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Cubic Intuitionistic Fuzzy Gamma-m-Normed Linear Space

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Abstract:

Objectives: The purpose of this research is to take the lead in foggiest idea of convergence in cubic intuitionistic fuzzy Gamma-m normed linear space(CCIFGMNLS). According to the theory of fuzzy m-Normed Linear Space(FMNLS),we offer the conception of Cauchy's sequence and convergence in cubic intuitionistic fuzzy Gamma-m Normed Linear Space (CIFGMNLS). We have reviewed the certain results, and this paper proposes the hypothesis of completeness in CIFGMNLS.

Methods: In this research paper we defined the intuitionistic fuzzy Gamma gamma ring, intuitionistic fuzzy gamma ideals, left and right intuitionistic fuzzy gamma vector space which are using to approach the theory of intuitionistic fuzzy gamma-2-normed linear space, intuitionistic fuzzy Gamma-m-Normed Linear Space and its axioms. And the CCIFGMNLS can be approached using the IFGMNLS.

Findings: In this research paper from CCIFGMNLS construct a norm function that satisfies the properties of IFGMNLS, and provided that example with proof of a sequence is cauchy sequence and convergence in IFGMNLS if and only it is cauchy sequence and convergence sequence in CCIFGMNLS. Also derived a theorem and its proof for completeness of a sequence in CCIFGMNLS.

Novelty: Already gamma ring and fuzzy n-normed linear space has been defined. we originated the notion of IFGMNLS using this also put forwarded the CCIFGMNLS and some results obtained from its properties . we provided a necessary axioms to completeness of a sequence in CCIFGMNLS.

Keywords: intuitionistic fuzzy Gamma gamma ring, intuitionistic fuzzy gamma ideals, left intuitionistic fuzzy $\Gamma-vector\ space$, intuitionistic fuzzy gamma normed linear space, 2-normed and m-normed right intuitionistic fuzzy gamma linear space. Intuitionistic fuzzy m-norm , intuitionistic fuzzy m-normed linear space.

1. Introduction:

Intuitionistic fuzzy sets (IFS) are a generalization of L.A.Zadeh's[1] fuzzy sets that consider both the degree of membership and non-membership of an element in a set. Intuitionistic fuzzy sets are used to represent uncertainty and vagueness in a more comprehensive way than traditional fuzzy sets, which were acquainted by Krassimir Atanassov in 1983[2,3,4,5]. Intuitionistic fuzzy sets (IFS) have numerous applications in various fields to handle uncertainty and vagueness in symptoms and diagnoses in Medical Diagnosis,in sensor readings and control actions in Robotics, in financial data and predictions in Financial Analysis, in weather patterns and predictions in Weather Forecasting, in decision-making systems, and is also used for image segmentation, edge detection, and image compression in Image Processing.[6,7,8,9]

On a linear space Gahler originated the results in the theory of n-norm. In the n-normed linear space an important conceptions convergent sequence and Cauchy sequence designed by Hendra Gunawan[10] . The first notion of fuzzy norm on a linear space introduced by Katsaras and Felbin, Chang and Mordeson initiated a new definition of a fuzzy norm on a linear space[11,12,13]. Bag and Samanta put forwarded a definition of fuzzy norm and proved the decomposition theorem of fuzzy norm to a family of crisp norms[14,15]. Azriel Rosenfeld is presented the supposition of fuzzy group[16],the notion of anti fuzzy subgroups established by R.Biswas[17]. W.Liu is introduced the concept of fuzzy ideals of rings[18], K.H.Kim and Y.B.Jun [19]were prepared the big idea of anti fuzzy ideals in near-rings and some results on it accoutered by many authors. An intuitionistic fuzzification of ideal of Γ-ring established by Kim et al in [20] and Palaniappan and Ramachandran were deliberated the abstraction of the intuitionistic fuzzy of prime ideal , semi-prime ideal [21]and impression of intuitionistic fuzzy prime spectrum of a commutative ring with identity is prefaced by P. K. Sharma et al. in [22,23]

The gamma ring is an algebraic tool used to study the relationship between the groups of homomorphisms of commutative groups .The gamma ring as an expansion of the idea of a classical ring which was first introduced by N. Nobusawa[24] and it was generalized by Barnes[25].The generalization of fuzzy rings and gamma rings is the fuzzy gamma ring and it is introduced by Bijan Davvaz[26].Many authors have established and designed the conceptions on the fuzzy normed linear space apart from this the work Vijayabalaji and Narayanan n-normed linear space extended into fuzzy n-normed linear space on this field[27,28,29]. The conceptualization of intuitionistic fuzzy normed linear space presented by Saadati and Park while Vijayabalaji originated the concept of intuitionistic fuzzy n-normed linear space also he and Reddy B S were introduced left gamma n-normed linear space and some results prepared on it[30,31,32,33,34]. The theory of cubic sets which comprehends of fuzzy set and interval-valued fuzzy set is initiated by Jun et al. Persuaded by above theory we present this research paper is to hypothesis of cubic intuitionistic fuzzy gamma-m-normed linear space and deliberated the results about convergence sequence ,Cauchy sequence in it.

2. Methodology:

Definition 2.1: Professor L A Zadeh generalization the concept of binary membership to accommodate various degrees of membership in the binary numbers 0 and 1 where 0 and 1 refers a “No membership” and “complete membership” respectively. Any kind of membership function values are bounded and whose in between 0 and 1.

An extension of classical set theory is a Fuzzy sets theory and in this set each element has changeable degree of membership value based on a logic two truth values 0 and 1. If U is the universe of discourse, a set F is said to be fuzzy set in U if there exists a function μ: F ⟶ [0, 1] and it is denoted by a set of ordered pairs as F={(u, μ(u)) /u ∈U }

Definition 2.2: suppose the fuzzy set $\mathcal{F}$ in a finite and discrete universe of discourse U and the tried set $F = \{(u, \mu_F(u), \eta_F(u)) / 0 \le \mu_F(u) + \eta_F(u)\} \le 1, \text{ for all } u \in U\}$ where $\mu_F : U \to [0,1]$ is a membership function with $\mu_F : U \to [0,1]$ *and* $\eta_F : U \to [0,1]$ with $0 \le \eta_F \le 1]$ is non-membership function is called as the intuitionistic fuzzy set.

Definition 2.3 : Let $F_1 = (u, \mu_{F_1}(u), \eta_{F_1}(u))$ and $F_2 = (u, \mu_{F_2}(u), \eta_{F_2}(u))$ be two intuitionistic fuzzy sets where F_1, F_2 are fuzzy sets in U then for every $u \in U$ the intuitionistic fuzzy set operations are defined as given bellow

(1) The union of two intuitionistic fuzzy sets F_1 *and* F_2 is defined as

$F_1 \cup F_2 = (u, \max\{\mu_{F_1}(u)\mu_{F_2}(u)\}, \min\{\eta_{F_1}(u), \eta_{F_2}(u)\})$

(2) The intersection of two intuitionistic fuzzy sets F_1 *and* F_2 is defined as

$F_1 \cap F_2 = (u, \min\{\mu_{F_1}(u)\mu_{F_2}(u)\}, \max\{\eta_{F_1}(u), \eta_{F_2}(u)\})$

(3) The compliment of F_1 is denoted by F_1^c and it is defined $F_1^c = (u, \eta_{F_1}(u), \mu_{F_1}(u))$

(4) Addition : The sum of the two intuitionistic fuzzy sets is

$F_1 + F_2 = (u, \mu_{F_1}(u) + \mu_{F_2}(u) - \mu_{F_1}(u).\mu_{F_2}(u), \eta_{F_1}(u).\eta_{F_2}(u)\})$

(5) Subtraction : The subtraction of the two intuitionistic fuzzy sets is

$F_1 - F_2 = (u, \min\{\mu_{F_1}(u), \eta_{F_2}(u)\}, \max\{\eta_{F_1}(u), \mu_{F_2}(u)\})$

(6) Multiplication: The multiplication of the two intuitionistic fuzzy sets is

$F_1.F_2 = (u, \mu_{F_1}(u).\mu_{F_2}(u), \eta_{F_1}(u) + \eta_{F_2}(u) - \eta_{F_1}(u).\eta_{F_2}(u)\})$

(7) Bounded sum : the bounded sum of the two intuitionistic fuzzy sets $\mu_{F1}(u)$, $\mu_{F2}(u)$ is $\mu_{F1}(u) \oplus \mu_{F2}(u)$ it is defined as $\mu_{F1\oplus F2}$ (u)= = min {1, $\mu_{F1}(u)$ + $\mu_{F2}(u)$}

(8) Bounded difference: the bounded difference $\ominus$ of the two intuitionistic fuzzy sets $\mu_{F1}(u)$, $\mu_{F2}(u)$ is $\mu_{F1}(u) \ominus \mu_{F2}(u)$ it is defined as $\mu_{F1\ominus F2}$ (u)= max{0, $\mu_{F1}(u)$ - $\mu_{F2}(u)$

Definition 2.4: Let (G,+) be an abelian group and let an intuitionistic fuzzy set $G = \{(u, \mu_G(u), \eta_G(u)) / 0 \le \mu_G(u) + \eta_G(u)\} \le 1, \textit{ for all } u \in U\}$ *where* $\mu_G : U \to [0,1]$ *and* $\eta_G : U \to [0,1]$ is said to be intuitionistic fuzzy group if it satisfies the following axioms

$(G-1)\mu_G(g_1 + g_2) \ge \min\{\mu_G(g_1), \mu_G(g_2)\}$ *and* $\mu_G(-g_1) = \mu_G(g_1)$

$(G-2)$ $\eta_G(g_1 + g_2) \le \max\{\eta_G(g_1), \eta_G(g_2)\}$ *and* $\eta_G(-g_1) = \eta_G(g_1)$ *for all* $g_1, g_2 \in G$

Definition 2.5: Let $M_\Gamma = \{m_1, m_2, m_3, ...\}$ *and* $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, ...\}$ be two additive abelian groups and for all $m_1, m_2, m_3 \in M_\Gamma$ *and* $\gamma_1, \gamma_2 \in \Gamma$ and M is called a Gamma-ring the following conditions holds

$(\Gamma-1) m_1\gamma_1 m_2 \in M_\Gamma$

$(\Gamma-2)(m_1 + m_2)\gamma_1 m_3 = m_1\gamma_1 m_3 + m_2\gamma_1 m_3$ *and* $(m_1\gamma_1(m_2 + m_3) = m_1\gamma_1 m_2 + m_1\gamma_1 m_3$

$(\Gamma-3) m_1(\gamma_1 + \gamma_2) m_2 = m_1\gamma_1 m_2 + m_1\gamma_2 m_2$

$(\Gamma-4)(m_1\gamma_1 m_2)\gamma_2 m_3 = m_1\gamma_1(m_2\gamma_2 m_3)$

Definition 2.6: A binary operation $\oplus : I \times I \to I$ where I is a unit closed interval [0,1] is continuous t- norm if it satisfies the following conditions for every t_1, t_2 ,t_3, & $t_4 \in$ I

$(T-1)$ $\oplus$ is associative and commutative,

$(T-2)$ $\oplus$ is continuous

$(T-3)$ $t_1 \oplus 1 = t_1$

$(T-4)$ $t_1 \oplus t_2 \le t_3 \oplus t_4$ *whenever* $t_1 \le t_3$ *and* $t_2 \le t_4$

Definition 2.7: A binary operation $\otimes : I \times I \to I$ is continuous S-norm if it satisfies the following axioms for every t_1, t_2, t_3, & $t_4 \in [0,1]$

$(S-1)$ $\otimes$ is associative and commutative,

$(S-2)$ $\otimes$ is continuous

$(S-3)$ $t_1 \otimes 1 = t_1$

$(S-4)$ $t_1 \otimes t_2 \le t_3 \otimes t_4$ *whenever* $t_1 \le t_3$ *and* $t_2 \le t_4$

Remark 2.8: For any $a, b \in (0,1)$ *with* $a > b$ $\exists$ $c, d \in (0,1) \ni a \oplus c > b$ *and* $b \otimes d < a$

Definition 2.9: Let $R = \{(u, \mu_R(u), \eta_R(u)) / 0 \le \mu_R(u) + \eta_R(u)\} \le 1,$ *for all* $u \in U\}$ *where* $\mu_R : U \to [0,1]$ *and* $\eta_R : U \to [0,1]$ be a intuitionistic fuzzy set of a $\Gamma - ring$ M_Γ is said to be intuitionistic fuzzy sub $\Gamma - ring$ *of* M_Γ if

$(R-1)\mu_R(m_{\gamma 1} - m_{\gamma 2}) \ge \min\{\mu_R(m_{\gamma 1}), \mu_R(m_{\gamma 2})\}$ *and* $\mu_R(m_{\gamma 1}\gamma m_{\gamma 2}) \ge \max\{\mu_R(m_{\gamma 1}), \mu_R(m_{\gamma 2})\}$

$(R-2)$ $\eta_R(m_{\gamma 1} - m_{\gamma 2}) \le \max\{\eta_R(m_{\gamma 1}), \eta_V(m_{\gamma 2})\}$*and* $\eta_R(m_{\gamma 1}\gamma m_{\gamma 2}) \le \min\{\eta_R(m_{\gamma 1}), \eta_V(m_{\gamma 2})\}$

for all $m_{\gamma 1}, m_{\gamma 2} \in M_\Gamma$ *for all* $\gamma \in \Gamma$

Definition 2.10: Let R be of a $\Gamma - ring$ *of* M_Γ and intuitionistic fuzzy set $L = \{(r, \mu_L(r), \eta_L(r)) /$ *for all* $r \in R\}$ is called an intuitionistic fuzzy left ideal of R if it has following axioms holds

(1)$\mu_L(r_1 + r_2) \ge \mu_L(r_1) \oplus \mu_L(r_2)$

(2)$\mu_L(r_1\gamma r_2) \ge (\mu_L(r_1\gamma r_2))\mu_L(r_2) \ge \mu_L(r_1)$

(3)$\eta_L(r_1 + r_2) \le \eta_L(r_1) \otimes \eta_L(r_2)$

(4)$\eta_L(r_1\gamma r_2) \le (\eta_L(r_1\gamma r_2))\eta_L(r_2) \le \eta_L(r_1)$ *for all* $r_1, r_2 \in R, \gamma \in \Gamma$

Similar way we can define the intuitionistic fuzzy right ideal of R.

Definition 2.11: Let an intuitionistic fuzzy set $L = \{(r, \mu_L(r), \eta_L(r)) /$ *for all* $r \in R\}$ is said to be an intuitionistic fuzzy ideal if it is both intuitionistic fuzzy left and right ideal of R in a $\Gamma - ring$ *of* M_Γ

Definition 2.12: Suppose D be an intuitionistic fuzzy $\Gamma - ring$ *of* M_Γ and is called as intuitionistic fuzzy $\Gamma - divison\ ring$ if it contains an identity element and its only non-zero ideal itself.

Definition 2.13: Let V be a vector space on a field F and let an intuitionistic fuzzy set $V=\{(u,\mu_V(u),\eta_V(u))/\mu_V(u)+\eta_V(u)\in[0,1],\ for\ all\, u\in U\}\ where\ \mu_V:U\to[0,1]\ and\ \eta_V:U\to[0,1]$ is said to be intuitionistic fuzzy vector spaces on V under t-norm and s-norm , if it satisfies the following conditions

$(1)\mu_V(w_1+w_2)\ge\mu_V(w_1)\oplus\mu_V(w_2)\ and\ \eta_V(w_1+w_2)\le\eta_V(w_1)\otimes\eta_V(w_2)$
$(2)\mu_V(-w_1)\ge\mu_V(w_1)\ and\ \eta_V(-w_1)\le\eta_V(w_1)$
$(3)\mu_V(\alpha w_1)\ge\mu_V(w_1)\ and\ \eta_V(\alpha w_1)\le\eta_V(w_1)\ for\ all\ w_1,w_2\in V,\alpha\in F$

Definition 2.14: Let (V,+) be an intuitionistic fuzzy abelian group and let D be an intuitionistic fuzzy $\Gamma-divison\ ring$ with identity element and the functionthen V be a left intuitionistic fuzzy $\Gamma-vector\ space$ over D if the functions $\mu_\gamma:D\times\Gamma\times V\to V\ and\ \eta_\gamma:D\times\Gamma\times V\to V$ holds the following conditions for all $for\ all\ w_1,w_2\in V,d_1,d_2\in D,\ \gamma_1,\gamma_2\in\Gamma$

$(1)\mu_\gamma(d_1\gamma_1(w_1+w_2))=\mu_\gamma(d_1\gamma_1w_1+d_1\gamma_1w_2)\ge\min\{\mu_\gamma(w_1),\mu_\gamma(w_2)\}$
$(2)\mu_\gamma((d_1+d_2)\gamma_1w_1)=\mu_\gamma(d_1\gamma_1w_1+d_1\gamma_1w_1)\ge\mu_\gamma(w_1)$
$(3)\mu_\gamma((d_1\gamma_1d_2)\gamma_2w_1)=\mu_\gamma(d_1\gamma_1(d_2\gamma_2w_1))\ge\mu_\gamma(w_1)$
$(4)\mu_\gamma(1.\gamma_1.w_1)=\mu_\gamma(w_1)$
$(5)\eta_\gamma(d_1\gamma_1(w_1+w_2))=\eta_\gamma(d_1\gamma_1w_1+d_1\gamma_1w_2)\le\max\{\eta_\gamma(w_1),\eta_\gamma(w_2)\}$
$(6)\eta_\gamma((d_1+d_2)\gamma_1w_1)=\eta_\gamma(d_1\gamma_1w_1+d_1\gamma_1w_1)\ge\eta_\gamma(w_1)$
$(7)\eta_\gamma((d_1\gamma_1d_2)\gamma_2w_1)=\eta_\gamma(d_1\gamma_1(d_2\gamma_2w_1))\ge\eta_\gamma(w_1)$
$(8)\eta_\gamma(1.\gamma_1.w_1)=\eta_\gamma(w_1)$

Definition 2.15: Let V is a fuzzy vector space over a field F a real valued norm function $\|,....,\|:V\times V\times V.....\times V(m\ times)\to[0,1]$ the pair $(V,\|.,....,.\|)$ is called as the fuzzy m-normed 0linear space(FMNLS), and is called the fuzzy m-norm on V if it has the following properties

$(V-1)\|w_1,w_2,w_3,.....,\mathrm{w}_{m-1},w_m\|=0$ if and only if $w_1,w_2,w_3,.....,\mathrm{w}_{m-1},w_m$ are linearly independent over F.

$(V-2)\|w_1,w_2,w_3,.....,\mathrm{w}_{m-1},w_m\|$ is invariant under any permutation $w_1,w_2,w_3,.....,\mathrm{w}_{m-1},w_m$.

$(V-3)\|w_1,w_2,w_3,.....,\mathrm{w}_{m-1},\sigma w_m\|=|\sigma|\|w_1,w_2,w_3,.....,\mathrm{w}_{m-1},w_m\|$

$(V-4)\|w_1,w_2,w_3,.....,\mathrm{w}_{m-1},w_m,w'_m\|\le\|w_1,w_2,w_3,.....,\mathrm{w}_{m-1},w_m\|+\|w_1,w_2,w_3,.....,\mathrm{w}_{m-1},w'_m\|$

Definition 2.16: A in a Let $(V,\|.,.....,.\|)$ be a FMNLS and the sequence $\{w_r\}_{r=1}^{\infty}$ said to convergence to $w\in V$ if. $\lim_{r\to\infty}\|w_1,w_2,w_3,.....,\mathrm{w}_{r-1},w_r\|=0$

A sequence $\{w_r\}_{r=1}^{\infty}$ in FMNLS $(V,\|.,.....,.\|)$ is a Cauchy sequence if $\lim_{r,q\to\infty}\|w_1,w_2,w_3,.....,\mathrm{w}_{r-1},w_r-w_q\|=0$

Definition 2.17: A fuzzy m-normed linear space is said to be complete If every Cauchy sequence in a $(V, \|.,.....,.\|)$ is convergent.

Definition 2.18: Let $V = \{(u, \mu_V(u), \eta_V(u)) / \mu_V(u) + \eta_V(u) \in [0,1],\ for\ all\ u \in U\}$

$where\ \mu_V : U \to [0,1]\ and\ \eta_V : U \to [0,1]$ is an intuitionistic fuzzy vector space. Suppose $\mu_I\ and\ \eta_I$ be two fuzzy subset of $V \times V \times .... \times V \times V (m - times) \times (-\infty, \infty)$ where $\mu_I\ and\ \eta_I$ are the degree of membership and degrees of non-membership of

$(w_1, w_2, w_3, ..., w_{m-1}, w_m, z) \in V^m \times (-\infty, \infty)$ then the cubic set (V, μ_I, η_I) is called the intuitionistic fuzzy m-normed linear space if it satisfies the following properties for all $(w_1, w_2, w_3, ..., w_{m-1}, w_m, z) \in V^m \times (-\infty, \infty)$

$(I-1) \mu_I((w_1, w_2, w_3, ..., w_{m-1}, w_m, z) = 0,\ \forall z \in (-\infty, 0],\ \forall (w_1, w_2, w_3, ..., w_{m-1}, w_m) \in V^m$

$(I-2)\ \mu_I(w_1, w_2, w_3, ..., w_{m-1}, w_m, z) = 1 \Leftrightarrow w_1, w_2, w_3, ..., w_{m-1}, w_m\ are\ linearly\ dependent.$

$(I-3)\ \mu_I(w_1, w_2, w_3, ..., w_{m-1}, w_m, z)\ is\ in\ \mathrm{var}\ iant\ under\ any\ permutation\ w_1, w_2, w_3, ..., w_{m-1}, w_m.$

$(I-4)\ \mu_I(w_1, w_2, w_3, ..., w_{m-1}, \rho w_m, z) = \mu_I(w_1, w_2, w_3, ..., w_{m-1}, w_m, \frac{z}{|\rho|}),\ when\ \rho \neq 0 \in F$

$(I-5)\ \mu_I(w_1, w_2, w_3, ..., w_{m-1}, w_m, z_1) \oplus \mu_I(w_1, w_2, w_3, ..., w_{m-1}, w'_m, z_2) \leq \mu_I(w_1, w_2, w_3, ..., w_{m-1}, w_m + w'_m, z_1 + z_2)$

$(I-6)\ \mu_I(w_1, w_2, w_3, ..., w_{m-1}, w_m, z) : (-\infty, \infty) \to [0,1]\ is\ a\ non-decrea\sin g\ function\ of\ z$

$, \lim_{z \to \infty} \mu_I(w_1, w_2, w_3, ..., w_{m-1}, w_m, z) = 1\ \ and\ \ \lim_{z \to 0} \mu_I(w_1, w_2, w_3, ..., w_{m-1}, w_m, z) = 0,$

$for\ all\ (w_1, w_2, w_3, ..., w_{m-1}, w_m) \in V^m$

$(I-7) \eta_I((w_1, w_2, w_3, ..., w_{m-1}, w_m, z) \leq 1,\ \forall z \in (0, \infty),\ \forall (w_1, w_2, w_3, ..., w_{m-1}, w_m) \in V^m$

$(I-8) \eta_I(w_1, w_2, w_3, ..., w_{m-1}, w_m, z) = 0 \Leftrightarrow w_1, w_2, w_3, ..., w_{m-1}, w_m\ are\ linearly\ dependent.$

$(I-9)\ \eta_I(w_1, w_2, w_3, ..., w_{m-1}, w_m, z)\ is\ in\ \mathrm{var}\ iant\ under\ any\ permutation\ w_1, w_2, w_3, ..., w_{m-1}, w_m.$

$(I-10) \eta_I(w_1, w_2, w_3, ..., w_{m-1}, \rho w_m, z) = \eta_I(w_1, w_2, w_3, ..., w_{m-1}, w_m, \frac{z}{|\rho|}),\ when\ \ \rho \neq 0 \in F$

$(I-11) \eta_I(w_1, w_2, w_3, ..., w_{m-1}, w_m, z_1) \otimes \eta_I(w_1, w_2, w_3, ..., w_{m-1}, w'_m, z_2) \geq \eta_I(w_1, w_2, w_3, ..., w_{m-1}, w_m + w'_m, z_1 + z_2)$

$(I-12)\ \eta_I(w_1, w_2, w_3, ..., w_{m-1}, w_m, z) : (-\infty, \infty) \to [0,1]\ is\ a\ non-increa\sin g\ \ function\ of\ z$

$, \lim_{z \to \infty} \eta_I(w_1, w_2, w_3, ..., w_{m-1}, w_m, z) = 0\ \ and\ \ \lim_{z \to 0} \eta_I(w_1, w_2, w_3, ..., w_{m-1}, w_m, z) = 1,$

$for\ all\ (w_1, w_2, w_3, ..., w_{m-1}, w_m) \in V^m$

$(I-13)\ \ 0 \leq \mu_I(w_1, w_2, w_3, ..., w_{m-1}, w_m, z) + \eta_I(w_1, w_2, w_3, ..., w_{m-1}, w_m, z) \leq 1,$

$\forall z \in (-\infty, \infty), and\ \forall (w_1, w_2, w_3, ..., w_{m-1}, w_m) \in V^m$

Example 2.19: Let $(V, \|.,.....,.\|)$ be an intuitionistic fuzzy m-normed linear space(FMNLS),define

$$\mu_I(w_1,w_2,w_3,...,w_{m-1},w_m,z)=\frac{z}{z+\|w_1,w_2,w_3,...,w_{m-1},w_m\|}=1, if\ \ \forall z\in(0,\infty),\ \forall(w_1,w_2,w_3,.....,w_{m-1},w_m)\in V^m$$

$$=0,\ if\ \ \forall\ z\in(-\infty,0),\forall(w_1,w_2,w_3,....,w_{m-1},w_m)\in V^m$$

$$\eta_I(w_1,w_2,w_3,...,w_{m-1},w_m,z)=\frac{\|w_1,w_2,w_3,...,w_{m-1},w_m\|}{z+\|w_1,w_2,w_3,...,w_{m-1},w_m\|}=0,\ if\ \forall\ z\in(-\infty,0),\ \forall(w_1,w_2,w_3,.....,w_{m-1},w_m)\in V^m$$

And

$$=1, if\ \ \forall z\in(0,\infty),\ \forall(w_1,w_2,w_3,.....,w_{m-1},w_m)\in V^m$$

then (V,μ_I,η_I) is an intuitionistic fuzzy m-normed linear space.

3. Results and discussions:

Definition 3.1: Let V be a intuitionistic fuzzy left $\Gamma-$*vector space* over D, a real valued function $\|.,....,.\|: V\times V\times V.....\times V(m\ times)\to[0,\infty)$ is called an intuitionistic fuzzy left Gamma-m-normed linear space in D if it satisfies the following properties for every $w_1,w_2,w_3,....,w_{m-1},w_m\in V\ and\ d_1,d_2,d_3,....,d_{m-1},d_m\in D,\gamma\in\Gamma$ it is represented by $(V,\|.,.....,.\|)$

$(I_{\gamma^m}-1)\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m\|=0\Leftrightarrow w_1,w_2,w_3,.....,\mathrm{w}_{m-1},w_m$ are linearly independent over D.

$(I_{\gamma^m}-2)\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m\|$ is invariant under any permutation $w_1,w_2,w_3,.....,\mathrm{w}_{m-1},w_m$

$(I_{\gamma^m}-3)\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},\rho(d_m\gamma w_m)\|=|\rho|\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m\|$

$(I_{\gamma^m}-4)\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m+d_m\gamma w'_m\|\le$
$\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m\|+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w'_m\|$

Similarly we can define an intuitionistic fuzzy right Gamma-m-normed linear space in D.

Definition 3.2: A real valued function $\|.,....,.\|: V\times V\times V.....\times V(m\ times)\to[0,\infty)$ is called an intuitionistic fuzzy Gamma-m-normed linear space in D if it is either an intuitionistic fuzzy left Gamma-m-normed linear space in D and an intuitionistic fuzzy right Gamma-m-normed linear space in D.

Definition 3.3: A cubic set C_I in a nonempty set K is the structure $C_I=\{<k,\mu_I(k),\eta_I(k)>/k\in K\}$ and it is denoted by $C_I=<\mu_I,\eta_I>$ where $\mu_I\in[\mu_{I^-},\mu_{I^+}]$ and $\eta_I\in[\eta_{I^-},\eta_{I^+}]$ are interval valued fuzzy sets in K.

Definition 3.4: Let V is a fuzzy vector space over a field F and (V,μ) *and* (V,η) be two the interval valued fuzzy vector spaces of V. A cubic set $C_I=<\mu_I,\eta_I>$ in V is represents a cubic linear space of V if for all $w_1,w_2\in V\ and\ f_1,f_2\in F$

$(1)\mu(f_1w_1,f_2w_2)\ge\min\{f_1w_1,f_2w_2\}$
$(2)\eta(f_1w_1,f_2w_2)\le\max\{f_1w_1,f_2w_2\}$

Definition 3.5: Let $(V,\|.,.....,.\|)$ be an intuitionistic fuzzy Gamma-m-normed linear space in D and the sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ said to convergence to $d\gamma w \in V$ if.

$$\lim_{r\to\infty}\| d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3,.....,d_{r-1}\gamma w_{r-1}, d_r\gamma w_r \| = d\gamma w$$

A sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ in an intuitionistic fuzzy Gamma-m-normed linear space in D $(V,\|.,.....,.\|)$ is a Cauchy sequence if

$$\lim_{r,q\to\infty}\| d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3,.....,d_{r-1}\gamma w_{r-1}, d_r\gamma w_r - d_q\gamma w_q \| = 0$$

Definition 3.6: An intuitionistic fuzzy Gamma-m-normed linear space in D ,is said to be complete If every Cauchy sequence in a $(V,\|.,....,.\|)$ is convergent in D.

Definition 3.7: Let V be an intuitionistic fuzzy left $\Gamma-$*vector space* in a D, a real valued function $\mu_\Gamma : V\times V\times V\times ......\times V(m\,times)\times[-\infty,\infty]\to[0,1]$ *and* $\eta_\Gamma : V\times V\times V.....\times V(m\ times)\times[-\infty,\infty]\to[0,1]$ be two intuitionistic fuzzy sets and the cubic structure $< V,\mu_\Gamma,\eta_\Gamma >$ is said to be cubic intuitionistic fuzzy Gamma-m-normed linear space in D if it is the following axioms holds for all $w_1,w_2,w_3,....,w_{m-1},w_m \in V$ *and* $d_1,d_2,d_3,....,d_{m-1},d_m \in D,\gamma\in\Gamma$

$(I_\Gamma-1)\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z)=0,\ \forall\, z\in(-\infty,0].$

$(I_\Gamma-2)\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z)=1,\Leftrightarrow w_1,w_2,w_3,.....,w_{m-1},w_m$ *are linearly dependent.*

$(I_\Gamma-3)\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z)$ *is invariant under any permutation* $w_1,w_2,w_3,.....,w_{m-1},w_m.$

$(I_\Gamma-4)\ \mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},(\rho d_m\gamma w_m),z)$
$=\mu_\Gamma(w_1\gamma d_1,w_2\gamma d_2,w_3\gamma d_3,...,w_{m-1}\gamma d_{m-1},w_m\gamma d_m,\frac{z}{|\rho|})$, *when* $\rho\neq 0\in F$

$(I_\Gamma-5)\ [\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z_1)\oplus$
$\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m,z_2)]$
$\leq \mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m+d_m\gamma w'_m,z_1+z_2)$

$(I_\Gamma-6)\ \mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z):(-\infty,\infty)\to[0,1]$
is a non-decreasing function of z, $\lim_{z\to\infty}\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,......,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z)=1$ *and*
$\lim_{z\to 0}\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z)=0,$

$(I_\Gamma-7)\ \eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z)\leq 1,\ \forall\, z\in(0,\infty),$

$(I_\Gamma-8)\eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z)=0$
$\Leftrightarrow w_1,w_2,w_3,....,w_{m-1},w_m$ *are linearly dependent.*

$(I_\Gamma-9)\ \eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z)$ *is invariant under any permutation* $w_1,w_2,w_3,.....,w_{m-1},w_m.$

$(I_\Gamma-10)\eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},\rho(d_m\gamma w_m),z)=$

$\eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m,\dfrac{z}{|\rho|}),when\ \rho\neq 0\in F$

$(I_\Gamma-11)\eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m,z_1)\otimes\eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w'_m,z_2)$
$\geq\eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m+d_m\gamma w'_m,z_1+z_2)$
$(I_\Gamma-12)\ \eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m,z):(-\infty,\infty)\to[0,1]\ \ is\ a\ non-increa\sin g\ \ function\ of\ \ z$
$,\lim\limits_{z\to\infty}\eta_\Gamma d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m,z)=0\ \ \ and$
$\lim\limits_{z\to 0}\eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m,z)=1,$

$(I_\Gamma-13)\ \ 0\leq\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m,z)+$
$\eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m,z)\leq 1,\forall z\in(-\infty,\infty),$

Definition 3.8: Let $(V,\|.,.....,.\|)$ be an intuitionistic fuzzy Gamma-m-normed linear space in D such that for all $t_1,t_2\in[0,1]\ then\ t_1\oplus t_2=\min\{t_1,t_2\}\ and\ t_1\otimes t_2=\max\{t_1,t_2\}$

Also define $\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z)=\dfrac{z}{z+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|}$ and

$\eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m,z)=\dfrac{\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m\|}{z+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m\|}$ then we prove that the cubic structure $<V,\mu_\Gamma,\eta_\Gamma>$ is the cubic intuitionistic fuzzy Gamma-m-normed linear space

$(I_\Gamma-1)Obiously\ \ for\,every\,z\leq 0\ \ \mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m,z)=0.$

$(I_\Gamma-2)\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m,z)=1$

$\Leftrightarrow\dfrac{z}{z+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m\|}=1$

$\Leftrightarrow z=z+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m\|$

$\Leftrightarrow\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m\|=0$

$\Leftrightarrow w_1,w_2,w_3,.....,w_{m-1},w_m\ are\,linearly\,dependent.$

$(I_\Gamma-3)\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m,z)=\dfrac{z}{z+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1},d_m\gamma w_m\|}$

$\Leftrightarrow\dfrac{z}{z+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,......,d_m\gamma w_m,d_{m-1}\gamma \mathrm{w}_{m-1}\|}=\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,......,d_m\gamma w_m,d_{m-1}\gamma \mathrm{w}_{m-1},z)=.....$
$it\ \ is\,in\,\mathrm{var}\,iant\,under\,any\ permutation\ \ w_1,w_2,w_3,.....,w_{m-1},w_m.$

$$(I_\Gamma-4)\ \mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},(\rho d_m\gamma w_m),z)$$

$$=\frac{z}{z+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},(\rho d_m\gamma w_m)\|}=\frac{z}{z+|\rho|\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|}$$

$$=\frac{\frac{z}{|\rho|}}{\frac{z}{|\rho|}+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|}$$

$$=\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,\frac{z}{|\rho|}),\ when\ \rho\neq 0\in F$$

$(I_\Gamma-5)$

now we consider $\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z_1)<\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m,z_2)$

$$\Leftrightarrow\frac{z_1}{z_1+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|}<\frac{z_2}{z_2+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m,z_2\|}$$

$$\Leftrightarrow z_1(z_2+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m\|)<z_2(z_1+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|)$$

$$\Leftrightarrow z_2\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|)<z_1\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m\|)\geq 0......(1)$$

again we consider $\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m+d_m\gamma w'_m,z_1+z_2)-\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z_1)$

$$\Leftrightarrow\frac{(z_1+z_2)}{(z_1+z_2)+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m+d_m\gamma w'_m\|}-\frac{z_1}{z_1+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|}$$

$$\Leftrightarrow\frac{(z_1+z_2)(z_1+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|)-z_1((z_1+z_2)+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m+d_m\gamma w'_m\|)}{(z_1+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|)((z_1+z_2)+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m+d_m\gamma w'_m\|)}$$

$$\Leftrightarrow\frac{z_2\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|-z_1\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m\|}{(z_1+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|)((z_1+z_2)+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m+d_m\gamma w'_m\|)}\geq 0\ by\ eqn(1)$$

hence $\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m+d_m\gamma w'_m,z_1+z_2)\geq\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z_1)$

$$\Leftrightarrow\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z_1)\oplus\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m,z_2)$$
$$\leq\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m+d_m\gamma w'_m,z_1+z_2)$$

$(I_\Gamma-6)$ *Clearly* $\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z)$ *is a non-decreasing function of* z *then*

$$\lim_{z\to\infty}\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,......,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z)$$

$$=\lim_{z\to\infty}\frac{z}{z+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|}=1\ and$$

$$\lim_{z\to 0}\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z)$$

$$=\lim_{z\to 0}\frac{z}{z+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|}=0,$$

$(I_\Gamma-7)$ *clearly* $\eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z)\leq 1$, *for any* $z\in(0,\infty)$,

$$(I_\Gamma-8)\eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z)=\eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z)=0$$

$$\Leftrightarrow\frac{\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|}{z+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|}=0\Leftrightarrow\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|=0$$

$\Leftrightarrow w_1,w_2,w_3,....,w_{m-1},w_m$ *are linearly dependent*.

$$(I_\Gamma-9)\ \eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z)=\frac{\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|}{z+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|}$$

$$\Leftrightarrow\frac{\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_m\gamma w_m,d_{m-1}\gamma w_{m-1}\|}{z+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_m\gamma w_m,d_{m-1}\gamma w_{m-1}\|}=\eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,......,d_m\gamma w_m,d_{m-1}\gamma w_{m-1},z)....so\,on$$

$hence\,it\,is\,in\mathrm{var}iant\ under\,any\ \ permutation\,w_1,w_2,w_3,....,w_{m-1},w_m.$

$$(I_\Gamma-10)\eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},\rho(d_m\gamma w_m),z)$$

$$=\frac{\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},\rho(d_m\gamma w_m)\|}{z+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},\rho(d_m\gamma w_m)\|}\Leftrightarrow\frac{|\rho|\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|}{z+|\rho|\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|}$$

$$\Leftrightarrow\frac{\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|}{\frac{z}{|\rho|}+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|}$$

$$=\eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,\frac{z}{|\rho|}),\rho\neq0\in F$$

$$(I_\Gamma-11)\,now\ we\ consider\ \eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z_1)$$
$$<\eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m,z_2)$$

$$\Leftrightarrow\frac{\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|}{z_1+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|}<\frac{\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|}{z_2+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m,z_2\|}$$

$$\Leftrightarrow\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|(z_2+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m\|)$$
$$<\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m\|(z_1+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|)$$

$$\Leftrightarrow z_2\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|)<z_1\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m\|)......(1)$$

$$again\ \ we\ consider\ \eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m+d_m\gamma w'_m,z_1+z_2)-$$
$$\eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m,z_2)$$

$$\Leftrightarrow\frac{\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m+d_m\gamma w'_m\|}{(z_1+z_2)+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m+d_m\gamma w'_m\|}-\frac{\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m\|}{z_2+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m\|}$$

$$\Leftrightarrow\frac{\begin{array}{c}\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m+d_m\gamma w'_m\|(z_2+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m\|)-\\ \|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m+d_m\gamma w'_m\|((z_1+z_2)+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m+d_m\gamma w'_m\|)\end{array}}{(z_2+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m\|)((z_1+z_2)+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m+d_m\gamma w'_m\|)}$$

$$\Leftrightarrow[\frac{z_2\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\|-z_1\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m\|}{\begin{array}{c}(z_2+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m\|)\\((z_1+z_2)+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m+d_m\gamma w'_m\|)\end{array}}]\leq0\ by\ equation(1)$$

$$hence\ \mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m+d_m\gamma w'_m,z_1+z_2)\leq\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z_1)$$
$$\Leftrightarrow[\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m,z_1)\otimes\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w'_m,z_2)]$$
$$\geq\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m+d_m\gamma w'_m,z_1+z_2)$$

$(I_\Gamma - 12)$ *Clearly* $\mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m, z)$ *is a non-increasing function of z then*

$$\lim_{z\to\infty}\eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m, z)$$

$$= \lim_{z\to\infty}\frac{\| d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m \|}{z + \| d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m \|} = 0 \ \ and$$

$$\lim_{z\to 0}\eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m, z)$$

$$= \lim_{z\to 0}\frac{\| d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m \|}{z + \| d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m \|} = 1,$$

$(I_\Gamma - 13)$ *Obviously* $\forall z \in (-\infty, \infty)$ *such that* $\mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m, z) +$
$\eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m, z) \in [0,1].$

Definition 3.9: Let $< V, \mu_\Gamma, \eta_\Gamma >$ is cubic intuitionistic fuzzy Gamma-m-normed linear space , a sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ is said to convergent to $d\gamma w$ in if given that for all $\theta \in (0,1)$ and z>0 there exists a positive integer k such that for all $m \geq k$

$$\mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m - d\gamma w, z) > 1-\theta, \eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m - d\gamma w, z) < \theta$$

Definition 3.10: Suppose $< V, \mu_\Gamma, \eta_\Gamma >$ be the cubic intuitionistic fuzzy Gamma-m-normed linear space and a sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ is a Cauchy sequence in $< V, \mu_\Gamma, \eta_\Gamma >$ if given that for all $\delta \in (0,1)$ and z>0 there exists a positive integer k such that for all $m, q \geq k$

$$\mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m - d_q\gamma w_q, z) > 1-\delta, \eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m - d_q\gamma w_q, z) < \delta$$

Definition 3.11: Suppose every Cauchy sequence is convergent in the cubic intuitionistic fuzzy Gamma-m-normed linear space $< V, \mu_\Gamma, \eta_\Gamma >$ then it is complete in $< V, \mu_\Gamma, \eta_\Gamma >$.

Theorem 3.12: Let $< V, \mu_\Gamma, \eta_\Gamma >$ is cubic intuitionistic fuzzy Gamma-m-normed linear space , a sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ is said to convergent to $d\gamma w$ in $< V, \mu_\Gamma, \eta_\Gamma >$ if and only if

$$\lim_{m\to\infty}\mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m - d\gamma w, z) = 1 \ and \ \lim_{m\to\infty}\eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m - d\gamma w, z) = 0$$

Proof: If $< V, \mu_\Gamma, \eta_\Gamma >$ is the cubic intuitionistic fuzzy Gamma-m-normed linear space and a sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ is convergent to $d\gamma w$, fix z given that for all $\theta \in (0,1)$ there exists a positive integer k such that for all $m \geq k$ $\mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m - d\gamma w, z) > 1-\theta$
$\eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m - d\gamma w, z) < \theta$

Therefore

$$1-\mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m - d\gamma w, z) < \theta \Rightarrow \eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m - d\gamma w, z) < \theta$$

Hence $\lim_{m\to\infty}\mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m - d\gamma w, z) = 1 \ and$

$$\lim_{m\to\infty}\eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m - d\gamma w, z) = 0$$

Conversely we have for each z >0 such that

$$\lim_{m\to\infty}\mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m - d\gamma w, z) = 1 \ and \ \lim_{m\to\infty}\eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m - d\gamma w, z) = 0$$

given that for very $\theta \in (0,1)$ and z> 0 there exists a positive integer k such that for all $m \geq k$

$$1-\mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, ..., d_{m-1}\gamma \text{w}_{m-1}, d_m\gamma w_m - d\gamma w, z) < \theta \Leftrightarrow$$

$\mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, \ldots, d_{m-1}\gamma w_{m-1}, d_m\gamma w_m - d\gamma w, z) > 1-\theta$ *and*
$\eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, \ldots, d_{m-1}\gamma w_{m-1}, d_m\gamma w_m - d\gamma w, z) < \theta$

Hence a sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ is to convergent to $d\gamma w$ in $< V, \mu_\Gamma, \eta_\Gamma >$.

Theorem 3.13: Every convergent sequence is Cauchy sequence in the cubic intuitionistic fuzzy Gamma-m-normed linear space $< V, \mu_\Gamma, \eta_\Gamma >$.

Proof: Let $\{d_r\gamma w_r\}_{r=1}^{\infty}$ be any sequence in the cubic intuitionistic fuzzy Gamma-m-normed linear space $< V, \mu_\Gamma, \eta_\Gamma >$ and is convergent to $d\gamma w$ in $< V, \mu_\Gamma, \eta_\Gamma >$ if for all $z > 0$ *and* $\delta \in (0,1)$

Let $z > 0$ *and* $\theta \in (0,1)$ *such that* $\theta \oplus \theta < \delta$ *and* $(1-\theta) \oplus (1-\theta) > 1-\delta$

Given that a sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ is to convergent to $d\gamma w$ in $< V, \mu_\Gamma, \eta_\Gamma >$ there is a positive integer $k \in \mathbb{N}$ such that for all $m \geq k$ $\mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, \ldots, d_{m-1}\gamma w_{m-1}, d_m\gamma w_m - d\gamma w, \frac{z}{2}) > 1-\theta$ *and*

$\eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, \ldots, d_{m-1}\gamma w_{m-1}, d_m\gamma w_m - d\gamma w, \frac{z}{2}) < \theta$

We have

$$\mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, \ldots, d_{m-1}\gamma w_{m-1}, d_m\gamma w_m - d_q\gamma w_q, z)$$
$$= \mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, \ldots, d_{m-1}\gamma w_{m-1}, d_m\gamma w_m + d\gamma w - d\gamma w - d_q\gamma w_q, \frac{z}{2} + \frac{z}{2})$$
$$\geq [\mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, \ldots, d_{m-1}\gamma w_{m-1}, d_m\gamma w_m - d\gamma w, \frac{z}{2}) \oplus$$
$$\mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, \ldots, d_{m-1}\gamma w_{m-1}, d\gamma w - d_q\gamma w_q, \frac{z}{2})] > [(1-\theta) \oplus (1-\theta)] > 1-\delta, \; \text{for all } m, q \geq k$$
$$\text{and} \quad \eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, \ldots, d_{m-1}\gamma w_{m-1}, d_m\gamma w_m - d_q\gamma w_q, z)$$
$$= \eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, \ldots, d_{m-1}\gamma w_{m-1}, d_m\gamma w_m + d\gamma w - d\gamma w - d_q\gamma w_q, \frac{z}{2} + \frac{z}{2})$$
$$\leq [\eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, \ldots, d_{m-1}\gamma w_{m-1}, d_m\gamma w_m - d\gamma w, \frac{z}{2}) \otimes$$
$$\eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3, \ldots, d_{m-1}\gamma w_{m-1}, d\gamma w - d_q\gamma w_q, \frac{z}{2})] < [\theta \otimes \theta)] < \delta, \; \text{for all } m, q \geq k$$

Therefore a sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ is a Cauchy sequence in $< V, \mu_\Gamma, \eta_\Gamma >$

Hence it is proved that every convergent sequence is Cauchy sequence in the cubic intuitionistic fuzzy Gamma-m-normed linear space $< V, \mu_\Gamma, \eta_\Gamma >$.

Remark 3.14: The following example proves that sequence in the cubic intuitionistic fuzzy Gamma-m-normed linear space $< V, \mu_\Gamma, \eta_\Gamma >$ may exist the Cauchy sequence which is not convergent.

Example 3.15: Suppose $<V,\mu_\Gamma,\eta_\Gamma>$ is a cubic intuitionistic fuzzy Gamma-m-normed linear space as in above example let a $\{d_r\gamma w_r\}_{r=1}^{\infty}$ is a sequence in $<V,\mu_\Gamma,\eta_\Gamma>$ then

(1) A sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ is a convergent sequence in an intuitionistic fuzzy Gamma-m-normed linear space $(V,\|.,....,.\|)$ if and only if a sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ is a convergent sequence in cubic intuitionistic fuzzy Gamma-m-normed linear space $<V,\mu_\Gamma,\eta_\Gamma>$.

(2) A sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ be a Cauchy sequence in an intuitionistic fuzzy Gamma-m-normed linear space $(V,\|.,....,.\|)$ if and only if a sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ be a Cauchy sequence in cubic intuitionistic fuzzy Gamma-m-normed linear space $<V,\mu_\Gamma,\eta_\Gamma>$.

Proof: (1) Suppose a sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ is a convergent sequence in an intuitionistic fuzzy Gamma-m-normed linear space $(V,\|.,....,.\|)$

$$\Leftrightarrow \lim_{m\to\infty}\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\| = d\gamma w$$

$$\Leftrightarrow \lim_{m\to\infty}\frac{z}{z+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m-d\gamma w\|}=1$$

$$\Leftrightarrow \lim_{m\to\infty}\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m-d\gamma w,z)=1$$

$$\Leftrightarrow \mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m-d\gamma w,z)>1-\theta$$

for all $\theta\in(0,1)$ there exists a positive integer k such that for all $m\geq k$

and a sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ is a convergent sequence in $(V,\|.,....,.\|)$

$$\Leftrightarrow \lim_{m\to\infty}\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m\| = d\gamma w$$

$$\Leftrightarrow \lim_{m\to\infty}\frac{\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m-d\gamma w\|}{z+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m-d\gamma w\|}=0$$

$$\Leftrightarrow \lim_{m\to\infty}\eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m-d\gamma w,z)=0$$

$$\Leftrightarrow \eta_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m-d\gamma w,z)<\theta$$

for all $\theta\in(0,1)$ there exists a positive integer k such that for all $m\geq k$

hence a sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ is a convergent sequence in cubic intuitionistic fuzzy Gamma-m-normed linear space $<V,\mu_\Gamma,\eta_\Gamma>$.

(2)Let a sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ be a Cauchy sequence in an intuitionistic fuzzy Gamma-m-normed linear space $(V,\|.,....,.\|)$

$$\Leftrightarrow \lim_{m,q\to\infty}\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m-d_q\gamma w_q\|=1$$

$$\Leftrightarrow \lim_{m,q\to\infty}\frac{z}{z+\|d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m-d_q\gamma w_q\|}=1$$

$$\Leftrightarrow \lim_{m,q\to\infty}\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m-d_q\gamma w_q,z)=1$$

$$\Leftrightarrow \lim_{m,q\to\infty}\mu_\Gamma(d_1\gamma w_1,d_2\gamma w_2,d_3\gamma w_3,.....,d_{m-1}\gamma w_{m-1},d_m\gamma w_m-d_q\gamma w_q,z)>1-\delta$$

for all $\delta \in (0,1)$ there exists a positive integer k such that for all $m,q \geq k$

and suppose a sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ be a Cauchy sequence in $(V,\|.,....,.\|)$

$\Leftrightarrow \lim\limits_{m,q\to\infty} \| d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3,......,d_{m-1}\gamma w_{m-1}, d_m\gamma w_m - d_q\gamma w_q \| = 0$

$\Leftrightarrow \lim\limits_{m,q\to\infty} \dfrac{\| d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3,......,d_{m-1}\gamma w_{m-1}, d_m\gamma w_m - d_q\gamma w_q \|}{z+\| d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3,......,d_{m-1}\gamma w_{m-1}, d_m\gamma w_m - d_q\gamma w_q \|} = 0$

$\Leftrightarrow \lim\limits_{m,q\to\infty} \eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3,......,d_{m-1}\gamma w_{m-1}, d_m\gamma w_m - d_q\gamma w_q, z) = 0$

$\Leftrightarrow \lim\limits_{m,q\to\infty} \eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3,.......,d_{m-1}\gamma w_{m-1}, d_m\gamma w_m - d_q\gamma w_q, z) < \delta$

for every $\delta \in (0,1)$ there exists a positive integer k such that for all $m,q \geq k$

hence a sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ is a Cauchy sequence in cubic intuitionistic fuzzy Gamma-m-normed linear space $< V, \mu_\Gamma, \eta_\Gamma >$.

Thus if there exists an intuitionistic fuzzy Gamma-m-normed linear space $(V,\|.,....,.\|)$ which is not complete, then the cubic intuitionistic fuzzy Gamma-m-normed linear space induced by such a fuzzy gamma –m-norm $\|.,....,.\|$ on an incomplete an intuitionistic fuzzy Gamma-m-normed linear space V is an incomplete the cubic intuitionistic fuzzy Gamma-m-normed linear space

Theorem 3.16: Every Cauchy sequence has a convergence sub sequence in the cubic intuitionistic fuzzy Gamma-m-normed linear space $< V, \mu_\Gamma, \eta_\Gamma >$ then it is complete.

Proof: Let $\{d_{r_j}\gamma w_{r_j}\}_{r_j=1}^{\infty}$ is a sub sequence of $\{d_r\gamma w_r\}_{r=1}^{\infty}$ in the cubic intuitionistic fuzzy Gamma-m-normed linear space $< V, \mu_\Gamma, \eta_\Gamma >$ which is convergent to $d\gamma w$ in $< V, \mu_\Gamma, \eta_\Gamma >$

Now we have to prove that $\{d_{r_j}\gamma w_{r_j}\}_{r_j=1}^{\infty}$ is a convergent to $d\gamma w$ if

$z > 0 \ and \ \theta \in (0,1) \ such that \ \theta \oplus \theta < \delta \ and \ (1-\theta) \oplus (1-\theta) > 1-\delta$

Since a sequence $\{d_r\gamma w_r\}_{r=1}^{\infty}$ is a cauchy sequence in $< V, \mu_\Gamma, \eta_\Gamma >$ there exists a positive integer k such that for all $m,q \geq k$ such that $\mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1}, d_m\gamma w_m - d\gamma w, \frac{z}{2}) > 1-\theta \ and$

$\eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3,.....,d_{m-1}\gamma \mathrm{w}_{m-1}, d_m\gamma w_m - d\gamma w, \frac{z}{2}) < \theta$

Also we have $\{d_{r_j}\gamma w_{r_j}\}_{r_j=1}^{\infty}$ is a convergent to $d\gamma w$ there exists a positive integer k such that for all $m_j \geq k$ such that $\mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3,.....,d_{m_j-1}\gamma w_{m_j-1}, d_{m_j}\gamma w_{m_j} - d\gamma w, \frac{z}{2}) > 1-\theta \ and$

$\eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3,.....,d_{m_j-1}\gamma w_{m_j-1}, d_{m_j}\gamma w_{m_j} - d\gamma w, \frac{z}{2}) < \theta$

$$\text{we have } \mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3,.....,d_{m_j-1}\gamma \mathrm{w}_{m_j-1}, d_{m_j}\gamma w_{m_j} - d\gamma w, z)$$

$$= \mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3,.....,d_{m_j-1}\gamma \mathrm{w}_{m_j-1}, d_{m_j}\gamma w_{m_j} - d\gamma w, \frac{z}{2}+\frac{z}{2})$$

$$\geq [\mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3,.....,d_{m_j-1}\gamma \mathrm{w}_{m_j-1}, d_{m_j}\gamma w_{m_j} - d\gamma w, \frac{z}{2}) \oplus$$

$$\mu_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3,.....,d_{m_j-1}\gamma \mathrm{w}_{m_j-1}, d_{m_j}\gamma w_{m_j} - d\gamma w, \frac{z}{2})] > [(1-\theta)\oplus(1-\theta)] > 1-\delta, \text{ for all } m_j \geq k$$

$$\text{and} \quad \eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3,.....,d_{m_j-1}\gamma \mathrm{w}_{m_j-1}, d_{m_j}\gamma w_{m_j} - d\gamma w, z)$$

$$= \eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3,.....,d_{m_j-1}\gamma \mathrm{w}_{m_j-1}, d_{m_j}\gamma w_{m_j} - d\gamma w, \frac{z}{2}+\frac{z}{2})$$

$$\leq [\eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3,.....,d_{m_j-1}\gamma \mathrm{w}_{m_j-1}, d_{m_j}\gamma w_{m_j} - d\gamma w, \frac{z}{2}) \otimes$$

$$\eta_\Gamma(d_1\gamma w_1, d_2\gamma w_2, d_3\gamma w_3,.....,d_{m_j-1}\gamma \mathrm{w}_{m_j-1}, d_{m_j}\gamma w_{m_j} - d\gamma w, \frac{z}{2})] < [\theta \otimes \theta] < \delta, \text{ for all } m_j \geq k$$

Therefore a sequence $\{d_{r_j}\gamma w_{r_j}\}_{r_j=1}^{\infty}$ is convergence to $d\gamma w$ in the cubic intuitionistic fuzzy Gamma-m-normed linear space $< V, \mu_\Gamma, \eta_\Gamma >$

Hence it is complete in $< V, \mu_\Gamma, \eta_\Gamma >$.

4. Conclusions:

In this research paper we premeditated algebraic properties of the intuitionistic fuzzy Gamma gamma ring, intuitionistic fuzzy gamma ideals, left and right intuitionistic fuzzy gamma vector space. Using these conceptualization we established the definitions with examples of intuitionistic fuzzy gamma-2-normed linear space, intuitionistic fuzzy Gamma-m-Normed Linear Space and cubic intuitionistic fuzzy Gamma-m-normed linear space. This work presents the properties with examples of theory of intuitionistic fuzzy Gamma-m-normed linear space and studied the results of convergence sequence and Cauchy sequence in the cubic intuitionistic fuzzy Gamma-m-normed linear space. Also we provided the theorem on completeness in in the cubic intuitionistic fuzzy Gamma-m-normed linear space. This research work can be extended to Banach spaces in in the cubic intuitionistic fuzzy Gamma-m-normed linear space.

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