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Common Fixed Point Results for Contractive Mappings in Bicomplex Valued B-Metric Spaces

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Abstract:
In this article, we extend and generalised the results of Ahmad et.al., and to establish the existence and uniqueness of common fixed points for pair of self mappings on a closed ball in bicomplex valued b-metric space. Our results generalised well known results in the literature.

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1. Introduction

The theory of bicomplex numbers have been studied for quite a long time, which probably began with the works [3,4,5].The algebra of bicomplex numbers are widely used in the literature as it becomes a viable commutative alternative [4,5] to the non commutative skew field of quaternions (both are four-dimensional and generalization of complex numbers).The commutativity in the former is gained at the cost of the fact that the ring of these numbers contains zero-divisors and so can not form a field .It is well known that the fixed point theory plays a very important role in theory and applications, in particular, whose importance comes from finding roots of algebraic equation and numerical analysis. Banach contraction principle in [15] gives appropriate and simple conditions to establish the existence and uniqueness of a solution of an operator equation $T x = x$. Later, a number of papers were devoted to the improvement and generalization of that result. Most of these results deal with the generalizations of the different contractive conditions in metric spaces [7,10,11,17]. There have been a number of generalizations of metric spaces such as vector valued metric spaces, G - metric spaces, pseudometric spaces, fuzzy metric spaces, D -metric spaces, cone metric spaces, and modular metric spaces. Bakhtin [14] introduced the notion of b -metric space which is a generalized form of metric spaces. Azam et al. [2,9] introduced the notion of complex-valued metric space which is a generalization of classical metric space and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive condition. The concept of complex valued b -metric spaces was introduced in 2013 by Rao et al. [18]. In sequel, Mukheimer [16] proved some common fixed point theorems in complex valued b -metric spaces. Recently Junesang Choi et al. [1] introduced the notion of bi-complex valued metric space which is a generalization of classical metric space and proved certain common fixed point theorems for a pair of weakly compatible mappings satisfying (CLRg) (or (E.A)) property in the bicomplex valued metric spaces. In 2019 Jebril et.al.[8] proved some important theorems on common fixed point theorems under rational contractions for pair of mappings in bicomplex valued metric spaces . In this article , we extend and generalised the results of Ahmad et.al.[13], Dubey et al. [17] and Rao et al. [18] and to establish the existence and uniqueness of common fixed points for pair of self mappings on a closed ball in bicomplex valued b-metric space which extends a recent results of, I.Beg, S.K.Datta and D.Pal [6] , Md. A.Hoque [12] and several others.Here in the following, the set of bicomplex numbers ,complex numbers and real numbers are denoted by $\mathbb{C}_2$, $\mathbb{C}$ and $\mathbb{C}_0$ respectively. $\mathbb{C}_2$ becomes a real commutative algebra with the identity

1=1+i_1. 0+i_2.0+$i_1 i_2$ .0, the set of bicomplex number is defined as

$\mathbb{C}_2$={ξ=a_0+$a_1 i_1$+i_2. a_2+$i_1 i_2$. a_3: a_0,a_1, a_2, $a_3 \in \mathbb{C}_0$ and $i_1{}^2 = i_2{}^2 = -1\}$.

Definition 1:Let ξ_1=u_1+$i_2\, u_2 \in \mathbb{C}_2$ and ξ_2=v_1+$i_2\, v_2 \in \mathbb{C}_2$ define partial order relation $\precsim_{i_2}$ on $\mathbb{C}_2$ as follows (see, e.g. [6]): $\xi_1 \precsim_{i_2} \xi_2$ if and only if $u_1 \precsim v_1$ and $u_2 \precsim v_2$ ....(1)

where $\precsim$ is the partial order on $\mathbb{C}_1$ (see,e.g [2]).

Thus $\xi_1 \precsim_{i_2} \xi_2$ if any one of the following properties holds:

[bo_1] if $u_1 = v_1$ and $u_2 = v_2$;

[bo_2] if $u_1 \prec v_1$ and $u_2 = v_2$;

[bo_3] if $u_1 = v_1$ and $u_2 \prec v_2$;

[bo_4] if $u_1 \prec v_1$ and $u_2 \prec v_2$.

We write $\xi_1 \not\precsim_{i_2} \xi_2$ if $\xi_1 \precsim_{i_2} \xi_2$ and $\xi_1 \neq \xi_2$ i.e , one of [bo_2], [bo_3] and [bo_4] is satisfied and we write $\xi_1 \prec_{i_2} \xi_2$ if only [bo_4] is satisfied.

The norm ||.||: $\mathbb{C}_2 \to \mathbb{C}_0{}^+$ (the set of all non negative real numbers) of a bicomplex number is defined as || ξ||=$\mathbb{C}_1\sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$ …………………(2)

For any two bicomplex numbers ξ_1 , $\xi_2 \in \mathbb{C}_2$, one can easily verify that

$0 \precsim_{i_2} \xi_1 \precsim_{i_2} \xi_2 \Rightarrow \left||\xi_1|\right| \leq \left||\xi_2|\right|$; $||\xi_1 + \xi_2|| \leq \left||\xi_1|\right| + \left||\xi_2|\right|$; $||\xi_1.\xi_2|| \leq \sqrt{2}\left||\xi_1|\right|.\left||\xi_2|\right|$ and

||a ξ||≤ a|| ξ|| where a∈ $\mathbb{C}_0{}^+$.

Bicomplex metric space:

Choi et al. [1] define the bicomplex valued metric space as :

Definition 2:Let X be a non empty set. Suppose the mapping d: X ×X → $\mathbb{C}_2$ satisfies the following conditions:

[1] $0 \precsim_{i_2}$ d(x,y) for all x,y∈ X;

[2] d(x,y)=0 if and only if x=y;

[3] d(x,y)=d(y,x) for all x,y∈ X;

[4] $d(x,y) \precsim_{i_2}$ d(x,z)+d(z,y) for all x,y∈ X .

Then (X,d) is called a bicomplex valued metric space.

Definition 3:[1] A sequence in a nonempty set X is a function x: $\mathbb{N} \to \mathbb{C}_2$, which is expressed by its range set $\{x_n\}$ where x(n)= x_n (n∈ ℕ). Let $\{x_n\}$ be a sequence in bicomplex valued metric space (X,d). The sequence $\{x_n\}$ is said to converge to x∈ Xif and only if for any $0 \prec_{i_2} \mathcal{E} \in \mathbb{C}_2$, there exists N∈ ℕ depending on $\mathcal{E}$ such that d(x_n,x) $\prec_{i_2} \mathcal{E}$ as n>N.. A sequence $\{x_n\}$ in a bicomplex valued metric space (X,d) is said to be Cauchy sequence if and only if for any $0 \prec_{i_2} \mathcal{E} \in \mathbb{C}_2$,there exists N ∈ ℕ depending on $\mathcal{E}$ such that d(x_n, x_m) $\prec_{i_2} \mathcal{E}$ as n,m>N. A bicomplex valued metric space is said to be complete if and only if every Cauchy sequence in X converges in X.

Definition 4. Let X be a nonempty set and let $s \geq 1$ be a given real number. A function

d: $X \times X \to \mathbb{C}_2$ is called a bicomplex valued b -metric on X if for all x , y , z $\in X$ the following conditions are satisfied: (i) $0 \precsim_{i_2} (x , y)$ and $d(x , y) = 0$ if and only if $x = y$;

(ii)d (x , y) = d(y , x);

(iii) d$(x , y) \precsim_{i_2}$ s[d(x , z) + d(z , y)].

The pair (X , d) is called a complex valued b -metric space.

Example : If $X = [0, 1]$, define the mapping d: $X \times X \to \mathbb{C}_2$ by

d(x , y) =$(1 + \mathrm{i}_1 + \mathrm{i}_2 + \mathrm{i}_1\mathrm{i}_2)|\mathrm{x} - \mathrm{y}|^2$, for all x , y $\in X$. Then (X , d) is bicomplex valued b -metric space with s =2.

2. Main Results

Theorem 1: Let (X,d) be a complete bicomplex valued metric space with coefficient s≥ 1 and $x_0 \in X$. $0 < r \in \mathbb{C}$ and A,B,C,D and E are non negative reals such that

A+$\sqrt{2}B + \sqrt{2}C + \sqrt{2}sD + \sqrt{2}sE < 1$. Let S,T:X→X are mapping satisfying

$$d(Sx,Ty) \precsim_{i_2} Ad(x,y)+B\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + C\frac{d(y,Sx)d(x,Ty)}{1+d(x,y)} + D\frac{d(x,Sx)d(x,Ty)}{1+d(x,y)} + E\frac{d(y,Sx)d(y,Ty)}{1+d(x,y)} \quad (1.1)$$

for all x,y∈ $\overline{B(x_0,r)}$. If $\|d(x_0, Sx_0\| \precsim_{i_2} (1-\lambda)|r|$ where λ= $\max\{\frac{A+\sqrt{2}sD}{1-B-\sqrt{2}sD}, \frac{A+\sqrt{2}sE}{1-\sqrt{2}B-\sqrt{2}sE}\}$,
(1.2)

Then there exist a unique point u∈ $\overline{B(x_0,r)}$ such that u=Su=Tu.

Proof: Let x_0 be an arbitrary point in X and define $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ where n=0,1,2…. We will prove that $x_n \in \overline{B(x_0,r)}$ for all n∈ℕ by mathematical induction . Using inequality (1.2) and the fact that λ= $\max\{\frac{A+\sqrt{2}sD}{1-B-\sqrt{2}sD}, \frac{A+\sqrt{2}sE}{1-\sqrt{2}B-\sqrt{2}sE}\} < 1$ we have $\|d(x_0, Sx_0\| \precsim_{i_2} |r|$. It implies that $x_1 \in \overline{B(x_0,r)}$. Let $x_2, x_3, \dots, x_k \in \overline{B(x_0,r)}$ for some kϵℕ.

If k=2n+1 where n=0,1,2…,$\frac{k-1}{2}$ or k=2n+2 where n=0,1,2,…., $\frac{k-2}{2}$, we obtain by using inequality (1.1)

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})$$

$$\precsim_{i_2} A\, d(x_{2n}, x_{2n+1}) + B\frac{d(x_{2n+1}, Tx_{2n+1})d(x_{2n}, Sx_{2n})}{1+d(x_{2n}, x_{2n+1})} + C\frac{d(x_{2n+1}, Sx_{2n})d(x_{2n}, Tx_{2n+1})}{1+d(x_{2n}, x_{2n+1})}$$

$$+D\frac{d(x_{2n},Tx_{2n+1})d(x_{2n},Sx_{2n})}{1+d(x_{2n},x_{2n+1})}+E\,\frac{d(x_{2n+1},Tx_{2n+1})d(x_{2n+1},Sx_{2n})}{1+d(x_{2n},x_{2n+1})}$$

$$\precsim_{i_2} A\,d(x_{2n},x_{2n+1})+B\,\frac{d(x_{2n+1},x_{2n+2})d(x_{2n},x_{2n+1})}{1+d(x_{2n},x_{2n+1})}+D\frac{d(x_{2n},x_{2n+2})d(x_{2n},x_{2n+1})}{1+d(x_{2n},x_{2n+1})}$$

This implies

$\|d(x_{2n+1},x_{2n+2})\| \leq A\|d(x_{2n},x_{2n+1})\|$

$+\sqrt{2}B\,\frac{\|d(x_{2n+1},x_{2n+2})\|\|d(x_{2n},x_{2n+1})\|}{\|1+d(x_{2n},x_{2n+1})\|}+\sqrt{2}D\,\frac{\|d(x_{2n},x_{2n+2})\|\|d(x_{2n},x_{2n+1})\|}{\|1+d(x_{2n},x_{2n+1})\|}$

Since $\|1+d(x_{2n},x_{2n+1})\| > \|d(x_{2n},x_{2n+1})\|$

Hence $\|d(x_{2n+1},x_{2n+2})\| \leq A\|d(x_{2n},x_{2n+1})\|+\sqrt{2}B\|d(x_{2n+1},x_{2n+2})\|+\sqrt{2}D\|d(x_{2n},x_{2n+2})\|$

$$\leq A\|d(x_{2n},x_{2n+1})\|+\sqrt{2}B\|d(x_{2n+1},x_{2n+2})\|$$

$$+\sqrt{2}sD\{\|d(x_{2n},x_{2n+1})\|+\|d(x_{2n+1},x_{2n+2})\|$$

$(1-\sqrt{2}B-\sqrt{2}sD)\|d(x_{2n+1},x_{2n+2})\| \leq (A+\sqrt{2}sD)\ \|d(x_{2n},x_{2n+1})\|$

$$\Rightarrow\|d(x_{2n+1},x_{2n+2})\| \leq \frac{A+\sqrt{2}sD}{1-\sqrt{2}B-\sqrt{2}sD}\ \|d(x_{2n},x_{2n+1})\| \tag{1.3}$$

Similarly we get,

$$\|d(x_{2n+2},x_{2n+3})\| \leq \frac{A+\sqrt{2}sE}{1-\sqrt{2}B-\sqrt{2}sE}\ \|d(x_{2n+1},x_{2n+2})\| \tag{1.4}$$

Putting $\lambda=\max\{\frac{A+\sqrt{2}sD}{1-B-\sqrt{2}sD},\frac{A+\sqrt{2}sE}{1-\sqrt{2}B-\sqrt{2}sE}\}$, we obtain

$$\|d(x_k,x_{k+1})\| \leq \lambda^k\|d(x_0,x_1\| \tag{1.5}$$

For all $k\epsilon\mathbb{N}$

$\|d(x_0,x_{k+1})\| \leq s\|d(x_0,x_1\|+s\|d(x_1,x_{k+1})\|$

$$\leq s\|d(x_0,x_1\|+s^2\|d(x_1,x_2\|+s^2\|d(x_2,x_{k+1}\|$$

$$\leq s\|d(x_0,x_1\|+s^2\|d(x_1,x_2\|+s^3\|d(x_2,x_3\|+\cdots+s^{k+1}\|d(x_k,x_{k+1}\|$$

$$\leq s\|d(x_0,x_1\|+s^2\lambda\|d(x_0,x_1\|+s^3\lambda^2\ \|d(x_0,x_1\|+\cdots+s^{k+1}\ \lambda^k\ \|d(x_0,x_1\|$$

$$=\|d(x_0,x_1\|\ [s+s^2\lambda+s^3\lambda^2+\cdots+s^{k+1}\ \lambda^k\ ]$$

$$\leq (1-\lambda)|r|s\,\frac{1-(s\lambda)^{k+1}}{1-s\lambda} \qquad [\text{as } \ \|d(x_0,x_1\| \ \leq (1-\lambda)|r|]$$

$$\leq |r| \ \text{ when } s=1$$

Gives $x_{k+1}\in B(x_0,r)$. hence $x_n\in\overline{B(x_0,r)}$ $\quad\forall n\in\mathbb{N}$

$$\text{and } \ \|d(x_n,x_{n+1})\| \leq \lambda^n\|d(x_0,x_1\|\ \forall n\in\mathbb{N} \tag{1.6}$$

without loss of generality , we take m$>n$, then

$$\|d(x_n, x_m)\| \le s\|d(x_n, x_{n+1})\| + s\|d(x_{n+1}, x_m)\|$$
$$\le s\|d(x_n, x_{n+1})\| + s^2\|d(x_{n+1}, x_{n+2}\| + s^2\|d(x_{n+2}, x_m\|$$
$$\le s\|d(x_n, x_{n+1})\| + s^2\|d(x_{n+1}, x_{n+2}\| + \cdots + s^{m-n-1}\|d(x_{m-2}, x_{m-1}\|$$
$$+s^{m-n}\|d(x_{m-1}, x_m\|$$

By using (1.6) we get,

$$\|d(x_n, x_m)\| \le s\lambda^n\|d(x_0, x_1\| + s^2\lambda^{n+1}\|d(x_0, x_1\|$$
$$+s^3\lambda^{n+2}\|d(x_0, x_1\| + \cdots + s^{m-n-1}\lambda^{m-2}\|d(x_0, x_1\| + s^{m-n}\lambda^{m-1}\|d(x_0, x_1\|$$
$$=\sum_{i=1}^{m-n} s^i\lambda^{n+i-1}\|d(x_0, x_1\|$$
$$\le \frac{s\lambda^n}{1-\lambda s}\|d(x_0, x_1\| \to 0 \text{ as } m,n\to\infty$$

This implies that the sequence $\{x_n\}$ as a cauchy sequence in $\overline{B(x_0, r)}$. Therefore there exists a point u∈ $\overline{B(x_0, r)}$ with $\lim_{n\to\infty} x_n = u$.

We prove that u=Su. Let us consider

$$\|d(u, Su)\| \le s\|d(u, x_{2n+2})\| + s\|d(x_{2n+2}, Su)\|$$
$$\le s\|d(u, x_{2n+2})\| + s\|d(Tx_{2n+1}, Su)\|$$
$$\le s\|d(u, x_{2n+2})\| + s\|d(Su, Tx_{2n+1})\|$$
$$\le s\|d(u, x_{2n+2})\| + As\|d(x_{2n+1}, u)\| + sB\sqrt{2}\frac{\|d(x_{2n+1}, Tx_{2n+1})\|\|d(u, Su)\|}{\|1+d(u, x_{2n+1})\|}$$
$$+sC\sqrt{2}\frac{\|d(x_{2n+1}, Su)\|\|d(u, Tx_{2n+1})\|}{\|1+d(x_{2n+1}), u)\|} + sD\sqrt{2}\frac{\|d(u, Su)\|\|d(u, Tx_{2n+1})\|}{\|1+d(u, x_{2n+1})\|} + sE\sqrt{2}\frac{\|d(x_{2n+1}, Tx_{2n+1})\|\|d(x_{2n+1}, Su)\|}{\|1+d(u, x_{2n+1})\|}$$

Notice that ,

$$\lim_{n\to\infty}\|d(u, x_{2n+2})\| = \lim_{n\to\infty}\|d(x_{2n+1}, u)\| + \lim_{n\to\infty}\|d(x_{2n+1}, Su)\| = 0$$

Hence $\|d(u, Su)\|$ =0 that is u=Su

Similarly , u=Tu

For uniqueness assume that u^* in $\overline{B(x_0, r)}$ is a another common fixed point of S and T. Then

$$\|d(u, u^*)\| \le \|d(Su, Tu^*)\|$$
$$\le A\|d(u, u^*)\| + B\sqrt{2}\frac{\|d(u, Su)\|\|d(u^*, Tu^*)\|}{\|1 + d(u, u^*)\|} + C\sqrt{2}\frac{\|d(u^*, Su)\|\|d(u, Tu^*)\|}{\|1 + d(u, u^*)\|}$$
$$+ D\sqrt{2}\frac{\|d(u, Su)\|\|d(u, Tu^*)\|}{\|1 + d(u, u^*)\|} + E\frac{\|d(u^*, Su)\|\|d(u^*, Tu^*)\|}{\|1 + d(u, u^*)\|}$$
$$\le A\|d(u, u^*)\| + C\sqrt{2}\|d(u^*, u)\| \qquad [\text{as } \|1 + d(u, u^*)\| > \|d(u, u^*)\|\ ]$$

$$\therefore\ \|d(u, u^*)\| \le (A + C\sqrt{2})\|d(u, u^*)\|$$

This is a contradiction because $(A + C\sqrt{2}) < 1$. Hence $u = u^*$. Therefore u is a unique common fixed point of T and S.

This completes the proof of the theorem .

Remark 1.1: The result of theorem 1.1 remains true if the condition (1.2) is replaced by the condition $\|d(x_0, Tx_0\| \leq (1 - \lambda)|r|$.

Corollary 1.1: Let (X,d) be a complete bi-complex valued b-metric space with coefficient s≥ 1 and degenerated 1+d(x,y) , $\|1 + d(x,y)\| \neq 0$ and $x_0 \in X$. Let $0 \leq r \in \mathbb{C}$ and A,B,C,D are non negative reals such that $A+\sqrt{2}B + \sqrt{2}C + \sqrt{2}sD < 1$. Let S,T:X→X are mappings satisfying:

$$d(Sx,Ty) \precsim_{i_2} Ad(x,y)+B\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + C\frac{d(y,Sx)d(x,Ty)}{1+d(x,y)} + D\frac{d(x,Sx)d(x,Ty)}{1+d(x,y)}$$

for all x,y∈ $\overline{B(x_0, r)}$. If $\|d(x_0, Sx_0\| \leq (1 - \lambda)|r|$, where λ= $\max\{\frac{A+\sqrt{2}sD}{1-\sqrt{2}B-\sqrt{2}sD}, \frac{A}{1-\sqrt{2}B}\}$,

then there exist a unique point u∈ $\overline{B(x_0, r)}$ such that u=Su=Tu.

Proof: We can prove this result by applying theorem 1.1 by setting E=0.

Corollary 1.2: Let (X,d) be a complete bi-complex valued b-metric space with coefficient s≥ 1 and degenerated 1+d(x,y) , $\|1 + d(x,y)\| \neq 0$ and $x_0 \in X$. Let $0 \leq r \in \mathbb{C}$ and A,B,C and E are non negative reals such that $A+\sqrt{2}B + \sqrt{2}C + \sqrt{2}sE < 1$. Let S,T:X→X are mappings satisfying:

$$d(Sx,Ty) \precsim_{i_2} Ad(x,y)+B\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + C\frac{d(y,Sx)d(x,Ty)}{1+d(x,y)} + E\frac{d(y,Sx)d(y,Ty)}{1+d(x,y)}$$

for all x,y∈ $\overline{B(x_0, r)}$. If $\|d(x_0, Sx_0\| \leq (1 - \lambda)|r|$, where λ= $\max\{\frac{A}{1-\sqrt{2}B}, \frac{A+\sqrt{2}sE}{1-\sqrt{2}B-\sqrt{2}sE}\}$,

then there exist a unique point u∈ $\overline{B(x_0, r)}$ such that u=Su=Tu.

Proof: We can prove this result by applying theorem 1.1 by setting D=0.

Corollary 1.3: Let (X,d) be a complete bi-complex valued b-metric space with coefficient s≥ 1 and degenerated 1+d(x,y) , $\|1 + d(x,y)\| \neq 0$ and $x_0 \in X$. Let $0 \leq r \in \mathbb{C}$ and A,B,C be three non negative reals such that $A+\sqrt{2}B + \sqrt{2}C < 1$. Let S,T:X→X are mappings satisfying:

$$d(Sx,Ty) \precsim_{i_2} Ad(x,y)+B\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + C\frac{d(y,Sx)d(x,Ty)}{1+d(x,y)}$$

for all x,y∈ $\overline{B(x_0, r)}$. If $\|d(x_0, Sx_0\| \leq (1 - \lambda)|r|$, where λ= $\frac{A}{1-\sqrt{2}B}$ then there exist a unique point u∈ $\overline{B(x_0, r)}$ such that u=Su=Tu.

Proof: We can prove this result by applying corollary 1.2 by setting E=0.

Corollary 1.4: Let (X,d) be a complete bi-complex valued b-metric space with coefficient s≥ 1 and degenerated 1+d(x,y) , $\|1 + d(x,y)\| \neq 0$ and $x_0 \in X$. Let $0 \leq r \in \mathbb{C}$ and A,B are non negative reals such that $A+\sqrt{2}B < 1$. Let S,T:X→X are mappings satisfying:

$d(Sx,Ty) \precsim_{i_2} Ad(x,y)+B\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)}$ for all x,y∈ $\overline{B(x_0, r)}$. If $\|d(x_0, Sx_0\| \leq (1 - \lambda)|r|$, where $\lambda=\frac{A}{1-\sqrt{2}B}$ then there exist a unique point u∈ $\overline{B(x_0,r)}$ such that u=Su=Tu.

Proof: We can prove this result by applying corollary 1.3 by setting C=0.Our result is the extension of theorem (3.1) of [6] to the closed ball in complex valued b-metric space.

Corollary 1.5: Let (X,d) be a complete bicomplex valued metric space with coefficient s≥ 1 and $x_0 \in X$. $0 \precsim r \in \mathbb{C}$ and A,B,C,D and E are non negative reals such that

$A+\sqrt{2}B+\sqrt{2}C+\sqrt{2}sD+\sqrt{2}sE<1$. Let T:X→X are mapping satisfying

$$d(T^n x, T^n y) \precsim_{i_2} Ad(x,y)+B\frac{d(x,T^n x)d(y,T^n y)}{1+d(x,y)}+C\frac{d(y,T^n x)d(x,T^n y)}{1+d(x,y)}$$
$$+D\frac{d(x,T^n x)d(x,T^n y)}{1+d(x,y)}+E\frac{d(y,T^n x)d(y,T^n y)}{1+d(x,y)}$$

for all x,y∈ $\overline{B(x_0,r)}$. If $\|d(x_0,T^n x_0\| \precsim_{i_2} (1-\lambda)|r|$ where $\lambda=\max\{\frac{A+\sqrt{2}sD}{1-B-\sqrt{2}sD},\frac{A+\sqrt{2}sE}{1-\sqrt{2}B-\sqrt{2}sE}\}$,

Then there exist a unique point u∈ $\overline{B(x_0,r)}$ such that u=Tu.

Proof: For some fixed n, we obtain u∈ $\overline{B(x_0,r)}$ such that $T^n u=u$.

The uniqueness follows from

$$d(Tu,u)=d(TT^n u,T^n u) \precsim_{i_2} A\, d(Tu,u)+B\frac{d(Tu,T^n Tu)d(u,T^n u)}{1+d(Tu,u)}+C\frac{d(u,T^n Tu)d(Tu,T^n u)}{1+d(Tu,u)}$$
$$+D\frac{d(Tu,T^n Tu)d(Tu,T^n u)}{1+d(Tu,u)}+E\frac{d(u,T^n Tu)d(u,T^n u)}{1+d(Tu,u)}$$
$$\precsim_{i_2} A\, d(Tu,u)+C\frac{d(u,T^n Tu)d(Tu,T^n u)}{1+d(Tu,u)}+D\frac{d(Tu,T^n Tu)d(Tu,T^n u)}{1+d(Tu,u)}$$
$$\precsim_{i_2} A\, d(Tu,u)+C\frac{d(u,Tu)d(Tu,u)}{1+d(Tu,u)}$$

Taking norm in above ,we get

$$\|d(Tu,u)\| \le A\|d(Tu,u)\|+C\sqrt{2}\frac{\|d(u,Tu)\|\|d(Tu,u)\|}{\|1+d(Tu,u)\|}$$
$$\le A\|d(Tu,u)\|+C\sqrt{2}\|d(Tu,u)\| \qquad [\text{as } \|1+d(u,u)\|>\|d(u,u)\|\ ]$$

$\therefore\ \|d(Tu,u)\| \le (A+C\sqrt{2})\|d(Tu,u)\|$

This is a contradiction . So u=$T^n u$=Tu . Therefore the fixed point of T is unique.

Theorem 2: Let (X,d) be a bi-complex valued complete metric space and T,S:X→X be a self map satisfying the following conditions $d(S(x),T(y)) \precsim_{i_2} \alpha \max[d(x,y),\frac{d(x,S(x))d(y,T(y))}{1+d(Sx,Ty)}]$ …. (2.1) for all x,y∈X, where α is a real with $0<\alpha<1$. Then S and T have a unique common fixed point.

Proof: Let x∈X be arbitrary. We define a sequence $\{x_n\}$ in X as follows x_{2k+1}=S(x_{2k}) and x_{2k+2}=T(x_{2k+1}) for k=0,1,2….

Then $d(x_{2k+1},x_{2k+2})=d(S(x_{2k}),T(x_{2k+1}))$

$$\precsim_{i_2} \alpha \max[d(x_{2k}, x_{2k+1}), \frac{d(x_{2k}, S(x_{2k}))d(x_{2k+1}, T(x_{2k+1}))}{1+d(Sx_{2k}, Tx_{2k+1})}$$

$\therefore$ $d(x_{2k+1}, x_{2k+2}) \precsim_{i_2} \alpha d(x_{2k}, x_{2k+1})$ (2.2)

Similarly, $d(x_{2k+2}, x_{2k+3}) = d(T(x_{2k+1}), S(x_{2k+2}))$

$=d(S(x_{2k+2}), T(x_{2k+1}))$

$\precsim_{i_2} \alpha \max[d(x_{2k+2}, x_{2k+1}), \frac{d(x_{2k+2}, S(x_{2k+2}))d(x_{2k+1}, T(x_{2k+1}))}{1+d(Sx_{2k+2}, Tx_{2k+1})}]$

$\precsim_{i_2} \alpha d(x_{2k+2}, x_{2k+1}) = \alpha d(x_{2k+1}, x_{2k+2})$ (2.3)

Then from (2.2) and (2.3), we get,

$d(x_{n+1}, x_{n+2}) \precsim_{i_2} \alpha\, d(x_n, x_{n+1}) \precsim_{i_2} \alpha^2 d(x_{n-1}, x_n) \ldots\ldots\ldots\ldots \precsim_{i_2} \alpha^{n+1} d(x_0, x_1) \quad \forall n \in \mathbb{N}.$

Now for all m,n$\in\mathbb{N}$ we have,

$d(x_n, x_{m+n}) \precsim_{i_2} d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m+n-1}, x_{m+n})$

$\precsim_{i_2} \alpha^n d(x_0, x_1) + \alpha^{n+1} d(x_0, x_1) + \cdots + \alpha^{m+n-1} d(x_0, x_1)$

$\therefore$ $d(x_n, x_{m+n}) \precsim_{i_2} \alpha^n (1 + \alpha + \alpha^2 + \cdots + \alpha^{m-1})\, d(x_0, x_1)$

$\therefore$ $\|d(x_n, x_{m+n})\| \leq \alpha^n \frac{1-\alpha^m}{1-\alpha} d(x_0, x_1) \to 0$ as m,n$\to\infty$

$\therefore$ $\{x_n\}$ is a Cauchy sequence in X. Since X is complete there exist $x \in X$ such that $x_n \to x$ as $n \to \infty$.

Thus $\lim_{n\to\infty} S(x_{2n}) = \lim_{n\to\infty} T(x_{2n+1}) = x$.

Thus from (2.1), we have $d(Sx,x) \precsim_{i_2} d(Sx,Tx_{2k+1})+d(Tx_{2k+1},x)$

$\precsim_{i_2} \alpha \max[d(x, x_{2k+1}), \frac{d(x,S(x))d(x_{2k+1},T(x_{2k+1}))}{1+d(Sx,Tx_{2k+1})}])+d(x_{2k+2},x)$

$\precsim_{i_2} \alpha d(x, x_{2k+1}) + d(x_{2k+2},x)$

$\therefore$ $\|d(Sx, x)\| \leq \alpha\|d(x, x_{2k+1})\| + \|d(x_{2k+2}, x)\| \to 0$ as n$\to\infty$.

Thus $\|d(Sx, x)\| = 0$. So $S(x) = x$.

similarly , We can prove T(x)=x.

Thus x is a common fixed point of S and T.

Now for uniqueness let us assume that $x^* \in X$ is another fixed point of S and T.

Then $d(x, x^*) = d(Sx, Tx) \precsim_{i_2} \alpha \max[d(x, x^*), \frac{d(x,S(x))d(x^*,T(x^*))}{1+d(Sx,Tx^*)}]$

$\precsim_{i_2} \alpha d(x, x^*)$

$\Rightarrow(1-\alpha)\, d(x, x^*) \precsim_{i_2} 0$

$\Rightarrow(1-\alpha)\|d(x, x^*)\| \leq 0 \Rightarrow d(x, x^*) = 0 \Rightarrow x = x^*.$

This completes the proof of the theorem.

Theorem 3: Let (X, d) be a complete bicomplex valued b-metric space with the coefficient $s \geq 1$ and let S, T: $X \to X$ be mappings satisfying

$$d(Sx,Ty) \precsim_{i_2} Ad(x,y)+B\frac{d(x,Sx)d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)} \qquad \ldots\ldots\ldots(3.1)$$

for all $x, y \in X$, such that $x \neq y$, $d(x, Ty)+d(y, Sx)+d(x, y) \neq 0$, where A,B are nonnegative reals with $A+\sqrt{2}s\,B < 1$ or $d(Sx, Ty) = 0$ if $d(x, Ty) + d(y, Sx) + d(x, y) = 0$. Then S and T have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X and define a sequence $\{x_n\}$ in X such that $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ where n=0,1,2.... ... … (3.2). Now, we show that the sequence $\{x_n\}$ is Cauchy.

Let $x = x_{2n}$ and $y = x_{2n+1}$ in (3.2); we have

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})$$

$$\precsim_{i_2} A\, d(x_{2n}, x_{2n+1}) + B\frac{d(x_{2n+1}, Tx_{2n+1})d(x_{2n}, Sx_{2n})}{d(x_{2n}, Tx_{2n+1})+d(x_{2n+1}, Sx_{2n})+d(x_{2n}, x_{2n+1})} \qquad (3.3)$$

$$\precsim_{i_2} A\, d(x_{2n}, x_{2n+1}) + B\frac{d(x_{2n+1}, x_{2n+2})d(x_{2n}, x_{2n+1})}{d(x_{2n}, x_{2n+2})+d(x_{2n+1}, x_{2n+1})+d(x_{2n}, x_{2n+1})}$$

This implies that

$$\|d(x_{2n+1}, x_{2n+2})\| \leq A\|d(x_{2n}, x_{2n+1})\|$$

$$+\sqrt{2}B\frac{\|d(x_{2n+1}, x_{2n+2})\|\|d(x_{2n}, x_{2n+1})\|}{\|d(x_{2n}, x_{2n+2})\|+\|d(x_{2n}, x_{2n+1})\|} \qquad (3.4)$$

As $\|d(x_{2n+1}, x_{2n+2})\| \leq s(\|d(x_{2n+1}, x_{2n})\| + \|d(x_{2n}, x_{2n+2})\|)$ (3.5)

Therefore $\|d(x_{2n+1}, x_{2n+2})\| \leq A\|d(x_{2n}, x_{2n+1})\| + \sqrt{2}sB\|d(x_{2n}, x_{2n+1})\|$

$$\leq (A + \sqrt{2}sB)\|d(x_{2n}, x_{2n+1})\| \qquad (3.6)$$

Similarly we get,

$$\|d(x_{2n+2}, x_{2n+3})\| \leq (A + \sqrt{2}sB)\ \|d(x_{2n+1}, x_{2n+2})\| \qquad (3.7)$$

Since $\left(A + \sqrt{2}sB\right) < 1$. Therefore with $\left(A + \sqrt{2}sB\right) = \lambda < 1$, and for all $n \geq 0$,and consequently,we have

$$\|d(x_{2n+1}, x_{2n+2})\| \leq \lambda\ \|d(x_{2n}, x_{2n+1})\| \leq \lambda^2\|d(x_{2n-1}, x_{2n})\| \leq \cdots \leq \lambda^{2n+1}\|d(x_0, x_1)\| \quad (3.8)$$

That is, $\|d(x_{n+1}, x_{n+2})\| \leq \lambda\ \|d(x_n, x_{n+1})\| \leq \lambda^2\|d(x_{n-1}, x_n)\| \leq \cdots \leq \lambda^{n+1}\|d(x_0, x_1)\|$ (3.9)

Thus, for any $m > n$, m, n $\in \mathbb{N}$,we have

$$\|d(x_n, x_m)\| \leq s\|d(x_n, x_{n+1})\| + s\|d(x_{n+1}, x_m)\|$$

$$\leq s\|d(x_n, x_{n+1})\| + s^2\|d(x_{n+1}, x_{n+2}\| + s^2\|d(x_{n+2}, x_m\|$$

$$\leq s\|d(x_n, x_{n+1})\| + s^2\|d(x_{n+1}, x_{n+2}\| + \cdots + s^{m-n-1}\|d(x_{m-2}, x_{m-1}\|$$

$$+s^{m-n}\|d(x_{m-1},x_m\| \quad (3.10)$$

By using (3.9) we get,

$$\|d(x_n,x_m)\| \leq s\lambda^n\|d(x_0,x_1\| + s^2\lambda^{n+1}\|d(x_0,x_1\|$$
$$+s^3\lambda^{n+2}\|d(x_0,x_1\| + \cdots + s^{m-n-1}\lambda^{m-2}\|d(x_0,x_1\| + s^{m-n}\lambda^{m-1}\|d(x_0,x_1\|$$
$$=\sum_{i=1}^{m-n} s^i\lambda^{n+i-1}\|d(x_0,x_1\|$$
$$\leq \frac{s\lambda^n}{1-\lambda s}\|d(x_0,x_1\| \to 0 \text{ as } m,n\to\infty$$

(3.11)

This implies that the sequence $\{x_n\}$ *as* a cauchy sequence in X.Since X is complete , there exists a point u$\in X$ with $\lim_{n\to\infty} x_n = u$.

Assume not, then there exists v$\in X$ such that $\|d(u,Su)\|$ =||v||>0 (3.12)

So by using the triangular inequality and (3.1), we get

v=d(u, Su) $\precsim_{i_2}$s $d(u,x_{2n+2})$+s $d(x_{2n+2},Su)$

$$\precsim_{i_2} s\, d(u,x_{2n+2}) + s\, d(Tx_{2n+1},Su)$$
$$\precsim_{i_2} s\, d(u,x_{2n+2}) + s\, Ad(u,x_{2n+1}) + sB\frac{d(u,Su)d(x_{2n+1},Tx_{2n+1})}{d(u,Tx_{2n+1})+d(x_{2n+1},Su)+d(u,x_{2n+1})} \quad (3.13)$$

which implies that

$$\|v\| = \|d(u,Su)\|$$
$$\leq s\|d(u,x_{2n+2})\| + As\|d(u,x_{2n+1})\| + sB\sqrt{2}\frac{\|d(x_{2n+1},x_{2n+2})\|\|d(u,Su)\|}{\|d(u,Tx_{2n+1})\|+\|d(x_{2n+1},Su)\|+\|d(u,x_{2n+1})\|}$$

Taking limit as n$\to\infty$,we get $\|v\| = \|d(u,Su)\| \leq 0$, a contradiction with (3.12). So $\|v\| = 0$

Hence $\|d(u,Su)\|$ =0 that is u=Su .

Similarly , we obtain u=Tu.

For uniqueness assume that u^* in X is another common fixed point of S and T. Then

$$d(u,u^*) = d(Su,Tu^*) \precsim_{i_2} A\, d(u,u^*) + \frac{d(u,Su)d(u^*,Tu^*)}{d(u,Tu^*) + d(u^*,Su) + d(u,u^*)}$$

So that $\|d(u,u^*)\| = \|d(Su,Tu^*)\| \leq A\|d(u,u^*)\| + B\sqrt{2}\frac{\|d(u,Su)\|\|d(u^*,Tu^*)\|}{\|d(u,Tu^*)\|+\|d(u^*,Su)\|+\|d(u,u^*)\|}$

$$\leq A\|d(u,u^*)\|$$

Hence $u = u^*$. Therefore u is a unique common fixed point of T and S.

Now , we consider the second case: $d(x,Ty) + d(y,Sx) + d(x,y) = 0$. Put x= x_{2n} and y=x_{2n+1} in this expression we get $(x_{2n}, Tx_{2n+1}) + (x_{2n+1}, Sx_{2n}) + (x_{2n}, x_{2n+1}) = 0$ (for any n) which implies $(Sx_{2n}, Tx_{2n+1})=0$ so that $x_{2n} = Sx_{2n} = x_{2n+1} = Tx_{2n+1} = x_{2n+2}$. Thus we have $x_{2n} = Sx_{2n} = x_{2n+1}$, So there exist K_1 and l_1 such that $K_1 = Sl_1 = l_1$ where $K_1 = x_{2n+1}$ and $l_1 = x_{2n}$. Using foregoing arguments , one can also show that there exists K_2 and l_2 such that $K_2 = Sl_2 = l_2$ where $K_2 = x_{2n+2}$ and $l_2 = x_{2n+1}$. As $(l_1, T\, l_2) + (l_2, S\, l_1) + (l_1, l_2) = 0$ (from definition) implies

$d\,(Sl_1, T\, l_2)=0$,therefore $K_1 = Sl_1 = Tl_2 = K_2$. Thus we obtain that $K_1 = Sl_1 = SK_1$. similarly ,one can also have $TK_2 = K_2$. As $K_1 = K_2$ implies $SK_1 = TK_1 = K_1$, therefore $K_1 = K_2$ is common fixed point of S and T. For uniqueness assume that K_1^* in X is another common fixed point of S and T. Then we have $SK_1^* = TK_1^* = K_1^*$

As $d\,(K_1, , T\, K_1^*) + d\,(K_1^*, S\, K_1,) + d\,(K_1, , K_1^*) = 0$, therefore $d\,(SK_1, TK_1^*) = d\,(K_1, , K_1^*) = 0$. This implies that $K_1, = K_1^*$.

This completes the proof of the theorem .

Corollary 3.1: Let (X , d) be a complete bicomplex valued b -metric space with the coefficient $s \geq 1$ and let T: $X \rightarrow X$ be mappings satisfying

$$d(Tx,Ty) \precsim_{i_2} Ad(x,y)+B\frac{d(x,Tx)d(y,Ty)}{d\,(x,Ty) + d\,(y,Tx) + d\,(x,y)} \quad \text{for all } x , y \in X \text{, such that } x \neq y ,$$

$d\,(x , T\, y) + d\,(y , Tx) + d\,(x , y) \neq 0$, where A,B are nonnegative reals with A+ $\sqrt{2}s$ B < 1 or $d\,(Tx , T\, y) = 0$ if $d\,(x , T\, y) + d\,(y , Tx) + d\,(x , y) = 0$. Then has a unique common fixed point.

Proof: We can prove this result by applying theorem 3 by setting S=T.

Corollary 3.2: Let (X , d) be a complete bicomplex valued b -metric space with the coefficient $s \geq 1$ and let T: $X \rightarrow X$ be mappings satisfying (for some fixed n)

$$d(T^n x, T^n y) \precsim_{i_2} Ad(x,y)+B\frac{d(x,T^n x)d(y,T^n y)}{d\,(x,T^n y) + d\,(y,T^n x) + d\,(x,y)} \quad \text{for all } x , y \in X \text{, such that } x \neq y ,$$

$d\,(x , T^n y) + d\,(y , T^n x) + d\,(x , y) \neq 0$, where A,B are nonnegative reals with A+ $\sqrt{2}s$ B < 1 or $d\,(T^n x , T^n y) = 0$ if $d\,(x , T^n y) + d\,(y , T^n x) + d\,(x , y) = 0$. Then has a unique common fixed point.

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