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Existence and Approximate Controllability of Random Impulsive Neutral Functional Differential Equation with Finite Delay

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Abstract:

This study investigates second-order neutral functional differential equations with delays, prevalent in various scientific and engineering fields. These equations, characterized by their neutral nature and delays, present unique challenges within Banach spaces. The research focuses on the existence and approximate controllability of solutions, using advanced mathematical tools like cosine family theory and the Leray-Schauder theorem to establish rigorous solution conditions. These theoretical results are empirically validated through practical examples, enhancing understanding of real-life behavior and bridging theory with practice. The study’s findings advance the understanding of delayed feedback systems, facilitating effective control strategies and practical engineering solutions, thereby contributing significantly to dynamical systems and control theory.

Keywords: Differential equation; Lerray-Schauder fixed point; mild solution; finite delay; semigroup theory; approximate controllability.

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1 Introduction

In the area of mathematical analysis and its interdisciplinary applications, a diverse array of theories, techniques, and models has emerged to address complex phenomena across various scientific domains. This introduction highlights a selection of seminal works and recent research contributions that delve into the intricate landscapes of neutral functional differential equations, impulsive systems, controllability theories, and interdisciplinary interactions bridging mathematics with physics. N. U. Ahmed’s seminal work, "Semigroup Theory with Applications to Systems and Control" [1], serves as a cornerstone in understanding the fundamental principles of semigroup theory and its versatile applications in systems and control theory. Ahmed’s text provides a comprehensive exploration of semigroups, offering insights into their algebraic structures and their pivotal role in analyzing dynamic systems and control processes.

The works by Baghli and Benchohra [2] delve into the uniqueness and existence results for partial and neutral functional differential equations in Frechet spaces, shedding light on the intricate dynamics of these equations with infinite delay. Additionally, Lupulescu and Lungan [5] contribute to the field by studying random integral equations on time scales, offering novel perspectives on the interplay between randomness and differential equations. Gunasekar et all [12,23,24] explore the existence results for nonlocal impulsive neutral functional integro-differential equations, unraveling the complexities of impulsive systems with nonlocal interactions. Furthermore, Baleanu et al. investigate the approximate controllability of second-order nonlocal impulsive functional integro-differential systems in Banach spaces, providing valuable insights into the controllability properties of such systems.

Recent research has also focused on exploring the synergies between physics, mathematics, and computer science. Hazra et al. [21] present a modeling framework that elucidates the interdisciplinary interactions among these fields, fostering a deeper understanding of complex phenomena. Similarly, Han et al. [22] delve into the formation of trade networks, highlighting the role of economies of scale and product differentiation in shaping global economic dynamics.

In mathematical and control theory research, various studies delve into the analysis and controllability of complex dynamical systems, aiming to understand their behavior and design effective control strategies. The research by Baleanu et al. focuses on the approximate controllability of second-order nonlocal impulsive functional integro-differential systems in Banach spaces. Their study investigates the ability to steer such systems arbitrarily close to desired states using control inputs. This research contributes to understanding the controllability properties of systems with impulsive and nonlocal behaviors.

Anguraj et al. explore the existence results for an impulsive neutral functional differential equation with state-dependent delay. By analyzing the existence of solutions to this equation, the study provides theoretical insights into the behavior of impulsive systems with state-dependent delays.

These references collectively contribute to advancing our understanding of the controllability properties of complex dynamical systems, particularly those involving impulses, delays, and nonlocal interactions. They provide valuable theoretical insights and mathematical techniques for analyzing and designing control strategies for such systems, with implications for various scientific and engineering applications.

The Second order impulsive neutral functional differential equation with delay and random effects is of the form.

$$\begin{aligned}
\frac{d}{dh}[\varphi'(h,\aleph)+\rho(h,\varphi_h(.,\aleph),\aleph)] &= A\varphi(h,\aleph)+\Upsilon(h,\varpi,\varphi_h(.,\aleph),\aleph);\;\; h\in J=(0,\varrho], h\neq h_\xi,\\
\varphi(0,\aleph) &= \phi_0(\aleph),\;\; \xi=1,2,3,\dots,m\\
\varphi'(0,\aleph) &= \phi'_0(\aleph),\\
\Delta\varphi(h_\xi,\aleph) &= I_\xi\left(\varphi(h_\xi,\aleph)\right),\\
\Delta'\varphi(h_\xi,\aleph) &= I_{\xi'}\left(\varphi(h_\xi,\aleph)\right).
\end{aligned} \tag{1}$$

The approximate controllability of random impulsive neutral functional differential equation with finite delay.

$$\begin{aligned}
\frac{d}{dh}[\varphi'(h,\aleph)+\rho(h,\varphi_h(.,\aleph),\aleph)] &= A\varphi(h,\aleph)+\Upsilon(\mathrm{h},\varphi_h(.,\aleph),\aleph)+By(h,\aleph); \quad h\in J\\
\varphi(0,\aleph) &= \phi_0(\aleph),\\
\varphi'(0,\aleph) &= \phi'_0(\aleph),\\
\Delta\varphi(h_\xi,\aleph) &= I_\xi(\varphi(h_\xi,\aleph)),\\
\Delta'\varphi(h_\xi,\aleph) &= I_{\xi'}(\varphi(h_\xi,\aleph)).
\end{aligned} \tag{2}$$

A symbolizes the infinitesimal source of a constantly evolving set of cosine transformations denoted by $\{T_1(h): h\in\mathbb{R}\}$, where these transformations are bounded linear operations occurring within a Banach Space $\mathcal{S}$, defined with the norm $||.||$ and $\Upsilon: J\times J\times\mathcal{D}\times\Omega\to\mathcal{S}, \rho: J*\mathcal{D}\times\Omega\to\mathcal{S}$ are continuous functions and $B:\mathcal{E}\to\mathbb{R}_+$ where $\mathcal{E}$ is the banach space and Ω is a random operator in a stochastic domain.

2 Preliminaries

In this section, we'll review fundamental concepts and terminology that are essential for understanding the key findings of our study. Lately, there's been more interest in studying a specific type of problem involving how things change over time, even when the speed of change isn't fixed.

$$\varphi''(h,\aleph)=A\varphi(h,\aleph)+\Upsilon(h,\aleph),\quad 0\le h\le\varrho\quad \varphi(0,\aleph)=x_0(\aleph),\quad \varphi'(0,\aleph)=y_0(\aleph) \tag{3}$$

Here, $A: D(A)\subseteq\mathcal{S}\to\mathcal{S}$, where $h\in J=[0,\varrho]$, denotes a closed operator that is densely defined. Furthermore, let $\Upsilon: J\times\Omega\to\mathcal{S}$ denote an appropriate function. Numerous studies have examined equations of this nature. Typically, the solutions of the problem is linked to the presence of an evolution operator $T_2(h,\varpi)$ for the corresponding homogeneous equation.

$$\varphi''(h,\aleph)=A\varphi(h,\aleph),\quad 0\le\varpi,h\le\varrho, \tag{4}$$

Definition 1 Let $(\mathcal{D},\|\cdot\|_{\mathcal{D}})$ be a seminormed linear space of functions defined on $(-\delta,0]$ and taking values in a Banach space $\mathcal{S}$. The space $\mathcal{D}$ satisfies the following axioms:

(A)For any continuous function $\varphi:(-\delta,0]\to\mathcal{S}$ and $\phi_0\in\mathcal{D}$, the following conditions hold for all $h\in J$

1. The function $\varphi_h\in\mathcal{D}$.
2. There exists a positive constant K such that

$$|\varphi(h,\aleph)|\le K\parallel\varphi_h(\cdot,\aleph)\parallel_{\mathcal{D}}.$$

Furthermore, there exist functions $U,\vartheta,\vartheta':\mathbb{R}_+\to\mathbb{R}_+$, where U is continuous and bounded, and ϑ,ϑ' are locally bounded and independent of φ, such that

$$\parallel\varphi_h(\cdot,\aleph)\parallel_X\le U(h)\sup\{|\varphi(m,\aleph)|: -\delta\le m\le 0\}+\vartheta\parallel\phi_0(\aleph)\parallel_{\mathcal{D}}+\vartheta'\parallel\phi_{0'}(\aleph)\parallel_{\mathcal{D}}.$$

(B)The function φ_h is $\mathcal{D}$-valued and continuous on J for the functions φ described in (A).

(C)The space $\mathcal{D}$ is complete.

Definition 2 A collection of bounded linear maps $\{T_1(h): h \in \mathcal{J}\}$ in the Banach space $\mathcal{S}$ is considered a strongly continuous cosine function when it meets these criteria

1. Addition Condition: $T_1(\varpi + h) + T_1(\varpi - h) = 2T_1(\varpi)T_1(h)$ for all $\varpi, h \in \mathcal{J}$.
2. Identity Property: $T_1(0) = I$, where I denotes the identity operator.
3. Continuity Requirement: $T_1(h)\varphi$ remains continuously dependent on h over $\mathcal{J}$ for any fixed $\varphi \in \mathcal{S}$.

In this scenario, A serves as the fundamental element behind a continuously evolving set of operations known as the strongly continuous cosine function, denoted by $\{T_1(h): h \in \mathcal{J}\}$. These operations involve bounded linear maps defined within the Banach space $\mathcal{S}$, where distances are measured using the norm $\|\cdot\|$. The associated sine function with $\{T_1(h): h \in \mathbb{R}\}$, denoted as $\{T_2(h): h \in \mathcal{J}\}$, is expressed as

$$T_2(h)\varphi = \int_0^h T_1(\varpi)\varphi \, d\varpi \quad \text{for} \varphi \in \mathcal{D} \text{and} h \in \mathcal{J}.$$

Additionally, ϑ and ϑ_a represent positive constants ensuring $\| T_1(h) \| \leq \vartheta$ and $\| T_2(h) \| \leq \vartheta_a$ for every $h \in J$.

Definition 3 Approximate controllability, an essential concept in control theory, addresses the capability to roughly guide a system from one state to another utilizing control inputs within a designated timeframe.

Formally, system represented by a state space $\mathcal{D}$, where we can influence its behavior through admissible control inputs from the space U. The evolution of this system is described by an equation:

$$\varphi'(h, \aleph) = A\varphi(h, \aleph) + By(h, \aleph),$$

where $\varphi(h, \aleph) \in \mathcal{D}$ denotes the system's state at time h, $y(h, \aleph) \in U$ denotes the control input, A is the system's operator or matrix, and B is the control operator or matrix.

Approximate controllability is achieved if, given any starting state x_0 and any desired terminal state x_f, there is a sequence of control inputs $\{\varphi_\xi(h, \aleph)\}$ such that the system's solution $\zeta(h, \aleph)$ of the dynamical system satisfies $x(0, \aleph) = \phi_0(\aleph)$ and $\lim_{\xi \to \infty} x_\xi(\varrho, \aleph) = \phi'_0(\aleph)$ for some finite time ϱ, where $x_\xi(h, \aleph)$ is the system"s solution resulting from the control input $\varphi_\xi(h, \aleph)$.

Lemma 1 (Leray-Schauder Nonlinear Alternative)

Let us denote a Banach space $\mathcal{S}$. Inside $\mathcal{S}$, there's a closed and convex subset Z. Within Z, there's a relatively open subset U containing the point 0. Then, there's a mapping $\Upsilon: U \to Z$ that's compact, meaning it preserves the "closeness" of points when mapping from U to Z. In that case, either

1. Υ possesses a fixed point in U, or
2. A point $\zeta \in \partial U$ satisfies $\zeta \in \lambda\Upsilon(\zeta)$ for some $\lambda \in (0,1)$.

Lemma 2 A set $\mathcal{D} \subset \mathcal{S}$ is relatively compact in $\mathcal{S}$ if and only $\overline{\mathcal{D}}_\xi$ is relatively compact in $C[(h_\xi, h_{\xi+1}]; \mathcal{S})$ for each $\xi = 0,1, \dots, n$.

Now, let's discuss how we can determine if the equations are approximately controllable within their interior, without involving impulses, delays, or nonlocal conditions. To do this, we consider the following scenario: For any starting point φ_0 within our space $\mathcal{S}$ and any function y belonging to the L^2 space over the interval $(0, \varrho]$ with values in U, we examine the initial-value problem:

$$\varphi'(h,\aleph) = A\varphi(h,\aleph) + By(h,\aleph), \quad \varphi \in \mathcal{S}, \qquad \varphi(0,\aleph)) = \varphi_0(\aleph), \tag{5}$$

where the control function φ belongs to $L^2(0,\varrho;U)$, has precisely one mild solution represented by

$$\varphi(h,\aleph) = T(h)\varphi_0(\aleph) + \int_0^h T(h-\varpi)By(\varpi,\aleph)\,d\varpi, \quad h \in (0,\varrho].$$

Definition 4 For the above system , the controllability mapping $G: L^2((0,\varrho];U) \to \mathcal{S}$ is defined for $h > 0$ as follows:

$$Gu = \int_0^h T(h-\varpi)By(\varpi,\aleph)\,d\varpi.$$

The corresponding adjoint operator $G^*: \mathcal{S} \to L^2((0,\varrho];\mathcal{S})$ is determined by the rule

$$(G^*\varphi)(\varpi) = B^*T^*(\varrho-\varpi)\varphi \quad \forall \varpi \in [0,\varrho], \forall z \in \mathcal{S}.$$

Consequently, the Grammian operator $W: \mathcal{S} \to \mathcal{S}$ is

$$k\varphi = GG^*\varphi = \int_0^\tau T(\varrho-\varpi)BB^*T^*(\varrho-\varpi)\,d\varpi.$$

Remark 1 The series of linear operators $(\Gamma(\aleph))_\alpha: \mathcal{S} \to L^2((0,\varrho];U)$, where $0 < \alpha \leq 1$, can be defined as follows:

$$(\Gamma(\aleph))_\alpha\varphi = B^*T^*(\cdot)(\alpha I + GG^*)^{-1}\varphi = G^*(\alpha I + GG^*)^{-1}\varphi, \quad (3.6)$$

This set of operators fulfills the condition:

$$\lim_{\alpha\to 0} G(\Gamma(\aleph))_\alpha = I,$$

in the strong topology.

3 Existence Results

In this section, we show that there are solutions to the problem described by equations (1.1). To do this, we list some conditions we'll need to consider.

Definition 5 If $\varphi_0 = \emptyset$ and the continuous function $x: (0,\varrho] \times \Omega \to \mathcal{S}, T > 0$ and $\mathcal{D} = C[(-\delta,\varrho],\mathcal{S}]$ solves the integral equation then it is considered a mild solution to equation (1.1).

$$\begin{aligned}\zeta(h,\aleph) = \; & T_1(h)\phi_0(\aleph) + T_2(h)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)] - \int_0^h T_1(h-\varpi)\rho(\varpi,\varphi_\varpi(.,\aleph),\aleph)d\varpi \\ & + \int_0^h T_2(h-\varpi)\Upsilon(\varpi,\varphi_\varpi(.,\aleph),\aleph)d\varpi + \sum_{0<h_\xi<h} T_1(h-h_\xi)I_\xi(\varphi(h_\xi,\aleph)) \\ & + \sum_{0<h_\xi<h} T_2(h-h_\xi)I'_\xi(\varphi(h_\xi,\aleph)).\end{aligned}$$

For your convenience, we have listed the hypotheses that will be discussed in the following section.

(G_1)There exist a continuous function $a_0, b_0, c_0, d_0: J \times \Omega \to \mathbb{R}$ such that

$$||\Upsilon(h,x,\aleph)|| \leq a_0(\aleph)||x,\aleph||_{\mathcal{D}}^{\alpha_0} + b_0(\aleph)$$

$$||\rho(h,x,\aleph)|| \le c_0(\aleph)||x,\aleph||_{\mathcal{D}}^{\beta_0} + d_0(\aleph)$$

for all $x \in \mathcal{S}, \aleph \in \Omega$

(G_2)(i)For all $h, \varpi \in J$, the function $\Upsilon(h,.,.) : \mathcal{D} \times \Omega \to \mathcal{S}$ is continuous and for all $(x,\aleph) \in \mathcal{D} \times \Omega$ the function $\Upsilon(.,x,\aleph): J \to \mathcal{S}$ is strongly measurable.

(ii) For all $h \in J$, the function $\Upsilon(h,.,.) : \mathcal{D} \times \Omega \to \mathcal{S}$ is continuous and for all $(x,\aleph) \in \mathcal{D} \times \Omega$ the function $\Upsilon(.,x,\aleph): J \to \mathcal{S}$ is strongly measurable.

(G_4)Let $I_\xi, I'_\xi \in C(\mathcal{S},\mathcal{S}), \xi = 1,2,3,\ldots,m$ are all compact operator

$$||I_\xi(h,x,\aleph)|| \le a_\xi(\aleph)||x,\aleph||_{\mathbb{R}}^{\alpha_\xi}$$

$$||I'_\xi(h,x,\aleph)|| \le a'_\xi(\aleph)||x,\aleph||_{\mathbb{R}}^{\alpha_\xi}$$

(G_5)The function $\Upsilon: J \times J \times \mathcal{D} \times \Omega \to \mathcal{S}, \rho: J \times \mathcal{D} \times \Omega \to \mathcal{S}$ is continuous, and a constant $\mathcal{L}$ exists for which

$$||\Upsilon(h,x_1,\aleph) - \Upsilon(h,x_2,\aleph)|| \le \mathcal{L}(\aleph)(||(x_1,\aleph) - (x_2,\aleph)||_{\mathcal{D}}^{\alpha_0})$$

$$||\rho(h,x_1,\aleph) - \rho(h,x_2,\aleph)|| \le \mathcal{L}1(\aleph)(||(x_1,\aleph) - (x_2,\aleph)||_{\mathcal{D}}^{\beta_0})$$

(G_6) There is a random funtion $R: \Omega \to \mathbb{R}^+$ such that

$$\vartheta||\phi_0|| + \vartheta_a||\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)|| + \vartheta_a h b_0(\aleph) + \vartheta h d_0(\aleph) +$$
$$\vartheta \sum_{0<h_\xi<h} a_\xi(\aleph)||\zeta(h_\xi,\aleph)||_{\mathbb{R}}^{\alpha_\xi} + \vartheta_a \sum_{0<h_\xi<h} a'_\xi(\aleph)||\zeta(h_\xi,\aleph)||_{\mathbb{R}}^{\alpha_\xi} +$$
$$\vartheta_a \int_0^h ||[\sup_{\varpi\in(0.\varrho]} ||\zeta_\varpi(.,\aleph),\aleph||_{\mathcal{D}}^{\alpha_0} d\varpi + \vartheta \int_0^h ||[\sup_{\varpi\in(0.\varrho]} ||\zeta_\varpi(.,\aleph),\aleph||_{\mathcal{D}}^{\beta_0} d\varpi \le R(\aleph).$$

Theorem 3.1 Assuming that conditions (G_1)-(G_6) are met, the problem described in (1.1) will have a mild random solution on the interval $(0,\varrho]$.

Proof: Let us consider a map $(\Gamma(\aleph)): \Omega \times PC_\delta = [(-\delta,\varrho],\mathcal{S}] \to PC_\delta$ be a random operator is defined by $(\Gamma(\aleph))\zeta(h)$ where $h \in (-\delta,\varrho]$

$$(\Gamma(\aleph))\zeta(h,\aleph) =$$
$$\begin{cases} \Upsilon(h,\aleph), & h \in (-\delta,\varrho] \\ T_1(h)\phi_0(\aleph) + T_2(h)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)] & \\ -\int_0^h T_1(h-\varpi)\rho(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi + \int_0^h T_2(h-\varpi)\Upsilon(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi & \\ +\sum_{0<h_\xi<h} T_1(h-h_\xi)I_\xi(\zeta(h_\xi,\aleph)) + \sum_{0<h_\xi<h} T_2(h-h_\xi)I'_\xi(\zeta(h_\xi,\aleph)), & h,\varpi \in J \end{cases} \quad (6)$$

We aim to illustrate that $(\Gamma(\aleph))$ satisfies all conditions outlined in Lemma 2.1. To enhance clarity, the proof will be broken down into many stages.

Step 1: The mapping $(\Gamma(\aleph))$ takes sets that are bounded and maps them into other bounded sets.

To demonstrate this, it's sufficient to establish that there is a $+_{ve}$ constant $r(\aleph)$ so that for every $\zeta \in \mathcal{B}_r(\delta)$, where δ is defined as follows:

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$$\mathcal{B}_{\mathrm{r}}(\delta) := \left\{ \zeta \in \mathrm{PC}_{\delta} : \sup_{\delta \le h \le \varrho} \| \zeta(\mathrm{h}, \aleph) \| \le \mathrm{r}(\aleph) \right\}$$

one has $\| (\Gamma(\aleph))\zeta \|_{\mathrm{PC}} \le \mathrm{R}(\aleph)$.

$$\begin{aligned}
||(\Gamma(\aleph))\zeta(h)|| \le\ & ||T_1(h)\phi_0(\aleph)|| + ||T_2(h)[\phi'_0(\aleph) + \rho(0, \phi_0(\aleph), \aleph)]|| \\
& + \int_0^h ||T_1(h-\varpi)\rho(\varpi, \zeta_\varpi(.,\aleph), \aleph)d\varpi|| + ||\int_0^h T_2(h-\varpi)\Upsilon(\varpi, \zeta_\varpi(.,\aleph), \aleph)||d\varpi \\
& + ||\sum_{0<h_\xi<h} T_1(h-h_\xi)I_\xi(\zeta(h_\xi, \aleph))|| + ||\sum_{0<h_\xi<h} T_2(h-h_\xi)I'_\xi(\zeta(h_\xi, \aleph))||d\varpi \\
& \le \vartheta||\phi_0(\aleph)|| + \vartheta_a||\phi'_0(\aleph) + \rho(0, \phi_0(\aleph), \aleph)|| + \vartheta\int_0^h ||\rho(\varpi, \zeta_\varpi(.,\aleph), \aleph)d\varpi|| \\
& + \vartheta_a\int_0^h ||\Upsilon(\varpi, \zeta_\varpi(.,\aleph), \aleph)||d\varpi + \vartheta\sum_{0<h_\xi<h} ||I_\xi(\zeta(h_\xi, \aleph))|| + \vartheta_a\sum_{0<h_\xi<h} ||I'_\xi(\zeta(h_\xi, \aleph))|| \\
& \le \vartheta||\phi_0|| + \vartheta_a||\phi'_0(\aleph) + \rho(0, \phi_0(\aleph), \aleph)|| + \vartheta\int_0^h [\sup_{\varpi\in(0.\varrho]} ||\zeta_\varpi(.,\aleph), \aleph||_{\mathcal{D}}^{\beta_0} + d_0(\aleph)]d\varpi \\
& + \vartheta_a\int_0^h [\sup_{\varpi\in(0.\varrho]} ||\zeta_\varpi(.,\aleph), \aleph||_{\mathcal{D}}^{\alpha_0} + b_0(\aleph)]d\varpi + \vartheta\sum_{0<h_\xi<h} a_\xi(\aleph)||\zeta(h_\xi, \aleph)||_{\mathbb{R}}^{\alpha_\xi} \\
& + \vartheta_a\sum_{0<h_\xi<h} a'_\xi(\aleph)||\zeta(h_\xi, \aleph)||_{\mathbb{R}}^{\alpha_\xi}
\end{aligned}$$

$$\begin{aligned}
& \le \vartheta||\phi_0|| + \vartheta_a||\phi'_0(\aleph) + \rho(0, \phi_0(\aleph), \aleph)|| + \vartheta_a h b_0(\aleph) + \vartheta h d_0(\aleph) + \vartheta\sum_{0<h_\xi<h} a_\xi(\aleph)||\zeta(h_\xi, \aleph)||_{\mathbb{R}}^{\alpha_\xi} \\
& + \vartheta_a\sum_{0<h_\xi<h} a'_\xi(\aleph)||\zeta(h_\xi, \aleph)||_{\mathbb{R}}^{\alpha_\xi} + \vartheta_a\int_0^h ||[\sup_{\varpi\in(0.\varrho]} ||\zeta_\varpi(.,\aleph), \aleph||_{\mathcal{D}}^{\alpha_0} d\varpi + \\
& \vartheta\int_0^h ||[\sup_{\varpi\in(0.\varrho]} ||\zeta_\varpi(.,\aleph), \aleph||_{\mathcal{D}}^{\beta_0} d\varpi \\
& \le R(\aleph)
\end{aligned}$$

Hence $(\Gamma(\aleph))$ is bounded set in PC_δ.

Step 2: We now show that $(\Gamma(\aleph))$ is continuous on $\mathcal{B}_r(\delta)$.

Let us consider that for $\zeta_1, \zeta_2 \in \mathcal{B}_r(\delta)$, $h \in J$,

$$
\begin{aligned}
||(\Gamma(\aleph))\zeta_1(h)-(\Gamma(\aleph))\zeta_2(h)|| \leq\ & ||\int_0^h T_1(h-\varpi)\rho(\varpi,\zeta_{1,\varpi}(.,\aleph),\aleph)-\rho(\varpi,\zeta_{2,\varpi}(.,\aleph),\aleph)||d\varpi \\
&+||\int_0^h T_2(h-\varpi)\Upsilon(\varpi,\zeta_{1,\varpi}(.,\aleph),\aleph)-\Upsilon(\varpi,\zeta_{2,\varpi}(.,\aleph),\aleph)||d\varpi \\
&+||\sum_{0<h_\xi<h} T_1(h-h_\xi)[I_\xi(\zeta_1(h_\xi,\aleph))-I_\xi(\zeta_2(h_\xi,\aleph))]|| \\
&+||\sum_{0<h_\xi<h} T_2(h-h_\xi)[I'_\xi(\zeta_1(h_\xi,\aleph))-I'_\xi(\zeta_2(h_\xi,\aleph))]||d\varpi \\
&\leq \vartheta\int_0^h ||\rho(\varpi,\zeta_{1,\varpi}(.,\aleph),\aleph)-\rho(\varpi,\zeta_{2,\varpi}(.,\aleph),\aleph)||d\varpi \\
&+\vartheta_a\int_0^h ||\Upsilon(\varpi,\zeta_{1,\varpi}(.,\aleph),\aleph)-\Upsilon(\varpi,\zeta_{2,\varpi}(.,\aleph),\aleph)||d\varpi \\
&+\vartheta\sum_{0<h_\xi<h} ||I_\xi(\zeta_1(h_\xi,\aleph))-I_\xi(\zeta_2(h_\xi,\aleph))||+ \\
&\vartheta_a\sum_{0<h_\xi<h} ||I'_\xi(\zeta_1(h_\xi,\aleph))-I'_\xi(\zeta_2(h_\xi,\aleph))|| \\
&\leq \vartheta\int_0^h \sup_{\varpi\in(0.\varrho]} ||(\zeta_{1,\varpi}(.,\aleph),\aleph)-(\zeta_{2,\varpi}(.,\aleph),\aleph)||_{\mathcal{D}}^{\beta_0}d\varpi \\
&+\vartheta_a\int_0^h \sup_{\varpi\in(0.\varrho]} ||(\zeta_{1,\varpi}(.,\aleph),\aleph)-(\zeta_{2,\varpi}(.,\aleph),\aleph)||_{\mathcal{D}}^{\alpha_0}d\varpi \\
&+\vartheta\sum_{0<h_\xi<h} a_\xi(\aleph)||\zeta_1(h_\xi,\aleph)-\zeta_2(h_\xi,\aleph)||_{\mathbb{R}}^{\alpha_\xi} \\
&+\vartheta_a\sum_{0<h_\xi<h} a'_\xi(\aleph)||\zeta_1(h_\xi,\aleph)-\zeta_2(h_\xi,\aleph)||_{\mathbb{R}}^{\alpha_\xi}
\end{aligned}
$$

$$
\begin{aligned}
||(\Gamma(\aleph))\zeta_1(h)-(\Gamma(\aleph))\zeta_2(h)|| \leq\ & \vartheta\textstyle\sum_{0<h_\xi<h} a_\xi(\aleph)||\zeta_1(h_\xi,\aleph)-\zeta_2(h_\xi,\aleph)||_{\mathbb{R}}^{\beta_\xi} \\
&+\vartheta\textstyle\int_0^h \sup_{\varpi\in(0.\varrho]} ||(\zeta_{1,\varpi}(.,\aleph),\aleph)-(\zeta_{2,\varpi}(.,\aleph),\aleph)||_{\mathcal{D}}^{\beta_0}d\varpi \\
&+\vartheta_a\textstyle\sum_{0<h_\xi<h} a'_\xi(\aleph)||\zeta_1(h_\xi,\aleph)-\zeta_2(h_\xi,\aleph)||_{\mathbb{R}}^{\alpha_\xi} \\
&+\vartheta_a\textstyle\int_0^h \sup_{\varpi\in(0.\varrho]} ||(\zeta_{1,\varpi}(.,\aleph),\aleph)-(\zeta_{2,\varpi}(.,\aleph),\aleph)||_{\mathbb{R}}^{\alpha_0}d\varpi
\end{aligned}
$$

for all $h\in(-\delta,\varrho]$, and their compactness for $h>0$ proves the uniform operator topology is continuous. Since $\zeta_1,\zeta_2\in\mathcal{B}_r(\delta)$, the righthand side of the above inequalities are independent.Therefore $||((\Gamma(\aleph))\zeta_1)(h)-((\Gamma(\aleph))\zeta_2)(h)||\to 0$ as $(\zeta_1-\zeta_2)\to 0$. Hence $(\Gamma(\aleph))$ is continuous.

Step 3: The operator $(\Gamma(\aleph))$ is compact.

To establish this, we decompose $(\Gamma(\aleph))$ into $(\Gamma_1(\aleph))+(\Gamma(\aleph))_2$, where both $(\Gamma_1(\aleph))$ and $(\Gamma_2(\aleph))$

are operators acting on $\mathcal{B}_r(\delta)$. Specifically, they are characterized as follows

$$(\Gamma_1(\aleph))\zeta(h) = \quad T_1(h)\phi_0(\aleph) + T_2(h)[\phi'_0(\aleph) + \rho(0, \phi_0(\aleph), \aleph)] - \int_0^h T_1(h-\varpi)\rho(\varpi, \zeta_\varpi(.,\aleph), \aleph)d\varpi$$

$$+ \int_0^h T_2(h-\varpi)\Upsilon(\varpi, \zeta_\varpi(.,\aleph), \aleph)d\varpi \quad \text{and}$$

$$(\Gamma_2(\aleph))\zeta(h) = \quad \sum_{0<h_\xi<h} T_1(h-h_\xi)I_\xi(\zeta(h_\xi, \aleph)) + \sum_{0<h_\xi<h} T_2(h-h_\xi)I'_\xi(\zeta(h_\xi, \aleph)), \text{ forall } h \in (-\delta, \varrho].$$

We will begin by demonstrating that $(\Gamma_1(\aleph))$ is a compact operator.

(i)The set $(\Gamma_1(\aleph))(\mathcal{B}_r(\delta))$ exhibits equicontinuity.

Now, consider $\delta \leq h_1 < h_2 \leq \varrho$ and let $\epsilon > 0$ be small. then

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$$\begin{aligned}
||(\Gamma_1(\aleph))\zeta(h_2) - \quad & (\Gamma_1(\aleph))\zeta(h_1)|| \leq ||[T_1(h_2) - T_1(h_2)]\phi_0(\aleph)|| + ||[T_2(h_2) - T_2(h_1)][,\aleph)]|| \\
& + ||\int_0^{h_1} T_1(h-\varpi)\rho(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi - \int_0^{h_2} T_1(h-\varpi)\rho(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi|| \\
& + ||\int_0^{h_1} T_2(h_2-\varpi)\Upsilon(\varpi,\zeta_\varpi(.,\aleph),\aleph) - \int_0^{h_2} T_2(h_1-\varpi)\Upsilon(\varpi,\zeta_\varpi(.,\aleph),\aleph)||d\varpi \\
& \leq ||[T_1(h_2) - T_1(h_2)]\phi_0(\aleph)|| + ||[T_2(h_2) - T_2(h_1)][\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)]|| \\
& + ||\int_0^{h_1-\epsilon} (T_1(h_2-\varpi) - T_1(h_1-\varpi))\rho(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi|| \\
& + ||\int_0^{h_1-\epsilon} (T_2(h_2-\varpi) - T_2(h_1-\varpi))\Upsilon(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi|| \\
& + ||\int_{h_1-\epsilon}^{h_1} T_1(h_2-\varpi) - T_1(h_1-\varpi))\rho(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi|| \\
& + ||\int_{h_1-\epsilon}^{h_1} T_2(h_2-\varpi) - T_2(h_1-\varpi))\Upsilon(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi|| \\
& + ||\int_{h_1}^{h_2} T_1(h_2-\varpi) - T_1(h_1-\varpi))\rho(\varpi,\zeta_\varpi(.,\aleph),\aleph)||d\varpi \\
& + ||\int_{h_1}^{h_2} T_2(h_2-\varpi) - T_2(h_1-\varpi))\Upsilon(\varpi,\zeta_\varpi(.,\aleph),\aleph)||d\varpi \\
& \leq ||[T_1(h_2) - T_1(h_2)]\phi_0(\aleph)|| + ||[T_2(h_2) - T_2(h_1)][\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)]|| \\
& + \int_0^{h_1-\epsilon} ||T_1(h_2-\varpi) - T_1(h_1-\varpi)||[\sup_{\varpi\in(0.\varrho]} ||\zeta_\varpi(.,\aleph),\aleph||_{\mathcal{D}}^{\beta_0} + d_0(\aleph)]d\varpi \\
& + \int_0^{h_1-\epsilon} ||T_2(h_2-\varpi) - T_2(h_1-\varpi)||[\sup_{\varpi\in(0.\varrho]} ||\zeta_\varpi(.,\aleph),\aleph||_{\mathcal{D}}^{\alpha_0} + b_0(\aleph)]d\varpi \\
& + \int_{h_1-\epsilon}^{h_1} ||T_1(h_2-\varpi) - T_1(h_1-\varpi)||[\sup_{\varpi\in(0.\varrho]} ||\zeta_\varpi(.,\aleph),\aleph||_{\mathcal{D}}^{\beta_0} + d_0(\aleph)]d\varpi \\
& + \int_{h_1-\epsilon}^{h_1} ||T_2(h_2-\varpi) - T_2(h_1-\varpi)||[\sup_{\varpi\in(0.\varrho]} ||\zeta_\varpi(.,\aleph),\aleph||_{\mathcal{D}}^{\alpha_0} + b_0(\aleph)]d\varpi \\
& + \int_{h_1}^{h_2} ||T_1(h_2-\varpi) - T_1(h_1-\varpi)||[\sup_{\varpi\in(0.\varrho]} ||\zeta_\varpi(.,\aleph),\aleph||_{\mathcal{D}}^{\beta_0} + d_0(\aleph)]d\varpi \\
& + \int_{h_1}^{h_2} ||T_2(h_2-\varpi) - T_2(h_1-\varpi)||[\sup_{\varpi\in(0.\varrho]} ||\zeta_\varpi(.,\aleph),\aleph||_{\mathcal{D}}^{\alpha_0} + b_0(\aleph)]d\varpi
\end{aligned}$$

We observe that as $h_2 - h_1$ approaches zero, $||(\Gamma_1(\aleph))\zeta(h_2) - (\Gamma(\aleph))\zeta(h_1)||$ tends to zero regardless of $\zeta \in \mathcal{B}_r(\delta)$. Because the operator $T_2(h)$ is compact for $h > 0$, it ensures continuity in the uniform operator topology. As a result, $(\Gamma_1(\aleph))$ transforms $\mathcal{B}_r(\delta)$ into a family of functions that are equicontinuous.

Next, we need to demonstrate that the set $(\Gamma_1(\aleph))(\mathcal{B}_r(\delta))(h)$ is precompact within $\mathcal{S}$.

Consider fixed values $\delta < h \leq \varpi \leq \varrho$, and let ϵ be a real number such that $0 < \epsilon < h$. For $\zeta \in$

$B_r(\delta)$, $((\Gamma_1(\aleph)),\epsilon)(h)$ is given by

$$T_1(h)\phi_0(\aleph)+T_2(h)[\phi'_0(\aleph)+\rho(0,\phi_0(\aleph),\aleph)]-\int_0^{h-\epsilon}T_2(h-\varpi)\rho(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi$$
$$+\int_0^{h-\epsilon}T_2(h-\varpi)\Upsilon(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi$$

Utilizing the compactness property of $T_2(h)$ for $h>0$, we establish that the set $\{((\Gamma(\aleph))_{1,\epsilon}\zeta)(h):\zeta\in B_r(\delta)\}$ is precompact for $\zeta\in B_r(\delta)$ and $0<\epsilon<h$. Additionally, for each $\zeta\in B_r(\delta)$, we ensure that

$$\begin{aligned}||((\Gamma_1(\aleph))\zeta)(h)-((\Gamma(\aleph))_{1,\epsilon}\zeta)(h)|| \leq\ & \int_{h-\epsilon}^{h}||T_1(h-\varpi)\rho(\varpi,\eta,\zeta_\varpi(.,\aleph),\aleph)|| \\ & +\int_{h-\epsilon}^{h}||T_2(h-\varpi)\Upsilon(\varpi,\zeta_\varpi(.,\aleph),\aleph)|| \\ \leq\ & \vartheta\int_{h-\epsilon}^{h}[\sup_{\varpi\in(0.\varrho]}||\zeta_\varpi(.,\aleph),\aleph||_{\mathcal{D}}^{\beta_0}+d_0(\aleph)]d\varpi \\ & +\vartheta_a\int_{h-\epsilon}^{h}[\sup_{\varpi\in(0.\varrho]}||\zeta_\varpi(.,\aleph),\aleph||_{\mathcal{D}}^{\alpha_0}+b_0(\aleph)]d\varpi\end{aligned}$$

Thus, there is a precompact sets that can be made arbitrarily close to the set $\{((\Gamma_1(\aleph))\zeta):\zeta\in B_r(\delta)\}$. Therefore, the set $\{((\Gamma_1(\aleph))\zeta):\zeta\in B_r(\delta)\}$ is relatively compact in $\mathcal{S}$. It's evident that $(\Gamma_1(\aleph))(B_r(\delta))$ is bounded uniformly.Since we have established that $(\Gamma_1(\aleph))(B_r(\delta))$ forms an equicontinuous family, the Arzelà-Ascoli theorem indicates that it is sufficient to show that $(\Gamma_1(\aleph))$ maps $B_r(\delta)$ into a relatively compact set in $\mathcal{S}$.

Now, we need to confirm that $(\Gamma(\aleph))_2$ is a compact operator as well. By applying Lemma 2.1, we establish its complete continuity. The property of $(\Gamma(\aleph))_2$ being continuous can be demonstrated by considering the state space. Conversely, for $r>0$, $h\in(h_\xi,h_{\xi+1}]\cap(0,\varrho]$, $i\geq 1$, and $\zeta\in\mathcal{B}_r=\mathcal{B}_r(0,\mathcal{B}_r(\delta))$, we observe that

$$(\Gamma(\aleph))\zeta(h)\in\begin{cases}\sum_{j=1}^{\xi}T(h-h_j)I_j(\mathcal{B}_{r^*}(0,\mathcal{S})), & h\in(h_\xi,h_{\xi+1}),\\ \sum_{j=0}^{\xi}T(h_{\xi+1}-h_j)I_j(\mathcal{B}_{r^*}(0,\mathcal{S})), & h=h_{\xi+1},\\ \sum_{j=0}^{\xi}T(h_\xi-h_j)I_j(\mathcal{B}_{r^*}(0,\mathcal{S}))+I_\xi(\mathcal{B}_{r^*}(0,\mathcal{S})), & h=h_\xi\end{cases} \quad (7)$$

This demonstrates that $[(\Gamma(\aleph))_2(B_r)]_\xi(h)$ is relatively compact in $\mathcal{S}$ for each $h\in[h_\xi,h_{\xi+1}]$, as the maps I_j are completely continuous. Additionally, by leveraging the compactness of the operators I_ξ along with the strong continuity of $(T(h))_{t_0}$, we can show that $[(\Gamma(\aleph))_2(B_r)]_\xi$ is uniformly continuous at h for every $h\in[h_\xi,h_{\xi+1}]$ and for each $\xi=1,2,\ldots,n$. Therefore, according to Lemma 2.2, $(\Gamma(\aleph))_2$ is completely continuous.

Step 4: Certainly, Our goal is to identify an open set $U\subseteq PC_\delta$ such that for any point ζ lying on the boundary of U, it won'h be in the set $\lambda(\Gamma(\aleph))(\zeta)$ for $\lambda\in(0,1)$. Therefore, for every $h\in(0,\varrho]$,

$$\begin{aligned}(\Gamma(\aleph))\zeta(h) = {} & \lambda\zeta(h,\aleph)\\ = {} & \lambda T_1(h)\phi_0(\aleph) + \lambda T_2(h)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)] - \lambda\int_0^{h-\epsilon} T_2(h)\rho(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi\\ & +\lambda\int_0^h T_2(h-\varpi)\Upsilon(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi + \lambda\sum_{0<h_\xi<h} T_1(h-h_\xi)I_\xi(\zeta(h_\xi,\aleph))\\ & +\lambda\sum_{0<h_\xi<h} T_2(h-h_\xi)I'_\xi(\zeta(h_\xi,\aleph))\end{aligned}$$

for each $h \in (0,\varrho]$, we have $||\zeta(h,\aleph)|| \le ||(\Gamma(\aleph))\zeta(h)||$ and

$$\begin{aligned}||(\Gamma(\aleph))\zeta(h)|| \le {} & ||T_1(h)\phi_0(\aleph)|| + ||T_2(h)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)]||\\ & + \int_0^h ||T_2(h-\varpi)\rho(\varpi,\zeta_\varpi(.,\aleph),\aleph)||d\varpi + \int_0^h ||T_2(h-\varpi)\Upsilon(\Pi,\zeta_\varpi(.,\aleph),\aleph)||d\varpi\\ & +||\sum_{0<h_\xi<h} T_1(h-h_\xi)I_\xi(\zeta(h_\xi,\aleph))|| + ||\sum_{0<h_\xi<h} T_2(h-h_\xi)I'_\xi(\zeta(h_\xi,\aleph))||\end{aligned}$$

By step 1, $||(\Gamma(\aleph))\zeta(h)|| \le R(\aleph)$ We can find a constant $R(\aleph)$ such that $\| \zeta \|_{PC} \neq R(\aleph)$. Set

$$U = \left\{\zeta \in PC([\delta,\varrho],\mathcal{S}) \mid \sup_{\delta\le h\le\varrho} \| \zeta(h,\aleph) \| < R(\aleph)\right\}$$

The results obtained from Steps 1-3 in Theorem 3.1 imply that it's enough to show that $(\Gamma(\aleph)): U \to PC_\delta$ is a compact mapping. With the selection of U, no $\varphi \in \partial U$ exists for which $\zeta \in \lambda(\Gamma(\aleph))(\zeta)$ for $\lambda \in (0,1)$. Based on Lemma 2.1, we assume that the operator $(\Gamma(\aleph))$ has a fixed point $\zeta^* \in U$.. Thus, we obtain

$$\begin{aligned}\zeta^*(h,\aleph) = {} & T_1(h)\phi_0(\aleph) + T_2(h)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)] + \int_0^h T_2(h-\varpi)\rho(\varpi,\zeta^*_\varpi(.,\aleph),\aleph)d\varpi\\ & + \int_0^h T_2(h-\varpi)\Upsilon(\Pi,\zeta^*_\varpi(.,\aleph),\aleph)d\varpi + \textstyle\sum_{0<h_\xi<h} T_1(h-h_\xi)I_\xi(\zeta(h_\xi,\aleph))\\ & + \textstyle\sum_{0<h_\xi<h} T_2(h-h_\xi)I'_\xi(\zeta^*(h_\xi,\aleph))\end{aligned} \tag{8}$$

This suggests that $\zeta^*(h,\aleph)$ possesses a fixed point and serves as a mild solution to problem (1.1). This concludes the proof of the theorem.

4 Approximate Contollability of Random Neutral Functional Differential Equation

Definition 6 The problem (1.2) is controllable on the interval $(0,\varrho]$ if, for any given final state $\zeta^1(\aleph)$, there is a control $y(h,\aleph)$ in $L^2(J,\Omega)$ such that the solution $\zeta(h,\aleph)$ of (1.2) reaches $\zeta^1(\aleph)$ at time ϱ.

We now present our primary existence result regarding problem (1.2). The definition of a mild random solution comes first.

If $\zeta_0 = \emptyset$ and the continuous function $\zeta: PC(J,\mathcal{S}) \times \Omega \to PC(J,\mathcal{S})$ and $\mathcal{D} = [(-\delta,\varrho],\mathcal{S}]$ solves the integral equation then it is referred to as a mild solution to equation (1.1).

Definition 7 A function $\zeta(\cdot,\aleph) \in PC(J,\mathcal{S})$ is considered a mild solution of problem (1.2) with initial conditions if it satisfies the following integral equation.

$$\begin{aligned}\zeta(h,\aleph) = \;& T_1(h)\phi_0(\aleph) + T_2(h)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)] - \int_0^h T_2(h-\varpi)\rho(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi \\ & + \int_0^h T_2(h-\varpi)[\Upsilon(\varpi,\zeta_\varpi(.,\aleph),\aleph) + By(h,\aleph)]d\varpi \\ & + \sum_{0<h_\xi<h} T_1(h-h_\xi)I_\xi(\zeta(h_\xi,\aleph)) + \sum_{0<h_\xi<h} T_2(h-h_\xi)I'_\xi(\zeta(h_\xi,\aleph))\end{aligned}$$

For your convenience, we have listed the additional hypotheses that will be discussed in the following section.

Let

(G_7) The linear operator $k: L^2(J,\mathcal{S}) \to \mathcal{S}$ given by $ky = \int_0^\varrho T_2(\varrho-\varpi)By(\varpi,\aleph)d\varpi$ has a pseudo-inverse operator k^{-1} in $L^2(J,S)/\ker k$

(G_8) There exist a random function $Q: \Omega \to \mathbb{R}_+$ where

$$\begin{aligned}& \vartheta_a B k^{-1} \int_0^h [||\zeta^1(\aleph)|| + \vartheta||\phi_0(\aleph)|| + \vartheta_a||\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)|| \\ & + \vartheta\int_0^\varrho [\sup_{\eta\in(0.\varrho]} ||\zeta_\eta(.,\aleph),\aleph||_{\mathcal{D}}^{\beta_0} + d_0(\aleph)]d\eta + \vartheta_a\int_0^\varrho [\sup_{\eta\in(0.\varrho]} ||\zeta_\eta(.,\aleph),\aleph||_{\mathcal{D}}^{\alpha_0} + b_0(\aleph)]d\eta \\ & + \vartheta\sum_{0<h_\xi<\varrho} a_\xi(\aleph)||\zeta(h_\xi,\aleph)||_{\mathbb{R}}^{\alpha_\xi} + \vartheta_a\sum_{0<h_\xi<\varrho} a'_\xi(\aleph)||\zeta(h_\xi,\aleph)||_{\mathbb{R}}^{\alpha_\xi}]d\varpi \le Q(\aleph)\end{aligned}$$

Theorem 4.1 *If (G_1) - (G_8) are fulfilled, then the problem (1.2) is approximately controllable on J.*

Proof: Let us specify the control:

$$\begin{aligned}y(h,\aleph) = \;& k^{-1}(\zeta^1(\aleph) - T_1(\varrho)\phi_0(\aleph) - T_2(\varrho)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)] + \int_0^\varrho T_1(\varrho-\varpi)\rho(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi \\ & - \int_0^\varrho T_2(\varrho-\varpi)\Upsilon(\Pi,\zeta_\varpi(.,\aleph),\aleph)d\varpi - \sum_{0<h_\xi<\varrho} T_1(\varrho-h_\xi)I_\xi(\zeta(h_\xi,\aleph)) \\ & - \sum_{0<h_\xi<\varrho} T_2(\varrho-h_\xi)I'_\xi(\zeta(h_\xi,\aleph))\end{aligned}$$

We define the operator $(\Gamma(\aleph))': PC_\delta = \Omega \times PC([\delta,\varrho],\mathcal{S}) \to PC_\delta$ be a random operator and is defined by

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$$((\Gamma(\aleph))'\zeta)(h) = T_1(h)\phi_0(\aleph) + T_2(h)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)] - \int_0^h T_1(h-\varpi)[\rho(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi$$
$$+\int_0^h T_2(h-\varpi)[\Upsilon(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi$$
$$+\int_0^h T_2(h-\varpi)Bk^{-1}[(\zeta^1(\aleph) - T_1(\varrho)\phi_0(\aleph) - T_2(\varrho)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)]$$
$$+\int_0^\varrho T_1(\varrho-\eta)\rho(\eta,\zeta_\eta(.,\aleph),\aleph)d\eta - \int_0^\varrho T_2(\varrho-\eta)\Upsilon(\mathrm{H},\zeta_\eta(.,\aleph),\aleph)d\eta$$
$$-\sum_{0<h_\xi<\varrho} T_1(\varrho-h_\xi)I_\xi(\zeta(h_\xi,\aleph)))] - \sum_{0<h_\xi<\varrho} T_2(\varrho-h_\xi)I'_\xi(\zeta(h_\xi,\aleph))]d\varpi$$
$$+\sum_{0<h_\xi<h} T_1(h-h_\xi)I_\xi(\zeta(h_\xi,\aleph)) + \sum_{0<h_\xi<h} T_2(h-h_\xi)I'_\xi(\zeta(h_\xi,\aleph))] \quad h\in(-\delta,\varrho].$$

$$(\Gamma(\aleph))' = (\Gamma(\aleph))'_1 + (\Gamma(\aleph))'_2$$

$$(\Gamma(\aleph))'_1\zeta(h) = T_1(h)\phi_0(\aleph) + T_2(h)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)] - \int_0^h T_2(h-\varpi)\rho(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi$$
$$+\int_0^h T_2(h-\varpi)\Upsilon(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi + \sum_{0<h_\xi<h} T_1(h-h_\xi)I_\xi(\zeta(h_\xi,\aleph))$$
$$+\sum_{0<h_\xi<h} T_2(h-h_\xi)I'_\xi(\zeta(h_\xi,\aleph))$$

$$(\Gamma(\aleph))'_2\zeta(h) = \int_0^h T_2(h-\varpi)Bk^{-1}[\zeta^1(\aleph) - T_1(\varrho)\phi_0(\aleph) - T_2(\varrho)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)]$$
$$+\int_0^\varrho T_1(\varrho-\eta)\rho(\eta,\zeta_\eta(.,\aleph),\aleph)d\eta - \int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta,\zeta_\eta(.,\aleph),\aleph)d\eta$$
$$-\sum_{0<h_\xi<\varrho} T_1(\varrho-h_\xi)I_\xi(\zeta(h_\xi,\aleph)) - \sum_{0<h_\xi<\varrho} T_2(\varrho-h_\xi)I'_\xi(\zeta(h_\xi,\aleph))]d\varpi$$

We have already outlined four scenarios for $(\Gamma_1(\aleph))$ in theorem(3.1). Hence, it suffices to validate the outcome for $(\Gamma(\aleph))_2$.

Step 1: $(\Gamma(\aleph))_2$ takes bounded sets and maps them to bounded sets.

Specifically, it is sufficient to establish that we can find a $+_{\mathrm{ve}}$ constant $q(\aleph)$ such that for every $\zeta \in \mathcal{B}_q(\delta)$, defined as:

$$\mathcal{B}_r(\delta) := \left\{\zeta \in PC_\delta : \sup_{\delta\le h\le\varrho} \| \zeta(h,\aleph) \| \le q(\aleph)\right\}$$

one has $\| (\Gamma(\aleph))_2\zeta \|_{PC} \le Q(\aleph)$.

$$\begin{aligned}||(\Gamma(\aleph))'_2\zeta(h)|| \leq\ & \int_0^h ||T_2(h-\varpi)Bk^{-1}[\zeta^1(\aleph) - T_1(\varrho)\phi_0(\aleph) - T_2(\varrho)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)] \\ & + \int_0^\varrho T_1(\varrho-\eta)\rho(\eta,\zeta_\eta(.,\aleph),\aleph)d\eta - \int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta,\zeta_\eta(.,\aleph),\aleph)d\eta \\ & - \sum_{0<h_\xi<\varrho} T_1(\varrho-h_\xi)I_\xi(\zeta(h_\xi,\aleph)) - \sum_{0<h_\xi<\varrho} T_2(\varrho-h_\xi)I'_\xi(\zeta(h_\xi,\aleph))]||d\varpi \\ & \leq \int_0^h ||T_2(h-\varpi)||Bk^{-1}[||\zeta^1(\aleph)|| + ||T_1(\varrho)\phi_0(\aleph)|| + ||T_2(\varrho)[\phi'_0(\aleph)|| \\ & +|| \int_0^\varrho T_1(\varrho-\eta)\rho(\eta,\zeta_\eta(.,\aleph),\aleph)d\eta|| + || \int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta,\zeta_\eta(.,\aleph),\aleph)d\eta|| \\ & + \sum_{0<h_\xi<\varrho} ||T_1(\varrho-h_\xi)I_\xi(\zeta(h_\xi,\aleph))|| + || \sum_{0<h_\xi<\varrho} T_2(\varrho-h_\xi)I'_\xi(\zeta(h_\xi,\aleph))||]d\varpi \\ & \leq \vartheta_a Bk^{-1}\int_0^h [||\zeta^1(\aleph)|| + \vartheta||\phi_0(\aleph)|| + \vartheta_a||\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)|| \\ & +\vartheta\int_0^\varrho [\sup_{\eta\in(0.\varrho]} ||\zeta_\eta(.,\aleph),\aleph||_{\mathcal{D}}^{\beta_0} + d_0(\aleph)]d\eta + \vartheta_a\int_0^\varrho [\sup_{\eta\in(0.\varrho]} ||\zeta_\eta(.,\aleph),\aleph||_{\mathcal{D}}^{\alpha_0} \\ & +\vartheta\sum_{0<h_\xi<\varrho} a_\xi(\aleph)||\zeta(h_\xi,\aleph)||_{\mathbb{R}}^{\alpha_\xi} + \vartheta_a\sum_{0<h_\xi<\varrho} a'_\xi(\aleph)||\zeta(h_\xi,\aleph)||_{\mathbb{R}}^{\alpha_\xi}]d\varpi \\ & \leq Q(\aleph)\end{aligned} \tag{9}$$

Hence $(\Gamma(\aleph))'_2$ is bounded in PC_δ

Step 2: We now demonstrate that $(\Gamma(\aleph))'_2$ is continuous on $\mathcal{B}_r(\delta)$. Let us consider $\zeta_1, \zeta_2 \in \mathcal{B}_r(\delta)$ and $h \in J$.

$$\begin{aligned}||(\Gamma(\aleph))'_2\zeta_1(h) - (\Gamma(\aleph))'_2\zeta_2(h)|| \leq\ & \int_0^h ||T_1(h-\varpi)Bk^{-1}[\int_0^\varrho T_1(\varrho-\eta)[\rho(\eta,(\zeta_{1,\eta}(.,\aleph),\aleph),\aleph) \\ & -\rho(\eta,(\zeta_{2,\eta}(.,\aleph),\aleph),\aleph)]d\eta \\ & -\int_0^\varrho T_2(\varrho-\eta)[\Upsilon(\eta,(\zeta_{1,\eta}(.,\aleph),\aleph),\aleph) - \Upsilon(\eta,(\zeta_{2,\eta}(.,\aleph),\aleph),\aleph)]d\eta \\ & -\sum_{0<h_\xi<\varrho} T_1(\varrho-h_\xi)[I_\xi(\zeta_1(h_\xi,\aleph)) - I_\xi(\zeta_2(h_\xi,\aleph))] \\ & -\sum_{0<h_\xi<\varrho} T_2(\varrho-h_\xi)[I'_\xi(\zeta_1(h_\xi,\aleph)) - I'_\xi(\zeta_2(h_\xi,\aleph))]||]d\varpi \\ & \leq \int_0^h \vartheta_a Bk^{-1}[\int_0^\varrho \vartheta \sup_{\eta\in(0.\varrho]} ||(\zeta_{1,\eta}(.,\aleph),\aleph) - (\zeta_{2,\eta}(.,\aleph),\aleph)||_{\mathcal{D}}^{\beta_0} d\eta \\ & + \int_0^\varrho \vartheta_a \sup_{\eta\in(0.\varrho]} ||(\zeta_{1,\eta}(.,\aleph),\aleph) - (\zeta_{2,\eta}(.,\aleph),\aleph)||_{\mathcal{D}}^{\alpha_0} d\eta \\ & +\vartheta\sum_{0<h_\xi<\varrho} a_\xi(\aleph)||\zeta_1(h_\xi,\aleph) - \zeta_2(h_\xi,\aleph)||_{\mathbb{R}}^{\alpha_\xi} \\ & +\vartheta_a\sum_{0<h_\xi<\varrho} a'_\xi(\aleph)||\zeta_1(h_\xi,\aleph) - \zeta_2(h_\xi,\aleph)||_{\mathbb{R}}^{\alpha_\xi}]d\varpi\end{aligned}$$

For all $h \in (-\delta, \varrho]$, and due to their compactness for $h > 0$, the uniform operator topology is continuous. Given $\zeta_1, \zeta_2 \in \mathcal{B}_r(\delta)$, the independence of the right side of the inequalities above is evident. Consequently, as $(\zeta_1 - \zeta_2) \to 0$, we have $\| ((\Gamma(\aleph))'_2\zeta_1)(h) - ((\Gamma(\aleph))'_2\zeta_2)(h) \| \to 0$. This implies that $(\Gamma(\aleph))$ is continuous.

Step 3: $(\Gamma(\aleph))'_2$ is a compact operator.

To establish this, we analyze the decomposition $(\Gamma(\aleph))'_2 = (\Gamma_a(\aleph))'_2 + (\Gamma_b(\aleph))'_2$, where $(\Gamma_1(\aleph))$ and $(\Gamma(\aleph))_2$ denote operators on $\mathcal{B}_r(\delta)$. They are defined as follows:

$$(\Gamma_a(\aleph))'_2 = \int_0^h ||T_2(h-\varpi)Bk^{-1}[\zeta^1(\aleph) - T_1(\varrho)\phi_0(\aleph) - T_2(\varrho)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)] + \int_0^\varrho T_1(\varrho-\eta)[\rho(\eta,\zeta_\eta(.,\aleph),\aleph)]d\eta - \int_0^\varrho T_2(\varrho-\eta)[\Upsilon(\eta,\zeta_\eta(.,\aleph),\aleph)]d\eta]d\varpi$$

$$(\Gamma_b(\aleph))'_2 = \int_0^h ||T_2(h-\varpi)Bk^{-1}[\sum_{0<h_\xi<\varrho} T_1(\varrho-h_\xi)I_\xi(\zeta(h_\xi,\aleph)) - \sum_{0<h_\xi<\varrho} T_2(\varrho-h_\xi)I'_\xi(\zeta(h_\xi,\aleph))]d\varpi$$

We first prove that $(\Gamma(\aleph))_{2,a}(\mathcal{B}_r(\delta))$ is equicontinuous.

Let $\delta \le h_1 < h_2 \le \varrho$ and $\epsilon > 0$ be small, then

$$\begin{aligned}
||(\Gamma_a&(\aleph))'_2\zeta(h_2) - (\Gamma_a(\aleph))'_2\zeta(h_1)|| \le \int_0^{h_1} ||T_2(h_2-\varpi)Bk^{-1}[\zeta^1(\aleph) - T_1(\varrho)\phi_0(\aleph) - T_2(\varrho)\\
&[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)] + \int_0^\varrho T_1(\varrho-\eta)\rho(\eta,\zeta_\eta(.,\aleph),\aleph)d\eta - \int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta,\zeta_\eta(.,\aleph),\aleph)d\eta]d\varpi\\
&-\int_0^{h_2} T_2(h_1-\varpi)Bk^{-1}[\zeta^1(\aleph) - T_1(\varrho)\phi_0(\aleph) - T_2(\varrho)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)]\\
&+\int_0^\varrho T_1(\varrho-\eta)\rho(\eta,\zeta_\eta(.,\aleph),\aleph)d\eta - \int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta,\zeta_\eta(.,\aleph),\aleph)d\eta]||d\varpi\\
&\le \int_0^{h_1-\theta} ||T_2(h_2-\varpi) - T_2(h_1-\varpi)Bk^{-1}[\zeta^1(\aleph) - T_1(\varrho)\phi_0(\aleph) - T_2(\varrho)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)]\\
&+\int_0^\varrho T_1(\varrho-\eta)\rho(\eta,\zeta_\eta(.,\aleph),\aleph)d\eta - \int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta,\zeta_\eta(.,\aleph),\aleph)d\eta]||d\varpi\\
&+||\int_{h_1-\theta}^{h_1} ||T_2(h_2-\varpi) - T_2(h_1-\varpi)Bk^{-1}[\zeta^1(\aleph) - T_1(\varrho)\phi_0(\aleph) - T_2(\varrho)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)]\\
&+\int_0^\varrho T_1(\varrho-\eta)\rho(\eta,\zeta_\eta(.,\aleph),\aleph)d\eta - \int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta,\zeta_\eta(.,\aleph),\aleph)d\eta]||d\varpi\\
&+\int_{h_1-\theta}^{h_1} ||T_2(h_2-\varpi) - T_2(h_1-\varpi)Bk^{-1}[\zeta^1(\aleph) - T_1(\varrho)\phi_0(\aleph) - T_2(\varrho)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)]\\
&+\int_0^\varrho T_1(\varrho-\eta)\rho(\eta,\zeta_\eta(.,\aleph),\aleph)d\eta + \int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta,\zeta_\eta(.,\aleph),\aleph)d\eta]||d\varpi\\
&\le \int_0^{h_1-\theta} ||T_2(h_2-\varpi) - T_2(h_1-\varpi)Bk^{-1}[\zeta^1(\aleph) - T_1(\varrho)\phi_0(\aleph) - T_2(\varrho)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)]\\
&+\int_0^\varrho T_1(\varrho-\eta)[\sup_{\eta\in(0.\varrho]} ||\zeta_\eta(.,\aleph),\aleph||_{\mathcal{D}}^{\beta_0} + d_0(\aleph)]d\eta + \int_0^\varrho T_2(\varrho-\eta)[\sup_{\eta\in(0.\varrho]} ||\zeta_\eta(.,\aleph),\aleph||_{\mathcal{D}}^{\alpha_0} + b_0(\aleph)]d\eta]||\\
&+||\int_{h_1-\theta}^{h_1} ||T_2(h_2-\varpi) - T_2(h_1-\varpi)Bk^{-1}[\zeta^1(\aleph) - T_1(\varrho)\phi_0(\aleph) - T_2(\varrho)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)]\\
&+\int_0^\varrho T_1(\varrho-\eta)[\sup_{\eta\in(0.\varrho]} ||\zeta_\eta(.,\aleph),\aleph||_{\mathcal{D}}^{\beta_0} + d_0(\aleph)]d\eta + \int_0^\varrho T_2(\varrho-\eta)[\sup_{\eta\in(0.\varrho]} ||\zeta_\eta(.,\aleph),\aleph||_{\mathcal{D}}^{\alpha_0} + b_0(\aleph)]d\eta]||\\
&+\int_{h_1-\theta}^{h_1} ||T_2(h_2-\varpi) - T_2(h_1-\varpi)Bk^{-1}[\zeta^1(\aleph) - T_1(\varrho)\phi_0(\aleph) - T_2(\varrho)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)]\\
&+\int_0^\varrho T_1(\varrho-\eta)[\sup_{\eta\in(0.\varrho]} ||\zeta_\eta(.,\aleph),\aleph||_{\mathcal{D}}^{\beta_0} + d_0(\aleph) + \int_0^\varrho T_2(\varrho-\eta)[\sup_{\eta\in(0.\varrho]} ||\zeta_\eta(.,\aleph),\aleph||_{\mathcal{D}}^{\alpha_0} + b_0(\aleph)]d\eta]||d\varpi
\end{aligned}$$

As $h_2 - h_1$ approaches zero, $||(\Gamma_a(\aleph))'_2\zeta(h_2) - (\Gamma_a(\aleph))'_2\zeta(h_1)||$ tends to zero for any $\zeta \in \mathcal{B}_r(\delta)$. This convergence is due to the operator's compactness $T_2(h)$ for $h > 0$, ensuring continuity in the uniform operator norm. Consequently, $(\Gamma_1(\aleph))$ maps $\mathcal{B}_r(\delta)$ into an uniformly continuous family of functions.

We now aim to demonstrate that the set $(\Gamma_a(\aleph))'_2(\mathcal{B}_r(\delta))(h)$ is precompact in $\mathcal{S}$.

Given $\delta < h \leq \varpi \leq \varrho$, let ϵ be a real number where $0 < \epsilon < h$. For $\zeta \in \mathcal{B}_r(\delta)$,we specify $((\Gamma(\aleph))'_{2,a,\epsilon}\zeta)(h)$ as

$$\int_0^{h-\epsilon} T_2(h_2-\varpi)Bk^{-1}[\zeta^1(\aleph) - T_1(\varrho)\phi_0(\aleph) - T_2(\varrho)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)] \\ - \int_0^{\varrho} T_1(\varrho-\eta)\rho(\eta,\zeta_\eta(.,\aleph),\aleph) + \int_0^{\varrho} T_2(\varrho-\eta)\Upsilon(\eta,\zeta_\eta(.,\aleph),\aleph)]d\eta$$

By leveraging the compactness of $T_2(h)$ for $h > 0$, we conclude that the set $\{((\Gamma(\aleph))'_{2,a,\epsilon}\zeta)(h): \zeta \in \mathcal{B}_r(\delta)\}$ is precompact for $\zeta \in \mathcal{B}_r(\delta)$ and $0 < \epsilon < h$. Moreover, for every $\zeta \in \mathcal{B}_r(\delta)$, we assert

$$\begin{aligned} ||((\Gamma_a(\aleph))_2\zeta)(h) - ((\Gamma(\aleph))_{2,a,\epsilon}\zeta)(h)|| \leq{}& \int_{h-\epsilon}^{h} ||T_2(h_2-\varpi)Bk^{-1}[\int_0^{\varrho} T_1(\varrho-\eta)\rho(\eta,\zeta_\eta(.,\aleph),\aleph)||d\eta \\ &+ \int_0^{\varrho} T_2(\varrho-\eta)\Upsilon(\eta,\zeta_\eta(.,\aleph),\aleph)||d\eta]d\varpi \\ \leq{}& \int_{h-\epsilon}^{h} \vartheta_a Bk^{-1}[\int_0^{\varrho} \vartheta[\sup_{\eta\in(0.\varrho]} ||\zeta_\eta(.,\aleph),\aleph||_{\mathcal{D}}^{\beta_0} + d_0(\aleph)]d\eta \\ &+ \int_0^{\varrho} \vartheta_a[\sup_{\eta\in(0.\varrho]} ||\zeta_\eta(.,\aleph),\aleph||_{\mathcal{D}}^{\alpha_0} + b_0(\aleph)]d\eta]d\varpi \end{aligned}$$

Thus, we can find sets that are precompact and close to $\{((\Gamma_a(\aleph))'_2\zeta): \zeta \in B_r(\delta)\}$. As a result, $\{((\Gamma_a(\aleph))'_2\zeta): \zeta \in B_r(\delta)\}$ itself becomes precompact within $\mathcal{S}$. It's clear that $(\Gamma_a(\aleph))'_2(B_r(\delta))$ is uniformly bounded. Given that we have demonstrated $(\Gamma_a(\aleph))'_2(B_r(\delta))$ constitutes an equicontinuous family, the Arzelà-Ascoli theorem implies that it is sufficient to show that $(\Gamma_a(\aleph))'_2$ maps $B_r(\delta)$ into a precompact set in $\mathcal{S}$.

Next, it is necessary to confirm that $(\Gamma_b(\aleph))'_2$ is also a compact operator.

By step 3 of theorem 3.1(above theorem) we prove that $(\Gamma_b(\aleph))'_2$ is compact.

Step 4: Next, we establish the existence of an open set $U \subseteq PC_\delta$ such that $\zeta \notin \lambda(\Gamma(\aleph))'(\zeta)$ for $\lambda \in (0,1)$ and $\zeta \in \partial U$. Consider $\lambda \in (0,1)$ and let $\zeta \in PC_\delta$ be a potential solution of $\zeta = \lambda(\Gamma(\aleph))'(\zeta)$ for some $0 < \lambda < 1$. Consequently, for every $h \in (0,\varrho]$, we have

$$\begin{aligned}\zeta(h,\aleph) = \;& \lambda T_1(h)\phi_0(\aleph) + \lambda T_2(h)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)] - \lambda \int_0^h T_1(h-\varpi)\rho(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi \\ &+\lambda \int_0^h T_2(h-\varpi)\Upsilon(\varpi,\zeta_\varpi(.,\aleph),\aleph)d\varpi + \lambda \int_0^h T_2(h-\varpi)Bk^{-1}[(\zeta^1(\aleph) - T_1(\varrho)\phi_0(\aleph) \\ &-T_2(\varrho)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)] + \int_0^\varrho T_1(\varrho-\eta)\rho(\eta,\zeta_\eta(.,\aleph),\aleph)d\eta \\ &-\int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta,\zeta_\eta(.,\aleph),\aleph)d\eta - \sum_{0<h_\xi<\varrho} T_1(\varrho-h_\xi)I_\xi(\zeta(h_\xi,\aleph))) \\ &-\sum_{0<h_\xi<\varrho} T_2(\varrho-h_\xi)I'_\xi(\zeta(h_\xi,\aleph))]d\varpi + \lambda\sum_{0<h_\xi<h} T_1(h-h_\xi)I_\xi(\zeta(h_\xi,\aleph)) \\ &+\lambda\sum_{0<h_\xi<h} T_2(h-h_\xi)I'_\xi(\zeta(h_\xi,\aleph))\end{aligned}$$

By step 1 of theorem 3.1 and 3.2, $||(\Gamma(\aleph))'\zeta(h)|| \leq R(\aleph) + Q(\aleph)$ We can find a constant $R(\aleph) + Q(\aleph)$ such that $\parallel \zeta \parallel_{PC} \neq R(\aleph) + Q(\aleph)$. Set

$$U = \left\{\zeta \in PC([\delta,\varrho],\mathcal{S}) \mid \sup_{\delta\leq h\leq\varrho} \parallel \zeta(h) \parallel < R(\aleph) + Q(\aleph)\right\}$$

Based on Steps 1-3 of Theorem 3.2, it is sufficient to show that $(\Gamma(\aleph))': U \to PC_\delta$ is a compact map. Given the choice of U, there is no $\varphi \in \partial U$ for which $\zeta \in \lambda(\Gamma(\aleph))(\zeta)$ with $\lambda \in (0,1)$. According to Lemma 2.1, we assume that the operator $(\Gamma(\aleph))$ has a fixed point $\zeta^* \in U$. Thus, we derive

$$\begin{aligned}\zeta^*(h,\aleph) = \;& T_1(h)\phi_0(\aleph) + T_2(h)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)] - \lambda \int_0^h T_1(h-\varpi)\rho(\varpi,\zeta^*_\varpi(.,\aleph),\aleph)d\varpi \\ &+\lambda \int_0^h T_2(h-\varpi)\Upsilon(\varpi,\zeta^*_\varpi(.,\aleph),\aleph)d\varpi + \int_0^h T_2(h-\varpi)Bk^{-1}[(\zeta^{*,1}(\aleph) - T_1(\varrho)\phi_0(\aleph) \\ &-T_2(\varrho)[\phi'_0(\aleph) + \rho(0,\phi_0(\aleph),\aleph)] + \int_0^\varrho T_1(\varrho-\eta)\rho(\eta,\zeta^*_\eta(.,\aleph),\aleph)d\eta \\ &-\int_0^\varrho T_2(\varrho-\eta)\Upsilon(\eta,\zeta^*_\eta(.,\aleph),\aleph)d\eta - \sum_{0<h_\xi<\varrho} T_1(\varrho-h_\xi)I_\xi(\zeta^*(h_\xi,\aleph))) \\ &-\sum_{0<h_\xi<\varrho} T_2(\varrho-h_\xi)I'_\xi(\zeta^*(h_\xi,\aleph))]d\varpi + \lambda\sum_{0<h_\xi<h} T_1(h-h_\xi)I_\xi(\zeta^*(h_\xi,\aleph)) \\ &+\lambda\sum_{0<h_\xi<h} T_2(h-h_\xi)I'_\xi(\zeta^*(h_\xi,\aleph))\end{aligned}$$

This implies, that $\zeta^*(h,\aleph)$ has a fixed point and $\zeta^*(h,\aleph)$ is a mild solution of problem (1.2). This completes the proof of this theorem.

5 Example

This section introduces an example to illustrate our findings. Before delving into the application of our abstract results, we must first establish some technical prerequisites. In what follows, let $\mathcal{S} = L^2([0,\pi])$, $D(A) = \{\varphi \in \mathcal{S}: x'' \in \mathcal{S}, \varphi(0) = \varphi(\pi) = 0\}$, and $A: D(A) \subseteq \mathcal{S} \to \mathcal{S}$ denote the linear operator defined by $A\varphi = \varphi''$.It's widely recognized that A acts as the infinitesimal generator of a strongly continuous cosine family $(T_1(h))_{h\in\mathbb{R}}$ on $\mathcal{S}$. Moreover, A has a discrete spectrum, with eigenvalues $-n^2$ for $n \in \vartheta$, each corresponding to the eigenvectors $z_n(\varrho) = (\frac{2}{\pi})^{1/2}$.

Consider the following impulsive partial neutral functional integro-differential equation of the form:

$$\frac{\partial}{\partial}[\frac{\partial}{\partial h}z(h,x,\aleph)-\rho(h,z(\cos h,x,\aleph),\aleph)=\frac{\partial^2}{\partial x^2}z(h,x,\aleph)+\Upsilon(\mathrm{h},z(\sin h,x,\aleph),\aleph),\quad \aleph\in(-\infty,0] \quad (11)$$

$$\Delta z(h_\xi,x,\aleph)=\int_0^\pi q_\xi(x,y)z(h_\xi,y,\aleph)dy \quad \text{and} \quad \Delta' z(h_\xi,x,\aleph)=\int_0^\pi q'_\xi(x,y)z(h_\xi,y,\aleph)dy,\quad \xi=1,\ldots,m, \quad (12)$$

$$z(h,0,\aleph)=z(h,\pi,\aleph)=0;\quad z(0,x,\aleph)=z_0(x,\aleph);z_h(0,x,\aleph)=z_1(x,\aleph),\quad h\in J=[0,1],\quad 0\le x\le\pi, \quad (13)$$

$$z(0,x,\aleph)=z_0(x,\aleph),\quad \text{and} \quad z_h(0,x,\aleph)=z_1(x,\aleph),\quad 0\le x\le\pi. \quad (14)$$

where we assume the following conditions:

The functions $\Upsilon(\cdot,\aleph)$ and are continuous on $[0,1]$ with $n=\sup_{0\le\varpi\le1}|\Upsilon(\varpi,\aleph)|<1$.

The functions q_ξ, $q'_\xi:[0,\pi]\times[0,\pi]\to\mathbb{R}$, $k=1,1,\ldots,m$, are continuously differentiable, and

$$\psi_\xi \quad =\left(\int_0^\pi\int_0^\pi\left(\frac{\partial}{\partial x}q_\xi(x,y)\right)^2dxdy\right)^{\frac{1}{2}}<\infty$$

$$\psi'_\xi \quad =\left(\int_0^\pi\int_0^\pi\left(\frac{\partial}{\partial x}q'_\xi(x,y)\right)^2dxdy\right)^{\frac{1}{2}}<\infty,$$

for every $\xi=1,2,\ldots,m$.

To address this system, we introduce the operators in the following manner $\Upsilon: J\times J\times\mathcal{D}\times\Omega\to\mathcal{S}$, and $\rho: J\times\mathcal{D}\times\Omega\to\mathcal{S}$,

$$\rho(h,z_h(.,\aleph),\aleph)(x)=\rho(h,z(\cos h,x,\aleph),\aleph)$$

$$\Upsilon(\mathrm{h},z_h(.,\aleph),\aleph)(x)=\Upsilon(\mathrm{h},z(\sin h,x,\aleph),\aleph)$$

$$I_\xi(z,\aleph)(x)=\int_0^\pi q_\xi(x,y)z(h_\xi,y,\aleph)dy \quad \xi=1,2,\ldots,m$$

$$I'_\xi(z,\aleph)(x)=\int_0^\pi q'_\xi(x,y)z(h_\xi,y,\aleph)dy \quad \xi=1,2,\ldots,m.$$

Sure, here's a simplified version:

The equations (5.13-5.16) can be transformed into a more general form, denoted as (1.1). By using the functions mentioned earlier, we meet the requirements stated in Theorem 3.1. Therefore, according to Theorem 3.1, we can conclude that the given nonlocal impulsive Cauchy problem (5.13-5.16) has a mild solution over the interval J.

6 Conclusion

This study delves into a specific class of mathematical problems concerning second-order equations with delays, a topic widespread in scientific and engineering disciplines. By situating these equations within the realm of Banach spaces, distinct challenges and pathways for analysis and control are uncovered. Through rigorous examination of the existence and approximate controllability of solutions, this research significantly contributes to understanding dynamical systems with delayed feedback. Mathematical tools such as cosine family theory and the Leray-Schauder theorem are leveraged to establish stringent conditions for solution existence, with implications for theoretical advancements and practical applications. Moreover, empirical validation through a practical example provides invaluable insights into the behavior of these equations in real-world scenarios, effectively bridging the gap between theory and application. This comprehensive investigation advances understanding of complex dynamical systems with delayed feedback and offers practical insights for developing robust control strategies and engineering solutions across various domains.

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