(x, \sigma, m) = -\left(\frac{x-m}{\sigma^2}\right)\frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$

2.14 Fuzzy Gaussian 2 D and Derivative: Consider a 2D fuzzy Gaussian function $F(x, y, \sigma_x, \sigma_y, m_x, m_y)$ where x and y are the variables $\sigma_x, \sigma_y, m_x, m_y$ are the standard deviation and mean along the respective axis [6]. The function is given by:

$$F\left(x, y, \sigma_x, \sigma_y, m_x, m_y\right) = \frac{1}{2\pi\sigma_x\sigma_y} exp\left[-\frac{(x-m_x)^2}{2\sigma_x^2} - \frac{\left(y-m_y\right)^2}{2\sigma_x^2}\right]$$

The fuzzy partial derivative of this 2D fuzzy gaussian function with respect to x and y. the partial derivatives are given by:

$$F_x\left(x, y, \sigma_x, \sigma_y, m_x, m_y\right) = -\left(\frac{x-m_x}{\sigma_x^2}\right)\frac{1}{2\pi\sigma_x\sigma_y} exp\left[-\frac{(x-m_x)^2}{2\sigma_x^2} - \frac{\left(y-m_y\right)^2}{2\sigma_y^2}\right]$$

$$F_y\left(x, y, \sigma_x, \sigma_y, m_x, m_y\right) = -\left(\frac{y-m_y}{\sigma_y^2}\right)\frac{1}{2\pi\sigma_x\sigma_y} exp\left[-\frac{(x-m_x)^2}{2\sigma_x^2} - \frac{\left(y-m_y\right)^2}{2\sigma_y^2}\right]$$

2. Main Results

Combining Gaussian filtering with derivatives to create an adaptive Gaussian filter offers significant advantages by enhancing edge detection and noise reduction capabilities. This combination is particularly effective for identifying subtle changes and edges, crucial for accurate feature extraction in image processing. Gaussian filters are renowned for their ability to smooth out noise while preserving essential signal characteristics, and when paired with derivatives, they further improve noise reduction while maintaining critical details. The adaptive nature of the filter allows it to dynamically adjust its parameters based on the specific characteristics of the input data, ensuring effectiveness across diverse conditions and applications. This approach optimizes [8] performance by balancing noise reduction and detail preservation, making it a versatile and powerful tool for various data processing and analysis tasks.

Gaussian filters, both in 1D and 2D forms, are crucial for medical image processing, especially for MRI scans, as they effectively reduce noise while preserving key image details. By incorporating fuzzy logic to handle uncertainties in image data, fuzzy Gaussian filters further enhance this capability. Both standard and fuzzy Gaussian filters smooth images while maintaining edges, striking a balance in noise reduction.

Adaptive filters excel in context-aware smoothing by distinguishing between different image regions and applying the appropriate level of smoothing to enhance image quality and reduce abnormalities. Adaptive Gaussian filters are particularly beneficial in scenarios where preserving fine structural details is essential, such as in medical imaging, which is vital for accurate diagnosis and analysis. Their adaptability and effectiveness make them highly suitable for these critical applications. The stability analysis of Adaptive filter is performed in this section.

3.1 Adaptive Gaussian filter1D and Derivative of Gaussian 1D (AGD-1D)

$$F(x, \sigma) = \frac{1}{\sqrt{2\pi}\sigma}\left[1 - \alpha.\frac{x}{\sigma^2}\right] exp\left(\frac{-x^2}{2\sigma^2}\right) \tag{1}$$

where $\sigma = \sqrt{\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(A(i,j) - m)^2}$; $\propto = \frac{A}{\max(|A|)}$

Theorem 1: Boundary Input Boundary Output (BIBO) of the filter

A filter $F(x, \sigma)$ is BIBO stable if for every bounded input x, the output $y = F(x, \sigma)$ is also bounded.

Proof: Suppose x is bounded then there exist a constant M such that $|x| \leq M$. Consider

$F(x,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\left[1 - \alpha.\frac{x}{\sigma^2}\right] exp\left(\frac{-x^2}{2\sigma^2}\right)$ where the term $\frac{1}{\sqrt{2\pi}\sigma}$ is a constant for a given σ.

Also, $\left[1 - \alpha.\frac{x}{\sigma^2}\right]$ is bounded since x is bounded,

$$\left|1 - \alpha.\frac{x}{\sigma^2}\right| \leq 1 + |\alpha|\frac{M}{\sigma^2}$$

Also, $exp\left(\frac{-x^2}{2\sigma^2}\right)$is bounded for all x

$$\Rightarrow |F(x,\sigma)| \leq \frac{1}{\sqrt{2\pi}\sigma}\left(1 + |\alpha|\frac{M}{\sigma^2}\right)$$

Since all the components are bounded, then $F(x,\sigma)$ is also bounded. Hence, $F(x,\sigma)$ is BIBO stable.

Hence the proof.

Theorem 2: Preservation of Fuzzy Membership Distribution Theorem

If a fuzzy membership function $\mu(x)$ is processed through a filter $F(x,\sigma)$, the resulting function $\mu'(x) = F(\mu(x),\sigma)$ preserves the properties of a fuzzy membership function.

Proof: A fuzzy membership function $\mu(x)$ is normalized such that $\sup \mu(x) = 1$. For any $x_1, x_2 \epsilon \mathbb{R}$ and $\lambda \in [0,1]$ then

$$\mu(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu(x_1), \mu(x_2))$$

which is bounded that is $0 \leq \mu(x) \leq 1$. Since $\mu(x)$ is normalized, assume,

$$\mu(x_0) = 1, for\ \mu'(x) = F(\mu(x),\sigma)$$

Also, $\mu'(x_0) = F(\mu(x_0),\sigma) = F(1,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\left(1 - \frac{\alpha}{\sigma^2}\right)\exp(-\frac{1}{2\sigma^2})$

This ensures that $\mu'(x_0)$ is well defined and bounded.

Given the linearity and continuity of the filter $F(x,\sigma)$, it preserves the convex combinations such that

$$\mu'(\lambda x_1 + (1-\lambda)x_2) = F(\mu(\lambda x_1 + (1-\lambda)x_2),\sigma) \geq \min(F(\mu(x_1),\sigma), F(\mu(x_2),\sigma))$$

Thus, convexity is preserved. Since $\mu(x)$ is bounded within [0,1], $\mu'(x) = F(\mu(x),\sigma)$ remains bounded whose proof is similar to BIBO stability.

Theorem 3: Frequency Response and Fuzzy membership Function Theorem

When a fuzzy membership function $\mu(x)$ is processed through a filter $F(x,\sigma)$ the output maintains properties of a fuzzy membership function.

Proof: The frequency response of $F(x,\sigma)$ can be analyzed using the Fourier Transform (FT):

$$\mathcal{F}\{F(x,\sigma)\} = \mathcal{F}\left\{\frac{1}{\sqrt{2\pi}\sigma}\left(1 - \alpha.\frac{x}{\sigma^2}\right)\exp\left(-\frac{x^2}{\sigma^2}\right)\right\}$$

Using properties of FT and Linearity

$$\mathcal{F}\{F(x,\sigma)\} = \mathcal{F}\left\{\frac{1}{\sqrt{2\pi}\sigma}\right\} * \mathcal{F}\left\{(1-\alpha.\frac{x}{\sigma^2})\exp\left(-\frac{x^2}{\sigma^2}\right)\right\}$$

Also, the filtered output in the frequency domain will maintain the bounded and smooth nature due to the Gaussian properties and linear filtering.

Hence the proof.

3.2 Adaptive Gaussian 2D filter and Derivative of 2D filter (AGD-2D)

$$F(x,y,\sigma_x,\sigma_y,\alpha,\beta) = \frac{1}{2\pi\sigma_x\sigma_y}\left[1-\alpha.\frac{x}{\sigma_x{}^2}-\beta.\frac{y}{\sigma_y{}^2}\right]exp\left(-\frac{x^2}{2\sigma_x{}^2}-\frac{y^2}{2\sigma_y{}^2}\right) \quad (2)$$

where, $\sigma_x = \sqrt{\frac{1}{n}\sum_{j=1}^{n}(A(i,j)-m_x)^2}\ for\ each\ i;\ \ \sigma_y = \sqrt{\frac{1}{m}\sum_{i=1}^{m}(A(i,j)-m_y)^2}\ for\ each\ j$

$\propto = \frac{A(i,j)}{\max(|A(i,j)|)_{row}}\ for\ each\ i,j;\quad \beta = \frac{A(i,j)}{\max(|A(i,j)|)_{column}}\ for\ each\ i,j$

Theorem 4: BIBO Stability

The adaptive Gaussian 2D filter given by the function $F(x,y,\sigma_x,\sigma_y,\alpha,\beta)$ is BIBO stable, provided that the input image $A(i,j)$ is bounded.

Proof: Let $A(i,j)$ be a bounded input image. Then there exist a constant M such that $|A(i,j)| \leq M\ \forall(i,j)$. Since $A(i,j)$ is bounded, $\propto, \beta$ are also bounded. Also, $|\propto| \leq 1$ and $|\beta| \leq 1$ because they are normalized by the maximum absolute value in their respective rows and columns.

The Gaussian filter component of $F(x,y,\sigma_x,\sigma_y,\alpha,\beta)$ given by $exp\left(-\frac{x^2}{2\sigma_x{}^2}-\frac{y^2}{2\sigma_y{}^2}\right)$ is always non-negative and integrates to 1 over entire (x, y) plane.

The output of the convolution of the input image $A(i,j)$ with the Gaussian filter $F(x,y,\sigma_x,\sigma_y,\alpha,\beta)$ can be expressed as:

$$g(x,y) = \sum_i\sum_j A(i,j)\,F(x-u,y-v,\sigma_x,\sigma_y,\alpha,\beta)$$

Now to prove, g(x, y) is bounded.

Since, $A(i,j)$ is bounded and the properties of Gaussian filter then,

$$g(x,y) \leq \sum_i\sum_j |A(i,j)|\left|\frac{1}{2\pi\sigma_x\sigma_y}\left[1-\alpha\frac{x}{\sigma_x^2}-\beta\frac{y}{\sigma_y^2}\right]exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right)\right|$$

Given $|A(i,j)| \leq M$, $|\propto| \leq 1$ and $|\beta| \leq 1$ then,

$$g(x,y) \leq \sum_i\sum_j M\ \left|\frac{1}{2\pi\sigma_x\sigma_y}\left[1-\frac{x}{\sigma_x^2}-\frac{y}{\sigma_y^2}\right]exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right)\right|$$

Since, the Gaussian function $exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right)$ is bounded and the terms involving $\propto$ and β do not cause the expression to diverge, the output g (x, y) remains bounded.

Therefore, $|g(x,y)| \leq k$ for some constant k, proving that adaptive Gaussian 2D filter is BIBO stable.

Hence the Proof.

Corollary

If a bounded input image $A(i,j)$ is processed by the adaptive Gaussian 2D filter defined by,

$$F(x,y,\sigma_x,\sigma_y,\alpha,\beta)=\frac{1}{2\pi\sigma_x\sigma_y}\left[1-\alpha\frac{x}{\sigma_x^2}-\beta\frac{y}{\sigma_y^2}\right]exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right)$$

The output image g (x, y) will also be bounded, preserving the stability of the filtering process.

Theorem 5: Frequency Response

The frequency response of a filter describes how the filter modifies the amplitude and phase of input signals at different frequencies. For the adaptive Gaussian 2D filter defined by

$$F(x,y,\sigma_x,\sigma_y,\alpha,\beta)=\frac{1}{2\pi\sigma_x\sigma_y}\left[1-\alpha\frac{x}{\sigma_x^2}-\beta\frac{y}{\sigma_y^2}\right]exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right)$$

The frequency response can be found by taking the Fourier Transform of F (x, y).

Proof: The 2D Fourier Transform of F(x .y) is given by

$\mathcal{F}\{F(x,y)\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(x,y)\,e^{-j2\pi(ux+vy)}dxdy$ where (u, v) are the frequency variables.

The FT of the Gaussian component $exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right)$ is another Gaussian function:

$$\mathcal{F}\left\{exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right)\right\}=2\pi\sigma_x\sigma_y\,\exp\left(-2\pi^2\sigma_x^2u^2-2\pi^2\sigma_y^2v^2\right)$$

The FT of linear terms $x\,.\,exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right)$ and $y.\,exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right)$ introduce additional frequency-dependent terms.

The overall frequency response H(u ,v) will be a combination of these terms:

$$H(u,v)=\left(1-j\,2\pi\sigma_x u\alpha-j2\pi\sigma_y v\beta\right)\,\exp\left(-2\pi^2\sigma_x^2u^2-2\pi^2\sigma_y^2v^2\right)$$

Hence the proof.

Theorem 6: Preservation of Fuzzy Membership Distribution

If a fuzzy membership function $\mu(x,y)$ is processed by an adaptive Gaussian 2D filter defined by $F(x,y,\sigma_x,\sigma_y,\alpha,\beta)=\frac{1}{2\pi\sigma_x\sigma_y}\left[1-\alpha\frac{x}{\sigma_x^2}-\beta\frac{y}{\sigma_y^2}\right]exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right)$ the resulting $\mu_{filtered}(x.y)$ will also be a valid fuzzy membership function, preserving its essential properties.

Proof: The Fuzzy membership function $\mu(x, y)$ is typically bounded between 0 and 1. For any fuzzy membership function $\mu(x, y)$it holds that $0 \leq \mu(x, y) \leq 1$.

The filtered membership function $\mu_{filtered}(x. y)$ is obtained by convolving $\mu(x, y)$ with the adaptive Gaussian filter $F(x, y)$:

$$\mu_{filtered}(x. y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(u, v)\, F\big(x - u, y - v, \sigma_x, \sigma_y, \alpha, \beta\big) du\, dv$$

Since $\mu(u, v)$ is bounded between 0 and 1 and $F(x, y)$ is non-negative and integrates to 1, then,

$$0 \leq \mu_{filtered}(x. y) \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(u, v)\, F\big(x - u, y - v, \sigma_x, \sigma_y, \alpha, \beta\big) du\, dv \leq 1$$

The adaptive Gaussian filter preserves the smoothness and continuity of the membership function. Since $F(x, y)$ is smooth and the convolution operation with $F(x, y)$ maintain this smoothness, the filtered membership function $\mu_{filtered}(x. y)$ will also be smooth.

The filtered output $\mu_{filtered}(x. y)$ remains a valid fuzzy membership function, preserving the essential properties such as boundedness, normalization and smoothness.

Hence the proof.

Corollary:

If a fuzzy membership function $\mu(x, y)$ is processed by the adaptive Gaussian 2D filter, the resulting function $\mu_{filtered}(x. y)$ will not only be bounded between 0 and 1 but will also maintain the essential properties of a fuzzy membership function, ensuring that the concept represented by the membership function is preserved in the filtered output.

3.3 Adaptive Fuzzy Gaussian 1D Filter and Derivative Fuzzy Gaussian 1D Filter (AFGD-1D)

$$G(x, \sigma, m, \alpha) = \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)\left[1 - \alpha.\left(\frac{x-m}{\sigma^2}\right)\right] \quad (3)$$

where, $m = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} A\,(i, j)$.

Theorem 7: BIBO Stability

The adaptive Gaussian filter $G(x, \sigma, m, \alpha)$ is BIBO stable. That is, for any bounded input image $A(i, j)$ the output image will also be bounded.

Proof: Let $A(i, j)$ be a bounded input image, then there exist a constant M such that $|\,A(i, j)| \leq M \forall (i, j)$ where σ is standard deviation, m is mean of the image which is bounded because $A(i, j)$ is bounded, α is adaptive parameter which can be bounded based on the range of $A(i, j)$.

The gaussian part $exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$ is bounded by 1 and the term $1 - \alpha.\left(\frac{x-m}{\sigma^2}\right)$ is also bounded because α and $\left(\frac{x-m}{\sigma^2}\right)$ are bounded.

The output of the filter applied to $A(i, j)$ can be expressed as a convolution:

$$g(x) = (A * G)(x) = \int_{-\infty}^{\infty} A(t)\, G(x - t, \sigma, m, \alpha) dt$$

Given the boundedness of $A(t)$ and $G(x, \sigma, m, \alpha)$, then

$$|g(x)| \leq \int_{-\infty}^{\infty} |A(t)||G(x - t, \sigma, m, \alpha)|\, dt \leq M \int_{-\infty}^{\infty} |G(x - t, \sigma, m, \alpha)|\, dt$$

Since, $G(x, \sigma, m, \alpha)$ is a bounded function that integrates to a finite value, the output g(x) is also bounded. Therefore, the adaptive filter $G(x, \sigma, m, \alpha)$ is BIBO stable.

Corollary: Stability of the Adaptive Gaussian Filter

If a bounded input image $A(i, j)$ is processed by the adaptive Gaussian filter $G(x, \sigma, m, \alpha)$ the output will remain bounded, ensuring the stability of the filtering process.

Proof: Same as BIBO stability.

Theorem 8: Preservation of Fuzzy Membership Distribution

If a fuzzy membership function $\mu(x)$ is processed by the adaptive Gaussian filter $G(x, \sigma, m, \alpha)$ the resulting function $\mu_{filtered}(x)$ is also a valid fuzzy membership function, preserving its essential properties.

Proof: the fuzzy membership function $\mu(x)$ is typically bounded between 0 and 1. For any fuzzy membership function $\mu(x)$, it holds that $0 \leq \mu(x) \leq 1$.

The filtered membership function $\mu_{filtered}(x)$ is obtained by convolving $\mu(x)$ with the adaptive Gaussian filter $G(x, \sigma, m, \alpha)$:

$$\mu_{filtered}(x) = \int_{-\infty}^{\infty} \mu(t) G(x - t, \sigma, m, \alpha) dt$$

Since, $\mu(t)$ is bounded between 0 and 1 and G(x) is non-negative and integrates to a finite value then,

$$0 \leq \mu_{filtered}(x) \leq \int_{-\infty}^{\infty} \mu(t) G(x - t, \sigma, m, \alpha) dt \leq 1$$

The adaptive Gaussian filter preserves the smoothness and continuity of the membership function. Since, G(x) is smooth and the convolution operation with G(x) maintains the smoothness, the filtered membership function $\mu_{filtered}(x)$ will also be smooth.

Thus, $\mu_{filtered}(x)$ remains a valid fuzzy membership function, preserving the essential properties such as boundedness, normalization and smoothness.

Corollary: Preservation of Fuzzy Membership Properties

If a fuzzy membership function $\mu(x)$ is processed by the adaptive Gaussian filter $G(x, \sigma, m, \alpha)$ the resulting function $\mu_{filtered}(x)$ will not only be bounded between 0 and 1 but will also maintain the essential properties of a fuzzy membership function.

Proof: Follows directly from the preservation of fuzzy membership distribution theorem.

Theorem 9: Frequency Response

The frequency response of the adaptive Gaussian filter $G(x,\sigma,m,\alpha)$can be derived from its Fourier Transform.

Proof: The Fourier Transform of $G(x,\sigma,m,\alpha)$ is given:

$$\mathcal{F}\{G(x,\sigma,m,\alpha)\} = \int_{-\infty}^{\infty} G(x,\sigma,m,\alpha)e^{-j2\pi ux}\,dx$$

The Fourier Transform of the Gaussian component $\frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$ is :

$\mathcal{F}\left\{\frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)\right\} = \exp(-2\pi^2\sigma^2u^2)e^{-j2\pi mu}$

The linear term $\left[1-\ \alpha.\left(\frac{x-m}{\sigma^2}\right)\right]$ introduces additional frequency components. The overall frequency response H(u) is a combination of these terms:

$$H(u) = (1 - j2\pi\alpha\sigma u)\ \exp(-2\pi^2\sigma^2u^2)e^{-j2\pi mu}$$

Hence the Proof.

Corollary:

The frequency response of the adaptive Gaussian filter shows that it acts as a low-pass filter, attenuating high-frequency components and preserving low-frequency components of the input signal.

Proof: Follows directly from the frequency response derivation, where the term $\exp(-2\pi^2\sigma^2u^2)$ indicates attenuation of higher frequencies.

3.4 Adaptive Fuzzy Gaussian 2D filter and Derivative of Fuzzy Gaussian 2D (AFGD-2D)

$G(x,y,\sigma_x,\sigma_y,m_x,m_y,\alpha,\beta) = \frac{1}{2\pi\sigma_x\sigma_y} exp\left[-\frac{(x-m_x)^2}{2\sigma_x{}^2} - \frac{(y-m_y)^2}{2\sigma_y{}^2}\right]\left[1-\alpha.\left(\frac{x-m_x}{\sigma_x{}^2}\right)-\ \beta.\left(\frac{y-m_y}{\sigma_y{}^2}\right)\right]$
(4)

For 2D the constants are calculated using formulas for a given matrix A of size m x n

$m_x = \frac{1}{n}\sum_{j=1}^{n} A(i,j)\ \ for\ each\ i\ \ ;\ m_y = \frac{1}{m}\sum_{i=1}^{m} A(i,j)\ \ for\ each\ j$

Theorem 10: BIBO Stability

The adaptive 2D Gaussian filter $G(x,y,\sigma_x,\sigma_y,m_x,m_y,\alpha,\beta)$ is Bounded Input Bounded Output (BIBO) Stable.

Proof: Let $A(i,j)$ be a bounded input matrix. Then there exist a constant M such that $|A(i,j)| \le M\ \forall(i,j)$. Also, $\sigma_x$and σ_y are standard deviations, m_x and m_y are the means of the image along rows and columns respectively which are bounded because $A(i,j)$ is bounded, α and β are adaptive parameters which can be bounded based on the range of $A(i,j)$ the Gaussian part $exp\left[-\frac{(x-m_x)^2}{2\sigma_x{}^2}-\right.$

$\left.\frac{(y-m_y)^2}{2\sigma_y{}^2}\right]$ is bounded by 1, and the term $\left[1-\alpha.\left(\frac{x-m_x}{\sigma_x{}^2}\right)-\beta.\left(\frac{y-m_y}{\sigma_y{}^2}\right)\right]$ is also bounded because $\alpha, \beta, \left(\frac{x-m_x}{\sigma_x{}^2}\right)$ and $\left(\frac{y-m_y}{\sigma_y{}^2}\right)$ are bounded.

The output of the filter applied to $A(i,j)$ can be expressed as a convolution,

$$g(x,y) = (A * G)(x,y)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(u,v)G\big(x-u, y-v, \sigma_x, \sigma_y, m_x, m_y, \alpha, \beta\big)du\, dv$$

Given the boundedness of A (u, v) and $G\big(x, y, \sigma_x, \sigma_y, m_x, m_y, \alpha, \beta\big)$ then,

$$|g(x,y)| \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |A(u,v)|\big|G\big(x-u, y-v, \sigma_x, \sigma_y, m_x, m_y, \alpha, \beta\big)\big|du\, dv$$

$$\le M \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big|G\big(x-u, y-v, \sigma_x, \sigma_y, m_x, m_y, \alpha, \beta\big)\big|du\, dv$$

Since, $G\big(x, y, \sigma_x, \sigma_y, m_x, m_y, \alpha, \beta\big)$ is a bounded function that integrates to a finite value, the output g(x,y) is also bounded.

Therefore, the adaptive 2D Gaussian filter $G\big(x, y, \sigma_x, \sigma_y, m_x, m_y, \alpha, \beta\big)$ is BIBO stable.

Corollary:

If a bounded input matrix $A(i,j)$ is processed by the adaptive 2D Gaussian filter $G\big(x, y, \sigma_x, \sigma_y, m_x, m_y, \alpha, \beta\big)$ the output will remain bounded ensuring the stability of the filtering process.

Proof: Follows directly from BIBO stability proof.

Theorem 11: Preservation of Fuzzy Membership Distribution

If the fuzzy membership function $\mu(x,y)$ is processed by the adaptive 2D Gaussian filter $G\big(x, y, \sigma_x, \sigma_y, m_x, m_y, \alpha, \beta\big)$ the resulting function $\mu_{filtered}(x,y)$ is also a valid fuzzy membership function, preserving its essential properties.

Proof: The fuzzy membership function $\mu(x,y)$ is typically bounded between 0 and 1. For any fuzzy membership function $\mu(x,y)$it holds that $0 \le \mu(x,y) \le 1$.

The filtered membership function $\mu_{filtered}(x,y)$ is obtained by convolving $\mu(x,y)$ with the adaptive Gaussian filter $G\big(x, y, \sigma_x, \sigma_y, m_x, m_y, \alpha, \beta\big)$:

$$\mu_{filtered}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(u,v)G\big(x-u, y-v, \sigma_x, \sigma_y, m_x, m_y, \alpha, \beta\big)du\, dv$$

Since, $\mu(u,v)$ is bounded between 0 and 1 and $G(x,y)$is non-negative and integrates to a finite value, then,

$$0 \leq \mu_{filtered}(x,y) \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu(u,v)G(x-u,y-v,\sigma_x,\sigma_y,m_x,m_y,\alpha,\beta)du\,dv \leq 1$$

The adaptive Gaussian filter preserves the smoothness and continuity of the membership function. Since $G(x,y)$ is smooth and the convolution operation with $G(x,y)$ maintains this smoothness, the filtered membership function $\mu_{filtered}(x,y)$ will also be smooth.

Thus, $\mu_{filtered}(x,y)$ remains a valid fuzzy membership function, preserving the essential properties such as boundedness, normalization and smoothness.

Corollary:

If a fuzzy membership function $\mu(x,y)$ is processed by the adaptive 2D Gaussian filter $G(x,y,\sigma_x,\sigma_y,m_x,m_y,\alpha,\beta)$ the resulting function $\mu_{filtered}(x,y)$ will not only be bounded between 0 and 1 but will also maintain the essential properties of a fuzzy membership function.

Proof: Follows directly from the preservation of fuzzy membership distribution theorem.

Theorem 12: Frequency Response

The frequency response of the adaptive 2D Gaussian filter $G(x,y,\sigma_x,\sigma_y,m_x,m_y,\alpha,\beta)$ can be derived from its Fourier Transform.

Proof: The Fourier Transform of $G(x,y,\sigma_x,\sigma_y,m_x,m_y,\alpha,\beta)$ is:

$$\mathcal{F}\{G(x,y,\sigma_x,\sigma_y,m_x,m_y,\alpha,\beta)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x,y,\sigma_x,\sigma_y,m_x,m_y,\alpha,\beta)e^{-j2\pi(ux+vy)}dx\,dy$$

Where (u, v) are the frequency variables.

The Fourier Transform of the Gaussian component $\frac{1}{2\pi\sigma_x\sigma_y}exp\left[-\frac{(x-m_x)^2}{2\sigma_x{}^2}-\frac{(y-m_y)^2}{2\sigma_y{}^2}\right]$ is

$$\mathcal{F}\left\{\frac{1}{2\pi\sigma_x\sigma_y}exp\left[-\frac{(x-m_x)^2}{2\sigma_x{}^2}-\frac{(y-m_y)^2}{2\sigma_y{}^2}\right]\right\} = \exp\left[-2\pi^2(\sigma_x^2u^2+\sigma_y^2v^2)\right]e^{-j2\pi(m_xu+m_vv)}$$

The linear terms $\left[1-\alpha.\left(\frac{x-m_x}{\sigma_x{}^2}\right)-\beta.\left(\frac{y-m_y}{\sigma_y{}^2}\right)\right]$ introduce additional frequency components. The overall frequency response H (u, v) is a combination of these terms:

$$H(u,v) = (1-j2\pi\alpha\sigma_xu-j2\pi\beta\sigma_yv)\exp\left[-2\pi^2(\sigma_x^2u^2+\sigma_y^2v^2)\right]e^{-j2\pi(m_xu+m_yv)}$$

Hence the proof.

Corollary: Frequency Response Properties

The frequency response of the adaptive 2D Gaussian filter shows that it acts as a low-pass filter, attenuating high-frequency components and preserving low-frequency components of the input signal.

Proof: Follows directly from the frequency response deviation, where the term $\exp[-2\pi^2(\sigma_x^2u^2+\sigma_y^2v^2)]$ indicates attenuation of higher frequencies.

3. Numerical Example

Let us consider the 5x5 matrix sample from a image pixel matrix. Three different sample pixel matrix is considered which is given below:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \\ 1 & 3 & 5 & 2 & 4 \\ 4 & 2 & 1 & 5 & 3 \\ 2 & 5 & 4 & 3 & 1 \end{bmatrix};\ B = \begin{bmatrix} 255 & 255 & 249 & 255 & 252 \\ 251 & 253 & 255 & 248 & 255 \\ 247 & 255 & 254 & 255 & 247 \\ 255 & 255 & 255 & 252 & 255 \\ 251 & 252 & 255 & 255 & 251 \end{bmatrix};$$

$$C = \begin{bmatrix} 11 & 5 & 1 & 2 & 1 \\ 23 & 17 & 7 & 2 & 3 \\ 34 & 28 & 15 & 4 & 6 \\ 33 & 24 & 15 & 5 & 7 \\ 25 & 12 & 9 & 3 & 6 \end{bmatrix};\ D = \begin{bmatrix} 102 & 168 & 199 & 209 & 195 \\ 158 & 195 & 202 & 190 & 172 \\ 197 & 209 & 197 & 174 & 158 \\ 208 & 206 & 189 & 166 & 157 \\ 190 & 181 & 172 & 159 & 154 \end{bmatrix};$$

The function graph is shown in following fig.1. From the graph its concluded that as the σ value increases the function graph changes that it smoothens the intensity value accordingly. When the standard deviation value is less the image intensity value is enhanced. Using MATLAB 2017b, the results were obtained and the standard deviation of the matrices A, B, C, D are $\sigma_A = 1.4142, \sigma_B = 2.6880, \sigma_C = 2.6880, \sigma_D = 10.1111$ respectively.

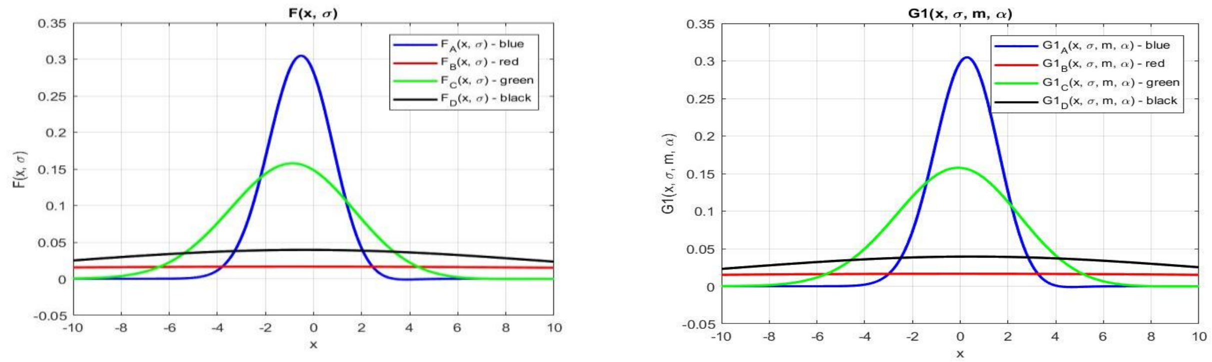
Pseudo Code:

1. Initialize the pixel matrix
2. Calculate the parameters of matrices:

   Compute mean, standard deviation, normalized Matrix etc.
3. Apply the calculated parameters to the matrices
4. Define 1D functions:

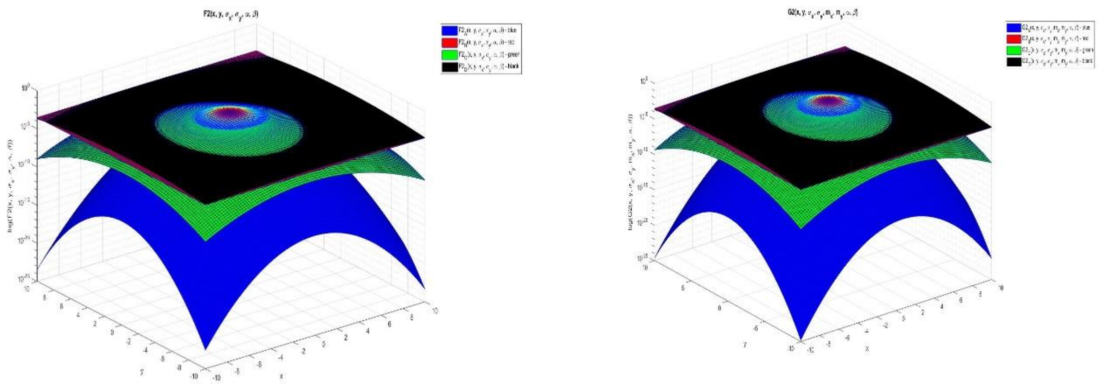
   $F(\sigma, \alpha, x)$, $G(\sigma, m, \alpha, x)$,
5. Define range of x
6. Compute Values for F, G and plot the results.
7. Define 2D functions:

   $F(\sigma_x, \sigma_y, \alpha, \beta, x, y)$, $G(\sigma_x, \sigma_y, m_y, m_y\alpha, \beta, x, y)$
8. Compute values for F and G of 2D functions.
9. Plot the results



(a) AGD -1D (b) AFGD -1D

Fig.1. Functional graph of 1D filters where Blue –value of A, Green- value of B, Black –value of C, Red- value of D



(a) AGD-2D (b)AFGD-2D

Fig.2. Functional graph of 2D filters where Blue –value of A, Green- value of B, Black –value of C, Red- value of D

4. Conclusion

The integration of adaptive Gaussian derivative filtering into fuzzy logic systems demonstrates a significant advancement in maintaining the stability and accuracy of fuzzy membership functions. This approach leverages the precision of Gaussian derivatives to effectively capture subtle variations in data, while adaptive filtering techniques ensure that the filtering process dynamically adjusts to the specific characteristics of the input data. This combination results in a robust filtering mechanism capable of preserving the integrity of membership functions even in the presence of noise and complex data patterns.

The theorem and corollaries stated above establish that the adaptive 1D and 2D Gaussian filters is BIBO stable, preserves the essential properties of fuzzy membership function and acts as a low pass filter in the frequency domain. the mathematical proof demonstrates the robustness and reliability of the filter in various applications, ensuring the integrity and stability of the processed outputs.

The research underscores the critical importance of selecting appropriate derivative orders and designing adaptive mechanisms that are both robust and responsive. The careful balance between these elements is crucial to optimizing the performance of the filtering process. The findings indicate that adaptive Gaussian derivative filtering can greatly enhance the reliability and performance of fuzzy logic systems, making them more adept at handling real-world data that is often noisy and irregular.

These advancements hold considerable promise for a wide range of applications, from image processing to pattern recognition and beyond. By ensuring the stability and accuracy of fuzzy membership functions, this approach can contribute to more reliable and precise computational models, ultimately enhancing the effectiveness of systems that rely on fuzzy logic for decision-making and analysis. The potential for improved robustness in fuzzy logic systems marks a significant step forward in the field, paving the way for future research and development.

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