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A Research in Bipolar Valued Vague Subfields of a Field

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Abstract:

Certain properties of bipolar valued vague subfield of a field are introduced and discussed.

Keywords: FS, VS, $B_{VF}S$, $B_{VV}S$, $B_{VV}SF$.

INTRODUCTION. First, fuzzy set had been introduced by Zadeh [14]. Succeeding years, fuzzy set was grown in different ways. The following are extension of fuzzy set, they are vague set, intuitionistic fuzzy set, bipolar valued fuzzy set and etc. Vague set by W. L. Gau and D. J. Buehrer [6]; F*uzzy group by* Azriel Rosenfeld [3]; Bipolar valued fuzzy subset by W.R.Zhang[15]; Vague group by RanjitBiswas [11]; Bipolar vague set by Cicily Flora. S and Arockiarani.I [5]; Bipolar valued fuzzy subgroup by Anitha.M.S., et.al.[2]; In similar way, [1], [4], [7], [8], [9], [10], [12] and [13] were useful to write this paper.

1.PRELIMINARIES.

Definition 1.1 [14] $A\ map\ \mathfrak{R}: \mathfrak{G} \to [0,1]\ is\ called\ a\ fuzzy\ subset\ of\ \mathfrak{G}$.

Definition 1.2 [6] $The\ ordered\ structure\ \mho = \{(\mathfrak{z}, [\mho_T(\mathfrak{z}), 1 - \mho_F(\mathfrak{z})]) : \mathfrak{z} \in \mathbb{W}\}\ is\ called$ a $vague\ set\ of\ \mathbb{W}, where\ \mho_T: \mathbb{W} \to [0,1]\ is\ a\ truth\ membership$ map and $\mho_F: \mathbb{W} \to [0, 1]$ is a false membership map, such that $\mho_T(\mathfrak{z}) + \mho_F(\mathfrak{z}) \leq 1, for\ all\ \mathfrak{z}\ in\ \mathbb{W}$.

Definition 1.3 [6] $The\ interval\ [\mho_T(\mathfrak{z}), 1 - \mho_F(\mathfrak{z})]\ is\ called\ the\ vague\ value\ of\ \mathfrak{z}\ in\ \mho$

$and\ it\ is\ denoted\ by\ \mho(\mathfrak{z}), i.e., \mho(\mathfrak{z}) = [\mho_T(\mathfrak{z}), 1 - \mho_F(\mathfrak{z})]$.

Example 1.4. $\mho$ = { < $\mathfrak{z}$, [0.04, 0.07] >, < $\mathfrak{v}$, [0.02, 0.06] >, < $\mathfrak{n}$, [0.03, 0.08] >} is a vague set of $\mathfrak{R} = \{\mathfrak{z}, \mathfrak{v}, \mathfrak{n}\}$.

Definition 1.5 [15] $The\ ordered\ structure\ \mathfrak{T} = \{(\mathfrak{z}, \mathfrak{T}^+(\mathfrak{z}), \mathfrak{T}^-(\mathfrak{z})) : \mathfrak{z} \in \mathbb{W}\}\ is\ called$ a bipolar $valued\ fuzzy\ subset\ of\ \mathbb{W}, where\ \mathfrak{T}^+: \mathbb{W} \to [0,1]\ is\ a\ positive\ membership$ map and $\mathfrak{T}^-: \mathbb{W} \to [-1,0]$ is a negative membership map.

Definition 1.6 [5] *The ordered structure* $\mho = \{(\mathfrak{z}, [\mho_T^+(\mathfrak{z}), 1-\mho_F^+(\mathfrak{z})], [-1-\mho_F^-(\mathfrak{z}), \mho_T^-(\mathfrak{z})]) : \mathfrak{z} \in \mathbb{W}\}$ *is called a bipolar valued vague subset of* $\mathbb{W}$, *where* $\mho_T^+ : \mathbb{W} \to [0,1], \mho_F^+ : \mathbb{W} \to [0,1], \mho_T^- : \mathbb{W} \to [-1, 0]$, *and* $\mho_F^- : \mathbb{W} \to [-1, 0]$ *are mapping such that*

$\mho_T^+(\mathfrak{z}) + \mho_F^+(\mathfrak{z}) \leq 1, -1 \leq \mho_T^-(\mathfrak{z}) + \mho_F^-(\mathfrak{z})$, *for all* $\mathfrak{z}$ *in* $\mathbb{W}$. Bipolar valued vague subset

$\mho$ is denoted as $\mho = \{(\mathfrak{z}, \mho^+(\mathfrak{z}), \mho^-(\mathfrak{z})) : \mathfrak{z} \in \mathbb{W}\}$, where $\mho^+(\mathfrak{z}) = [\mho_T^+(\mathfrak{z}), 1-\mho_F^+(\mathfrak{z})]$ and $\mho^-(\mathfrak{z}) = [-1-\mho_F^-(\mathfrak{z}), \mho_T^-(\mathfrak{z})]$. It is denoted as $B_{VV}S$.

Example 1.7. $\mho$ = { < $\mathfrak{z}$, [0.05, 0.07], [–0.05, –0.02] >, < $\mathfrak{v}$, [0.04, 0.08], [–0.06, –0.03] >, < $\mathfrak{n}$, [0.14, 0.19], [–0.25, –0.22] >} is a $B_{VV}S$ of $\mathfrak{R} = \{\mathfrak{z}, \mathfrak{v}, \mathfrak{n}\}$.

Definition 1.8 [5] Let $\mho = \langle \mho^+, \mho^- \rangle$ and $\mathfrak{G} = \langle \mathfrak{G}^+, \mathfrak{G}^- \rangle$ be *two* $B_{VV}Ss$ *of a set* $\mathcal{W}$. *Then*

(i) $\mho \subset \mathfrak{G}$ *if and only if* $\mho^+(z) \leq \mathfrak{G}^+(z)$ and $\mho^-(z) \geq \mathfrak{G}^-(z)$, *for all* $z \in \mathcal{W}$.

(ii) $\mho \cap \mathfrak{G}$ = {⟨z, rmin($\mho^+(z)$, $\mathfrak{G}^+(z)$), rmax($\mho^-(z)$, $\mathfrak{G}^-(z)$) ⟩ / $z \in \mathcal{W}$}.

Definition 1.9 [5] *A* $B_{VV}S$ $\mathfrak{C} = \langle \mathfrak{C}^+, \mathfrak{C}^- \rangle$ *of a field* $\mathfrak{Z}$ *is said to ba a bipolar* valued vague subfieldg of $\mathfrak{Z}$ ($B_{VV}SF$) *if* $\mathfrak{C}$ *has*,

(i) $\mathfrak{C}^+(\mathfrak{y} - \mathfrak{w}) \geq rmin\{\mathfrak{C}^+(\mathfrak{y}), \mathfrak{C}^+(\mathfrak{w})\}$,

(ii) $\mathfrak{C}^+(\mathfrak{y}\mathfrak{w}) \geq rmin\{\mathfrak{C}^+(\mathfrak{y}), \mathfrak{C}^+(\mathfrak{w})\}$,

(iii) $\mathfrak{C}^-(\mathfrak{y} - \mathfrak{w}) \leq rmax\{\mathfrak{C}^-(\mathfrak{y}), \mathfrak{C}^-(\mathfrak{w})\}$,

(iv) $\mathfrak{C}^-(\mathfrak{y}\mathfrak{w}) \leq rmax\{\mathfrak{C}^-(\mathfrak{y}), \mathfrak{C}^-(\mathfrak{w})\}$, *for all* $\mathfrak{y}, \mathfrak{w} \in \mathfrak{Z}$,

(v) $\mathfrak{C}^+(\mathfrak{y}^{-1}) \geq \mathfrak{C}^+(\mathfrak{y}), \forall\, \mathfrak{y} \neq \sigma \in \mathfrak{Z}$

(vi) $\mathfrak{C}^-(\mathfrak{y}^{-1}) \leq \mathfrak{C}^-(\mathfrak{y}), \forall\, \mathfrak{y} \neq \sigma \in \mathfrak{Z}$,

where σ is an first operation identity element of $\mathfrak{Z}$,

$rmin\{[\mathfrak{r}, \mathfrak{s}], [\mathfrak{t}, \mathfrak{u}]\} = [\min\{\mathfrak{r}, \mathfrak{t}\}, \min\{\mathfrak{s}, \mathfrak{u}\}]$ and

$rmax\{[\mathfrak{r}, \mathfrak{s}], [\mathfrak{t}, \mathfrak{u}]\} = [\max\{\mathfrak{r}, \mathfrak{t}\}, \max\{\mathfrak{s}, \mathfrak{u}\}]$.

Example 1.10. $\mathfrak{H} = \{ < 0, [0.36, 0.38], [-0.39, -0.36] >, < 1, [0.35, 0.37], [-0.38, -0.35] >, < 2, [0.35, 0.37], [-0.38, -0.35] > \}$ *is a* $B_{VV}SF$ of the field $Z_3 = \{0, 1, 2\}$.

Definition 1.11. [5] *Let* $\acute{K} = \langle \acute{K}^+, \acute{K}^- \rangle$ *be* $B_{VV}S$ of the set $\mathfrak{Z}_1$, the strongest B_{VV} *relation on* $\mathfrak{Z}_1$, that is a B_{VV} *relation* on $\acute{K}$ *is* $\mathfrak{W}$ = {⟨(ä, ζ), $\breve{И}^+$(ä, ζ), $\breve{И}^-$(ä, ζ)⟩ / for all ä, $\zeta \in \mathfrak{Z}_1$}, where $\breve{И}^+$(ä, ζ) = rmin{$\acute{K}^+$(ä), $\acute{K}^+(\zeta)$} and $\breve{И}^-$(ä, ζ) = rmax{$\acute{K}^-$(ä), $\acute{K}^-(\zeta)$}, for all ä, $\zeta \in \mathfrak{Z}_1$.

Definition 1.12. [5]*Let* $\tilde{A} = \langle \tilde{A}^+, \tilde{A}^- \rangle$ and $\bar{W} = \langle \bar{W}^+, \bar{W}^- \rangle$ *be* $B_{VV}S$s of the sets $\mathfrak{V}_1$ and $\mathfrak{V}_2$

respectively. The product of $\tilde{A}$ and $\bar{W}$, denoted by $\tilde{A} \times \bar{W}$, is defined as $\tilde{A} \times \bar{W}$ = {⟨($\varkappa$, ζ), $(\tilde{A}\times\bar{W})^+(\varkappa, \zeta)$, $(\tilde{A}\times\bar{W})^-(\varkappa, \zeta)$⟩ / for all $(\varkappa, \zeta) \in \mathfrak{V}_1 \times \mathfrak{V}_2$}, where $(\tilde{A}\times\bar{W})^+(\varkappa, \zeta)$ = rmin{$\tilde{A}^+(\varkappa)$, $\bar{W}^+(\zeta)$} and $(\tilde{A}\times\bar{W})^-(\varkappa, \zeta)$ = rmax{$\tilde{A}^-(\varkappa)$, $\bar{W}^-(\zeta)$}.

2 – THEOREMS.

Theorem 2.1. *If* $\H{U} = \langle \H{U}^+, \H{U}^- \rangle$ is a $B_{VV}SF$ of a field $\mathfrak{F}_1$, then

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(i) $\H{U}^{+}(-\mathcal{Z}) = \H{U}^{+}(\mathcal{Z})$, $\H{U}^{-}(-\mathcal{Z}) = \H{U}^{-}(\mathcal{Z})$, $\forall\ \mathcal{Z} \in \mathfrak{F}_1$;

(ii) $\H{U}^{+}(\mathcal{Z}^{-1}) = \H{U}^{+}(\mathcal{Z})$, $\H{U}^{-}(\mathcal{Z}^{-1}) = \H{U}^{-}(\mathcal{Z})$, $\forall\ \mathfrak{o} \neq \mathcal{Z} \in \mathfrak{F}_1$;

(iii) $\H{U}^{+}(\mathfrak{o}) \geq \H{U}^{+}(\mathcal{Z})$, $\H{U}^{-}(\mathfrak{o}) \leq \H{U}^{-}(\mathcal{Z})$, $\forall\ \mathcal{Z} \in \mathfrak{F}_1$;

(iv) $\H{U}^{+}(\mathfrak{z}) \geq \H{U}^{+}(\mathcal{Z})$, $\H{U}^{-}(\mathfrak{z}) \leq \H{U}^{-}(\mathcal{Z})$, $\forall\ \mathfrak{o} \neq \mathcal{Z} \in \mathfrak{F}_1$;

where $\mathfrak{o}, \mathfrak{z}$ are first, second operation identity elements of $\mathfrak{F}_1$.

Proof. (i) Let $\mathcal{Z} \in \mathfrak{F}_1$. Then $\H{U}^{+}(\mathcal{Z}) = \H{U}^{+}\left(-(-\mathcal{Z})\right) \geq \H{U}^{+}(-\mathcal{Z}) \geq \H{U}^{+}(\mathcal{Z})$.

That is $\H{U}^{+}(\mathcal{Z}) = \H{U}^{+}(-\mathcal{Z})$, $\forall\ \mathcal{Z} \in \mathfrak{F}_1$. And $\H{U}^{-}(\mathcal{Z}) = \H{U}^{-}\left(-(-\mathcal{Z})\right) \leq \H{U}^{-}(-\mathcal{Z}) \leq \H{U}^{-}(\mathcal{Z})$.

Thus $\H{U}^{-}(\mathcal{Z}) = \H{U}^{-}(-\mathcal{Z})$, $\forall\ \mathcal{Z} \in \mathfrak{F}_1$.

(ii) Let $\mathfrak{o} \neq \mathcal{Z} \in \mathfrak{F}_1$. Then $\H{U}^{+}(\mathcal{Z}) = \H{U}^{+}\left((\mathcal{Z}^{-1})^{-1}\right) \geq \H{U}^{+}(\mathcal{Z}^{-1}) \geq \H{U}^{+}(\mathcal{Z})$. That is

$\H{U}^{+}(\mathcal{Z}) = \H{U}^{+}(\mathcal{Z}^{-1})$, $\forall\ \mathfrak{o} \neq \mathcal{Z} \in \mathfrak{F}_1$. And $\H{U}^{-}(\mathcal{Z}) = \H{U}^{-}\left((\mathcal{Z}^{-1})^{-1}\right) \leq \H{U}^{-}(\mathcal{Z}^{-1}) \leq \H{U}^{-}(\mathcal{Z})$.

That is $\H{U}^{-}(\mathcal{Z}) = \H{U}^{-}(\mathcal{Z}^{-1})$, $\forall\ \mathfrak{o} \neq \mathcal{Z} \in \mathfrak{F}_1$.

(iii) Also $\H{U}^{+}(\mathfrak{o}) = \H{U}^{+}(\mathcal{Z} - \mathcal{Z}) \geq$ rmin$\{\H{U}^{+}(\mathcal{Z}), \H{U}^{+}(\mathcal{Z})\} = \H{U}^{+}(\mathcal{Z})$, $\forall\ \mathcal{Z} \in \mathfrak{F}_1$.

And $\H{U}^{-}(\mathfrak{o}) = \H{U}^{-}(\mathcal{Z} - \mathcal{Z}) \leq$ rmax$\{\H{U}^{-}(\mathcal{Z}), \H{U}^{-}(\mathcal{Z})\} = \H{U}_i^{-}(\mathcal{Z})$, $\forall\ \mathcal{Z} \in \mathfrak{F}_1$.

(iv) Also $\H{U}^{+}(\mathfrak{z}) = \H{U}^{+}\left(\mathcal{Z}\mathcal{Z}^{-1}\right) \geq$ rmin$\{\H{U}^{+}(\mathcal{Z}), \H{U}^{+}(\mathcal{Z})\} = \H{U}^{+}(\mathcal{Z})$, $\forall\ \mathfrak{o} \neq \mathcal{Z} \in \mathfrak{F}_1$.

And $\H{U}^{-}(\mathfrak{z}) = \H{U}^{-}\left(\mathcal{Z}\mathcal{Z}^{-1}\right) \leq$ rmax$\{\H{U}^{-}(\mathcal{Z}), \H{U}^{-}(\mathcal{Z})\} = \H{U}_i^{-}(\mathcal{Z})$, $\forall\ \mathfrak{o} \neq \mathcal{Z} \in \mathfrak{F}_1$.

Theorem 2.2. *If* $\mathfrak{K} = \langle \mathfrak{K}^{+}, \mathfrak{K}^{-} \rangle$ and $\mathfrak{V} = \langle \mathfrak{V}^{+}, \mathfrak{V}^{-} \rangle$ *are two* $B_{VV}SF$ of a field $\mathfrak{F}_1$, then their intersection $\mathfrak{K} \cap \mathfrak{V}$ *is also a* $B_{VV}SF$ of $\mathfrak{F}_1$.

Proof. Let $\varrho, \mathfrak{x}$ be in $\mathfrak{F}_1$. Let $\mathfrak{K} \cap \mathfrak{V} = \mathfrak{U}$. Then $\mathfrak{U}^{+}(\varrho - \mathfrak{x})$ = rmin$\{\mathfrak{K}^{+}(\varrho - \mathfrak{x}), \mathfrak{V}^{+}(\varrho - \mathfrak{x})\}$ $\geq$ rmin{rmin$\{\mathfrak{K}^{+}(\varrho), \mathfrak{K}^{+}(\mathfrak{x})\}$, rmin$\{\mathfrak{V}^{+}(\varrho), \mathfrak{V}^{+}(\mathfrak{x})\}$} = rmin{rmin$\{\mathfrak{K}^{+}(\varrho), \mathfrak{V}^{+}(\varrho)\}$, rmin$\{\mathfrak{K}^{+}(\mathfrak{x}), \mathfrak{V}^{+}(\mathfrak{x})\}$} = rmin$\{\mathfrak{U}^{+}(\varrho), \mathfrak{U}^{+}(\mathfrak{x})\}$, $\forall\ \varrho, \mathfrak{x} \in \mathfrak{F}_1$. And $\mathfrak{U}^{+}(\varrho\mathfrak{x}^{-1})$ = rmin$\{\mathfrak{K}^{+}(\varrho\mathfrak{x}^{-1}), \mathfrak{V}^{+}(\varrho\mathfrak{x}^{-1})\}$ $\geq$ rmin{rmin$\{\mathfrak{K}^{+}(\varrho), \mathfrak{K}^{+}(\mathfrak{x})\}$, rmin$\{\mathfrak{V}^{+}(\varrho), \mathfrak{V}^{+}(\mathfrak{x})\}$} = rmin{rmin$\{\mathfrak{K}^{+}(\varrho), \mathfrak{V}^{+}(\varrho)\}$, rmin$\{\mathfrak{K}^{+}(\mathfrak{x}), \mathfrak{V}^{+}(\mathfrak{x})\}$} = rmin$\{\mathfrak{U}^{+}(\varrho), \mathfrak{U}^{+}(\mathfrak{x})\}$, $\forall\ \varrho, \mathfrak{o} \neq \mathfrak{x} \in \mathfrak{F}_1$. Also $\mathfrak{U}^{-}(\varrho - \mathfrak{x})$ = rmax$\{\mathfrak{K}^{-}(\varrho - \mathfrak{x}), \mathfrak{V}^{-}(\varrho - \mathfrak{x})\}$ $\leq$ rmax{rmax$\{\mathfrak{K}^{-}(\varrho), \mathfrak{K}^{-}(\mathfrak{x})\}$, rmax$\{\mathfrak{V}^{-}(\varrho), \mathfrak{V}^{-}(\mathfrak{x})\}$} = rmax{rmax$\{\mathfrak{K}^{-}(\varrho), \mathfrak{V}^{-}(\varrho)\}$, rmax$\{\mathfrak{K}^{-}(\mathfrak{x}), \mathfrak{V}^{-}(\mathfrak{x})\}$} = rmax$\{\mathfrak{U}^{-}(\varrho), \mathfrak{U}^{-}(\mathfrak{x})\}$, $\forall\ \varrho, \mathfrak{x} \in \mathfrak{F}_1$. And $\mathfrak{U}^{-}(\varrho\mathfrak{x}^{-1})$ = rmax$\{\mathfrak{K}^{-}(\varrho\mathfrak{x}^{-1}), \mathfrak{V}^{-}(\varrho\mathfrak{x}^{-1})\}$ $\leq$ rmax{rmax$\{\mathfrak{K}^{-}(\varrho), \mathfrak{K}^{-}(\mathfrak{x})\}$, rmax$\{\mathfrak{V}^{-}(\varrho), \mathfrak{V}^{-}(\mathfrak{x})\}$} = rmax{rmax$\{\mathfrak{K}^{-}(\varrho), \mathfrak{V}^{-}(\varrho)\}$, rmax$\{\mathfrak{K}^{-}(\mathfrak{x}), \mathfrak{V}^{-}(\mathfrak{x})\}$} = rmax$\{\mathfrak{U}^{-}(\varrho), \mathfrak{U}^{-}(\mathfrak{x})\}$, $\forall\ \varrho, \mathfrak{o} \neq \mathfrak{x} \in \mathfrak{F}_1$. Hence $\mathfrak{K} \cap \mathfrak{W} = \mathfrak{U}$ *is also a* $\mathbb{B}_{\mathbb{VV}}\mathbb{SF}$ of $\mathfrak{F}_1$.

Theorem 2.3. *If* $\mathfrak{P}_1 = \langle \mathfrak{P}_1{}^{+}, \mathfrak{P}_1{}^{-} \rangle$, $\mathfrak{P}_2 = \langle \mathfrak{P}_2{}^{+}, \mathfrak{P}_2{}^{-} \rangle$, … and $\mathfrak{P}_m = \langle \mathfrak{P}_m{}^{+}, \mathfrak{P}_m{}^{-} \rangle$ *are* $B_{VV}SFs$ of a field $\mathfrak{F}_1$, then their intersection $\mathfrak{P}_1 \cap \mathfrak{P}_2 \cap \ldots \cap \mathfrak{P}_m$ is also a $B_{VV}SF$ of $\mathfrak{F}_1$.

Proof. By Theorem 2.2, it can be easily shown.

Theorem 2.4. $If\ \mathfrak{P}_1 = \langle \mathfrak{P}_1{}^{+},\ \ \mathfrak{P}_1{}^{-} \rangle,\ \ \mathfrak{P}_2 = \langle \mathfrak{P}_2{}^{+},\ \ \mathfrak{P}_2{}^{-} \rangle,\ \ \ldots\ \ are\ \ B_{VV}SFs$ of a field $\mathfrak{F}_1$, then their intersection $\mathfrak{P}_1 \cap \mathfrak{P}_2 \cap \ldots$ is also a $B_{VV}SF$ of $\mathfrak{F}_1$.

Proof. By Theorem 2.3, it can be easily shown.

Theorem 2.5. $If\ \mathfrak{K} = \langle \mathfrak{K}^{+}, \mathfrak{K}^{-} \rangle$ and $\mathfrak{B} = \langle \mathfrak{B}^{+}, \mathfrak{B}^{-} \rangle\ are\ two\ \ B_{VV}SFs$ of a field $\mathfrak{F}_1$, then their union $\mathfrak{K} \cup \mathfrak{B}\ need\ not\ be\ a\ \ B_{VV}SF$ of $\mathfrak{F}_1$.

Proof. It can be easily shown.

Theorem 2.6. $If\ \mathfrak{K} = \langle \mathfrak{K}^{+}, \mathfrak{K}^{-} \rangle$ and $\mathfrak{B} = \langle \mathfrak{B}^{+}, \mathfrak{B}^{-} \rangle\ are\ two\ \ B_{VV}SFs$ of a field $\mathfrak{F}_1$ and one $one\ is\ contained\ other$, then their union $\mathfrak{K} \cup \mathfrak{B}\ is\ the\ \ B_{VV}SF$ of $\mathfrak{F}_1$.

Proof. It can be easily shown.

Theorem 2.7. $If\ \mathfrak{N} = \langle \mathfrak{N}^{+}, \mathfrak{N}^{-} \rangle$ and $\mathfrak{B} = \langle \mathfrak{B}^{+}, \mathfrak{B}^{-} \rangle\ are\ \ B_{VV}SF$s of the fields $\mathfrak{F}_1$ and $\mathfrak{F}_2$ respectively, then $\mathfrak{N} \times \mathfrak{B}$ is a $B_{VV}SF$ of the field $\mathfrak{F}_1 \times \mathfrak{F}_2$.

Proof. Let $\varrho, \mathfrak{d} \in \mathfrak{F}_1$ and $\zeta, \xi \in \mathfrak{F}_2$. Then $(\varrho, \zeta), (\mathfrak{d}, \xi) \in \mathfrak{F}_1 \times \mathfrak{F}_2$. Then $(\mathfrak{N}\times\mathfrak{B})^{+}[(\varrho, \zeta)-(\mathfrak{d}, \xi)] = (\mathfrak{N}\times\mathfrak{B})^{+}(\varrho-\mathfrak{d}, \zeta-\xi) = \mathrm{rmin}\{\mathfrak{N}^{+}(\varrho-\mathfrak{d}), \mathfrak{B}^{+}(\zeta-\xi)\} \geq \mathrm{rmin}\{\mathrm{rmin}\{\mathfrak{N}^{+}(\varrho), \mathfrak{N}^{+}(\mathfrak{d})\}, \mathrm{rmin}\{\mathfrak{B}^{+}(\zeta), \mathfrak{B}^{+}(\xi)\}\} = \mathrm{rmin}\{\mathrm{rmin}\{\mathfrak{N}^{+}(\varrho), \mathfrak{B}^{+}(\zeta)\}, \mathrm{rmin}\{\mathfrak{N}^{+}(\mathfrak{d}), \mathfrak{B}^{+}(\xi)\}\} = \mathrm{rmin}\{(\mathfrak{N}\times\mathfrak{B})^{+}(\varrho, \zeta), (\mathfrak{N}\times\mathfrak{B})^{+}(\mathfrak{d}, \xi)\}, \forall\ (\varrho, \zeta), (\mathfrak{d}, \xi) \in \mathfrak{F}_1\times\mathfrak{F}_2$. And $(\mathfrak{N}\times\mathfrak{B})^{+}[(\varrho, \zeta)(\mathfrak{d}, \xi)^{-1}] = (\mathfrak{N}\times\mathfrak{B})^{+}(\varrho\mathfrak{d}^{-1}, \zeta\,\xi^{-1}) = \mathrm{rmin}\{\mathfrak{N}^{+}(\varrho\mathfrak{d}^{-1}), \mathfrak{B}^{+}(\zeta\,\xi^{-1})\} \geq \mathrm{rmin}\{\mathrm{rmin}\{\mathfrak{N}^{+}(\varrho), \mathfrak{N}^{+}(\mathfrak{d})\}, \mathrm{rmin}\{\mathfrak{B}^{+}(\zeta), \mathfrak{B}^{+}(\xi)\}\} = \mathrm{rmin}\{\mathrm{rmin}\{\mathfrak{N}^{+}(\varrho), \mathfrak{B}^{+}(\zeta)\}, \mathrm{rmin}\{\mathfrak{N}^{+}(\mathfrak{d}), \mathfrak{B}^{+}(\xi)\}\} = \mathrm{rmin}\ \{(\mathfrak{N}\times\mathfrak{B})^{+}(\varrho, \zeta), (\mathfrak{N}\times\mathfrak{B})^{+}(\mathfrak{d}, \xi)\}, \forall\ (\varrho, \zeta), (\mathfrak{d}, \xi) \in \mathfrak{F}_1\times\mathfrak{F}_2$. Also $(\mathfrak{N}\times\mathfrak{B})^{-}[(\varrho, \zeta)-(\mathfrak{d}, \xi)] = (\mathfrak{N}\times\mathfrak{B})^{-}(\varrho-\mathfrak{d}, \zeta-\xi) = \mathrm{rmax}\{\mathfrak{N}^{-}(\varrho-\mathfrak{d}), \mathfrak{B}^{-}(\zeta-\xi)\} \leq \mathrm{rmax}\{\mathrm{rmax}\{\mathfrak{N}^{-}(\varrho), \mathfrak{N}^{-}(\mathfrak{d})\}, \mathrm{rmax}\{\mathfrak{B}^{-}(\zeta), \mathfrak{B}^{-}(\xi)\}\} = \mathrm{rmax}\{\mathrm{rmax}\{\mathfrak{N}^{-}(\varrho), \mathfrak{B}^{-}(\zeta)\}, \mathrm{rmax}\{\mathfrak{N}^{-}(\mathfrak{d}), \mathfrak{B}^{-}(\xi)\}\} = \mathrm{rmax}\{(\mathfrak{N}\times\mathfrak{B})^{-}(\varrho, \zeta), (\mathfrak{N}\times\mathfrak{B})^{-}(\mathfrak{d}, \xi)\}, \forall\ (\varrho, \zeta), (\mathfrak{d}, \xi) \in \mathfrak{F}_1\times\mathfrak{F}_2$. And $(\mathfrak{N}\times\mathfrak{B})^{-}[(\varrho, \zeta)(\mathfrak{d}, \xi)^{-1}] = (\mathfrak{N}\times\mathfrak{B})^{-}(\varrho\mathfrak{d}^{-1}, \zeta\xi^{-1}) = \mathrm{rmax}\{\mathfrak{N}^{-}(\varrho\mathfrak{d}^{-1}), \mathfrak{B}^{-}(\zeta\xi^{-1})\} \leq \mathrm{rmax}\ \{\mathrm{rmax}\{\mathfrak{N}^{-}(\varrho), \mathfrak{N}^{-}(\mathfrak{d})\}, \mathrm{rmax}\{\mathfrak{B}^{-}(\zeta), \mathfrak{B}^{-}(\xi)\}\} = \mathrm{rmax}\{\mathrm{rmax}\{\mathfrak{N}^{-}(\varrho), \mathfrak{B}^{-}(\zeta)\}, \mathrm{rmax}\{\mathfrak{N}^{-}(\mathfrak{d}), \mathfrak{B}^{-}(\xi)\}\} = \mathrm{rmax}\{(\mathfrak{N}\times\mathfrak{B})^{-}(\varrho, \zeta), (\mathfrak{N}\times\mathfrak{B})^{-}(\mathfrak{d}, \xi)\}, \forall\ (\varrho, \zeta), (\mathfrak{d}, \xi) \in \mathfrak{F}_1\times\mathfrak{F}_2$. Hence $\mathfrak{N}\times\mathfrak{B}$ is a $B_{VV}SF$ of $\mathfrak{F}_1\times\mathfrak{F}_2$.

Theorem 2.8. $If\ \mathfrak{D}_1 = \langle \mathfrak{D}_1{}^{+},\ \ \mathfrak{D}_1{}^{-} \rangle,\ \ \mathfrak{D}_2 = \langle \mathfrak{D}_2{}^{+},\ \ \mathfrak{D}_2{}^{-} \rangle,\ \ \ldots$ and $\mathfrak{D}_m = \langle \mathfrak{D}_m{}^{+},\ \ \mathfrak{D}_m{}^{-} \rangle$ $are\ \ B_{VV}SFs$ of a fields $\mathfrak{F}_1, \mathfrak{F}_2, \ldots \mathfrak{F}_m$, then $\mathfrak{D}_1 \times \mathfrak{D}_2 \times \ldots \times \mathfrak{D}_m$ is also a $B_{VV}SF$ of $\mathfrak{F}_1 \times \mathfrak{F}_2 \times \ldots \times \mathfrak{F}_m$.

Proof. By Theorem 2.7, it can be easily shown.

Theorem 2.9. $Let\ \mathfrak{D}_1 = \langle \mathfrak{D}_1{}^{+},\ \ \mathfrak{D}_1{}^{-} \rangle,\ \ \mathfrak{D}_2 = \langle \mathfrak{D}_2{}^{+},\ \ \mathfrak{D}_2{}^{-} \rangle$ be two $B_{VV}Ss$ of a $fields$ $\mathfrak{F}_1, \mathfrak{F}_2$, respectively. If $\mathfrak{D}_1 \times \mathfrak{D}_2$ is a $B_{VV}SF$ of the field $\mathfrak{F}_1 \times \mathfrak{F}_2$, then atleast the following one holds, where $\mathfrak{o}, \mathfrak{k}$ are identity elements of $\mathfrak{F}_1$, $\mathfrak{h}, \mathfrak{d}$ are identity elements $of\ \mathfrak{F}_2$,

$(i)\mathfrak{D}_2{}^{+}(\mathfrak{h}) \geq \mathfrak{D}_1{}^{+}(\mathscr{Z}), \forall\ \mathscr{Z} \in \mathfrak{F}_1, \mathfrak{D}_2{}^{+}(\mathfrak{d}) \geq \mathfrak{D}_1{}^{+}(\mathscr{Z}), \forall\ \mathscr{Z} \neq \mathfrak{o} \in \mathfrak{F}_1, \mathfrak{D}_2{}^{-}(\mathfrak{h}) \leq \mathfrak{D}_1{}^{-}(\mathscr{Z}), \forall\ \mathscr{Z} \in \mathfrak{F}_1$ and $\mathfrak{D}_2{}^{-}(\mathfrak{d}) \leq \mathfrak{D}_1{}^{-}(\mathscr{Z}), \forall\ \mathscr{Z} \neq \mathfrak{o} \in \mathfrak{F}_1$,

$(ii)\mathfrak{D}_1{}^{+}(\mathfrak{o}) \geq \mathfrak{D}_2{}^{+}(\mathscr{Z}), \forall\ \mathscr{Z} \in \mathfrak{F}_2, \mathfrak{D}_1{}^{+}(\mathfrak{k}) \geq \mathfrak{D}_2{}^{+}(\mathscr{Z}), \forall\ \mathscr{Z} \neq \mathfrak{h} \in \mathfrak{F}_2, \mathfrak{D}_1{}^{-}(\mathfrak{o}) \leq \mathfrak{D}_2{}^{-}(\mathscr{Z}), \forall\ \mathscr{Z} \in \mathfrak{F}_2$ and $\mathfrak{D}_1{}^{-}(\mathfrak{k}) \leq \mathfrak{D}_2{}^{-}(\mathscr{Z}), \forall\ \mathscr{Z} \neq \mathfrak{h} \in \mathfrak{F}_2$,

Proof. Let $\mathfrak{D}_1 \times \mathfrak{D}_2$ is a $B_{VV}SF$ of the field $\mathfrak{F}_1 \times \mathfrak{F}_2$. $Assume\ that\ (i)\ and\ (ii)\ are\ not$

$true. Let\ \mathcal{Z} \neq \mathfrak{o} \in \mathfrak{F}_1\ and\ \omega \neq \mathfrak{h} \in \mathfrak{F}_2$ and $(\mathcal{Z}, \omega) \in \mathfrak{F}_1 \times \mathfrak{F}_2$. Then $(\mathfrak{D}_1 \times \mathfrak{D}_2)^+(\mathcal{Z}, \omega)$ = rmin$\{\mathfrak{D}_1^+(\mathcal{Z}), \mathfrak{D}_2^+(\omega)\}$ > rmin$\{\mathfrak{D}_1^+(\mathfrak{o}), \mathfrak{D}_2^+(\mathfrak{h})\} = (\mathfrak{D}_1 \times \mathfrak{D}_2)^+(\mathfrak{o}, \mathfrak{h})$, which is $\Rightarrow\Leftarrow$ to $\mathfrak{D}_1 \times \mathfrak{D}_2$ is a $B_{VV}SF$ of the field $\mathfrak{F}_1 \times \mathfrak{F}_2$. Hence atleast one of the two (i) and (ii) are true.

Theorem 2.10. $Let\ \mathfrak{D}_1 = \langle \mathfrak{D}_1^+, \mathfrak{D}_1^- \rangle$, $\mathfrak{D}_2 = \langle \mathfrak{D}_2^+, \mathfrak{D}_2^- \rangle$ be two $B_{VV}SFs$ of a $fields$ $\mathfrak{F}_1, \mathfrak{F}_2$, respectively. If $\mathfrak{D}_1 \times \mathfrak{D}_2$ is a $B_{VV}SF$ of the field $\mathfrak{F}_1 \times \mathfrak{F}_2$,

$(i)\ if\ \mathfrak{D}_2^+(\mathfrak{h}) \geq \mathfrak{D}_1^+(\mathcal{Z}), \forall\ \mathcal{Z} \in \mathfrak{F}_1, \mathfrak{D}_2^+(\mathfrak{d}) \geq \mathfrak{D}_1^+(\mathcal{Z}), \forall\ \mathcal{Z} \neq \mathfrak{o} \in \mathfrak{F}_1, \mathfrak{D}_2^-(\mathfrak{h}) \leq \mathfrak{D}_1^-(\mathcal{Z}), \forall\ \mathcal{Z} \in \mathfrak{F}_1$ and $\mathfrak{D}_2^-(\mathfrak{d}) \leq \mathfrak{D}_1^-(\mathcal{Z}), \forall\ \mathcal{Z} \neq \mathfrak{o} \in \mathfrak{F}_1, then\ \mathfrak{D}_1\ is\ a\ B_{VV}SF\ of\ \mathfrak{F}_1$, where $\mathfrak{o}, \mathfrak{k}$ are identity elements of $\mathfrak{F}_1$, $\mathfrak{h}, \mathfrak{d}$ are identity elements $of\ \mathfrak{F}_2$;

$(ii) \mathfrak{D}_1^+(\mathfrak{o}) \geq \mathfrak{D}_2^+(\varsigma), \forall\ \varsigma \in \mathfrak{F}_2, \mathfrak{D}_1^+(\mathfrak{k}) \geq \mathfrak{D}_2^+(\varsigma), \forall\ \varsigma \neq \mathfrak{h} \in \mathfrak{F}_2, \mathfrak{D}_1^-(\mathfrak{o}) \leq \mathfrak{D}_2^-(\varsigma), \forall\ \varsigma \in \mathfrak{F}_2$ and $\mathfrak{D}_1^-(\mathfrak{k}) \leq \mathfrak{D}_2^-(\varsigma), \forall\ \varsigma \neq \mathfrak{h} \in \mathfrak{F}_2, then\ \mathfrak{D}_2\ is\ a\ B_{VV}SF\ of\ \mathfrak{F}_2$.

Proof. Let $\mathcal{Z}, \mathfrak{s}$ be in $\mathfrak{F}_1$. Then $(\mathcal{Z}, \mathfrak{h})$ and $(\mathfrak{s}, \mathfrak{h})$ are in $\mathfrak{F}_1 \times \mathfrak{F}_2$. Then

(i) $\mathfrak{D}_1^+(\mathcal{Z}-\mathfrak{s})$ = rmin$\{\mathfrak{D}_1^+(\mathcal{Z}-\mathfrak{s}), \mathfrak{D}_2^+(\mathfrak{h}-\mathfrak{h})\} = (\mathfrak{D}_1 \times \mathfrak{D}_2)^+(\mathcal{Z}-\mathfrak{s}, \mathfrak{h}-\mathfrak{h}) = (\mathfrak{D}_1 \times \mathfrak{D}_2)^+[(\mathcal{Z}, \mathfrak{h})-(\mathfrak{s}, \mathfrak{h})] \geq$ rmin$\{(\mathfrak{D}_1 \times \mathfrak{D}_2)^+(\mathcal{Z}, \mathfrak{h}), (\mathfrak{D}_1 \times \mathfrak{D}_2)^+(\mathfrak{s}, \mathfrak{h})\}$=rmin{rmin$\{\mathfrak{D}_1^+(\mathcal{Z}), \mathfrak{D}_2^+(\mathfrak{h})\}$, rmin$\{\mathfrak{D}_1^+(\mathfrak{s}), \mathfrak{D}_2^+(\mathfrak{h})\}$}= rmin$\{\mathfrak{D}_1^+(\mathcal{Z}), \mathfrak{D}_1^+(\mathfrak{s})\}$, $\forall \mathcal{Z}, \mathfrak{s}$ in $\mathfrak{F}_1$. And $\mathfrak{D}_1^+(\mathcal{Z}\mathfrak{s}^{-1})$ = rmin$\{\mathfrak{D}_1^+(\mathcal{Z}\mathfrak{s}^{-1}), \mathfrak{D}_2^+(\mathfrak{h}\,\mathfrak{h})\}$ = $(\mathfrak{D}_1 \times \mathfrak{D}_2)^+(\mathcal{Z}\mathfrak{s}^{-1}, \mathfrak{h}\mathfrak{h}) = (\mathfrak{D}_1 \times \mathfrak{D}_2)^+[(\mathcal{Z}, \mathfrak{h})(\mathfrak{s}, \mathfrak{h})^{-1}] \geq$ rmin$\{(\mathfrak{D}_1 \times \mathfrak{D}_2)^+(\mathcal{Z}, \mathfrak{h}), (\mathfrak{D}_1 \times \mathfrak{D}_2)^+(\mathfrak{s}, \mathfrak{h})\}$ = rmin{rmin$\{\mathfrak{D}_1^+(\mathcal{Z}), \mathfrak{D}_2^+(\mathfrak{h})\}$, rmin$\{\mathfrak{D}_1^+(\mathfrak{s}), \mathfrak{D}_2^+(\mathfrak{h})\}$} = rmin$\{\mathfrak{D}_1^+(\mathcal{Z}), \mathfrak{D}_1^+(\mathfrak{s})\}$, $\forall \mathcal{Z}, \mathfrak{s} \neq \mathfrak{o}$ in $\mathfrak{F}_1$. Also $\mathfrak{D}_1^-(\mathcal{Z}-\mathfrak{s})$ = rmax$\{\mathfrak{D}_1^-(\mathcal{Z}-\mathfrak{s}), \mathfrak{D}_2^-(\mathfrak{h}-\mathfrak{h})\} = (\mathfrak{D}_1 \times \mathfrak{D}_2)^-(\mathcal{Z}-\mathfrak{s}, \mathfrak{h}-\mathfrak{h}) = (\mathfrak{D}_1 \times \mathfrak{D}_2)^-[(\mathcal{Z}, \mathfrak{h})-(\mathfrak{s}, \mathfrak{h})] \leq$ rmax$\{(\mathfrak{D}_1 \times \mathfrak{D}_2)^-(\mathcal{Z}, \mathfrak{h}), (\mathfrak{D}_1 \times \mathfrak{D}_2)^-(\mathfrak{s}, \mathfrak{h})\}$ = rmax {rmax$\{\mathfrak{D}_1^-(\mathcal{Z}), \mathfrak{D}_2^-(\mathfrak{h})\}$, rmax$\{\mathfrak{D}_1^-(\mathfrak{s}), \mathfrak{D}_2^-(\mathfrak{h})\}$}= rmax$\{\mathfrak{D}_1^-(\mathcal{Z}), \mathfrak{D}_1^-(\mathfrak{s})\}$, $\forall \mathcal{Z}, \mathfrak{s}$ in $\mathfrak{F}_1$. And $\mathfrak{D}_1^-(\mathcal{Z}\mathfrak{s}^{-1})$ =rmax$\{\mathfrak{D}_1^-(\mathcal{Z}\mathfrak{s}^{-1}), \mathfrak{D}_2^-(\mathfrak{h}\,\mathfrak{h})\} = (\mathfrak{D}_1 \times \mathfrak{D}_2)^-(\mathcal{Z}\mathfrak{s}^{-1}, \mathfrak{h}\mathfrak{h}) = (\mathfrak{D}_1 \times \mathfrak{D}_2)^-[(\mathcal{Z}, \mathfrak{h})(\mathfrak{s}, \mathfrak{h})^{-1})] \leq$ rmax$\{(\mathfrak{D}_1 \times \mathfrak{D}_2)^-(\mathcal{Z}, \mathfrak{h}), (\mathfrak{D}_1 \times \mathfrak{D}_2)^-(\mathfrak{s}, \mathfrak{h})\}$= rmax{rmax$\{\mathfrak{D}_1^-(\mathcal{Z}), \mathfrak{D}_2^-(\mathfrak{h})\}$, rmax$\{\mathfrak{D}_1^-(\mathfrak{s}), \mathfrak{D}_2^-(\mathfrak{h})\}$}= rmax$\{\mathfrak{D}_1^-(\mathcal{Z}), \mathfrak{D}_1^-(\mathfrak{s})\}$, $\forall \mathcal{Z}, \mathfrak{s} \neq \mathfrak{o} \in \mathfrak{F}_1$. Hence $\mathfrak{D}_1\ is\ a\ B_{VV}SF\ of\ \mathfrak{F}_1$.

Let $\varsigma, \mathfrak{s}$ be in $\mathfrak{F}_2$. Then $(\mathfrak{o}, \varsigma)$ and $(\mathfrak{o}, \mathfrak{s})$ are in $\mathfrak{F}_1 \times \mathfrak{F}_2$. Then

(ii) $\mathfrak{D}_2^+(\varsigma-\mathfrak{s})$ = rmin$\{\mathfrak{D}_2^+(\varsigma-\mathfrak{s}), \mathfrak{D}_1^+(\mathfrak{o}-\mathfrak{o})\} = (\mathfrak{D}_1 \times \mathfrak{D}_2)^+(\mathfrak{o}-\mathfrak{o}, \varsigma-\mathfrak{s}) = (\mathfrak{D}_1 \times \mathfrak{D}_2)^+[(\mathfrak{o}, \varsigma)-(\mathfrak{o}, \mathfrak{s})] \geq$ rmin$\{(\mathfrak{D}_1 \times \mathfrak{D}_2)^+(\mathfrak{o}, \varsigma), (\mathfrak{D}_1 \times \mathfrak{D}_2)^+(\mathfrak{o}, \mathfrak{s})\}$= rmin{rmin$\{\mathfrak{D}_1^+(\mathfrak{o}), \mathfrak{D}_2^+(\varsigma)\}$, rmin$\{\mathfrak{D}_1^+(\mathfrak{o}), \mathfrak{D}_2^+(\mathfrak{s})\}$} = rmin$\{\mathfrak{D}_2^+(\varsigma), \mathfrak{D}_2^+(\mathfrak{s})\}$, $\forall \varsigma, \mathfrak{s} \in \mathfrak{F}_2$. And $\mathfrak{D}_2^+(\varsigma\mathfrak{s}^{-1})$ = rmin$\{\mathfrak{D}_2^+(\varsigma\mathfrak{s}^{-1}), \mathfrak{D}_1^+(\mathfrak{o}\mathfrak{o})\}$ = $(\mathfrak{D}_1 \times \mathfrak{D}_2)^+(\mathfrak{o}\mathfrak{o}, \varsigma\mathfrak{s}^{-1}) = (\mathfrak{D}_1 \times \mathfrak{D}_2)^+[(\mathfrak{o}, \varsigma)(\mathfrak{o}, \mathfrak{s})^{-1}] \geq$ rmin$\{(\mathfrak{D}_1 \times \mathfrak{D}_2)^+(\mathfrak{o}, \varsigma), (\mathfrak{D}_1 \times \mathfrak{D}_2)^+(\mathfrak{o}, \mathfrak{s})\}$ = rmin{rmin$\{\mathfrak{D}_1^+(\mathfrak{o}), \mathfrak{D}_2^+(\varsigma)\}$, rmin$\{\mathfrak{D}_1^+(\mathfrak{o}), \mathfrak{D}_2^+(\mathfrak{s})\}$}= rmin$\{\mathfrak{D}_2^+(\varsigma), \mathfrak{D}_2^+(\mathfrak{s})\}$, $\forall \varsigma, \mathfrak{s} \neq \mathfrak{h} \in \mathfrak{F}_2$. Also $\mathfrak{D}_2^-(\varsigma-\mathfrak{s})$ = rmax$\{\mathfrak{D}_2^-(\varsigma-\mathfrak{s}), \mathfrak{D}_1^-(\mathfrak{o}-\mathfrak{o})\}$= $(\mathfrak{D}_1 \times \mathfrak{D}_2)^-(\mathfrak{o}-\mathfrak{o}, \varsigma-\mathfrak{s}) = (\mathfrak{D}_1 \times \mathfrak{D}_2)^-[(\mathfrak{o}, \varsigma)-(\mathfrak{o}, \mathfrak{s})] \leq$ rmax$\{(\mathfrak{D}_1 \times \mathfrak{D}_2)^-(\mathfrak{o}, \varsigma), (\mathfrak{D}_1 \times \mathfrak{D}_2)^-(\mathfrak{o}, \mathfrak{s})\}$= rmax{rmax$\{\mathfrak{D}_1^-(\mathfrak{o}), \mathfrak{D}_2^-(\varsigma)\}$, rmax$\{\mathfrak{D}_1^-(\mathfrak{o}), \mathfrak{D}_2^-(\mathfrak{s})\}$}= rmax$\{\mathfrak{D}_2^-(\varsigma), \mathfrak{D}_2^-(\mathfrak{s})\}$, $\forall \varsigma, \mathfrak{s} \in \mathfrak{F}_2$. And $\mathfrak{D}_2^-(\varsigma\mathfrak{s}^{-1})$ = rmax$\{\mathfrak{D}_2^-(\varsigma\mathfrak{s}^{-1}), \mathfrak{D}_1^-(\mathfrak{o}\mathfrak{o})\}$ = $(\mathfrak{D}_1 \times \mathfrak{D}_2)^-(\mathfrak{o}\mathfrak{o}, \varsigma\mathfrak{s}^{-1}) = (\mathfrak{D}_1 \times \mathfrak{D}_2)^-[(\mathfrak{o}, \varsigma)(\mathfrak{o}, \mathfrak{s})^{-1}] \leq$ rmax$\{(\mathfrak{D}_1 \times \mathfrak{D}_2)^-(\mathfrak{o}, \varsigma), (\mathfrak{D}_1 \times \mathfrak{D}_2)^-(\mathfrak{o}, \mathfrak{s})\}$ =rmax{rmax$\{\mathfrak{D}_1^-(\mathfrak{o}), \mathfrak{D}_2^-(\varsigma)\}$, rmax$\{\mathfrak{D}_1^-(\mathfrak{o}), \mathfrak{D}_2^-(\mathfrak{s})\}$}= rmax$\{\mathfrak{D}_2^-(\varsigma), \mathfrak{D}_2^-(\mathfrak{s})\}$, $\forall \varsigma, \mathfrak{s} \neq \mathfrak{h} \in \mathfrak{F}_2$.

Hence $\mathfrak{D}_2\ is\ a\ B_{VV}SF\ of\ \mathfrak{F}_2$.

Theorem 2.11. $Let\ \mathfrak{D}_1 = \langle \mathfrak{D}_1^+, \mathfrak{D}_1^- \rangle$, $\mathfrak{D}_2 = \langle \mathfrak{D}_2^+, \mathfrak{D}_2^- \rangle$, ..., $\mathfrak{D}_m = \langle \mathfrak{D}_m^+, \mathfrak{D}_m^- \rangle$ be $B_{VV}SFs$ of the $fields$ $\mathfrak{F}_1, \mathfrak{F}_2, \dots, \mathfrak{F}_m$, respectively. If $\mathfrak{D}_1 \times \mathfrak{D}_2 \times \dots \times \mathfrak{D}_m$ is a $B_{VV}SF$ of the field $\mathfrak{F}_1 \times \mathfrak{F}_2 \times \dots \times \mathfrak{F}_m$, for each $\mathfrak{D}_i$, $if\ \mathfrak{D}_j^+(\mathfrak{h}_j) \geq \mathfrak{D}_i^+(\mathcal{Z}), \forall\ \mathcal{Z} \in \mathfrak{F}_i, \mathfrak{D}_j^+(\mathfrak{d}_j) \geq \mathfrak{D}_i^+(\mathcal{Z}), \forall\ \mathcal{Z} \neq \mathfrak{o}_i \in \mathfrak{F}_i$,

$\mathfrak{O}_j^{-}(\mathfrak{h}_j) \leq \mathfrak{O}_i^{-}(\mathcal{J}), \forall\, \mathcal{J} \in \mathfrak{F}_i$ and $\mathfrak{O}_j^{-}(\mathfrak{d}_j) \leq \mathfrak{O}_i^{-}(\mathcal{J}), \forall\, \mathcal{J} \neq \mathfrak{o}_i \in \mathfrak{F}_i$, then $\mathfrak{O}_i$ is a $B_{VV}SF$ of $\mathfrak{F}_i$, where $\mathfrak{o}_i, \mathfrak{k}_i$ are identity elements of $\mathfrak{F}_i$, $\mathfrak{h}_j, \mathfrak{d}_j$ are identity elements of $\mathfrak{F}_j$.

Proof. By Theorem 2.10, it can be easily shown.

Theorem 2.12. *Let* Д *be a* $B_{VV}S$ of a field $\mathfrak{C}_1$ and Ћ be the stronget B_{VV} relation of $\mathfrak{C}_1$. Then Д is a $B_{VV}SF$ of $\mathfrak{C}_1$ if and only if Ћ is a $B_{VV}SF$ of $\mathfrak{C}_1 \times \mathfrak{C}_1$.

Proof. Let $\varkappa, \tilde{\omega}$ be in $\mathfrak{C}_1$ and ζ, ξ be in $\mathfrak{C}_1$. Then $(\varkappa, \zeta)$ and $(\tilde{\omega}, \xi)$ are in $\mathfrak{C}_1 \times \mathfrak{C}_1$. If Д is a $B_{VV}SF$ of $\mathfrak{C}_1$, then $Ћ^{+}[(\varkappa, \zeta)-(\tilde{\omega}, \xi)] = Ћ^{+}(\varkappa-\tilde{\omega}, \zeta-\xi) = \text{rmin}\{Д^{+}(\varkappa-\tilde{\omega}), Д^{+}(\zeta-\xi)\} \geq \text{rmin}\{\text{rmin}\{Д^{+}(\varkappa), Д^{+}(\tilde{\omega})\}, \text{rmin}\{Д^{+}(\zeta), Д^{+}(\xi)\}\} = \text{rmin}\{\text{rmin}\{Д^{+}(\varkappa), Д^{+}(\zeta)\}, \text{rmin}\{Д^{+}(\tilde{\omega}), Д^{+}(\xi)\}\} = \text{rmin}\{Ћ^{+}(\varkappa, \zeta), Ћ^{+}(\tilde{\omega}, \xi)\}, \forall\, (\varkappa, \zeta), (\tilde{\omega}, \xi) \in \mathfrak{C}_1 \times \mathfrak{C}_1$. And $Ћ^{+}[(\varkappa, \zeta)(\tilde{\omega}, \xi)^{-1}] = Ћ^{+}(\varkappa\tilde{\omega}^{-1}, \zeta\xi^{-1}) = \text{rmin}\{Д^{+}(\varkappa\tilde{\omega}^{-1}), Д^{+}(\zeta\xi^{-1})\} \geq \text{rmin}\{\text{rmin}\{Д^{+}(\varkappa), Д^{+}(\tilde{\omega})\}, \text{rmin}\{Д^{+}(\zeta), Д^{+}(\xi)\}\} = \text{rmin}\{\text{rmin}\{Д^{+}(\varkappa), Д^{+}(\zeta)\}, \text{rmin}\{Д^{+}(\tilde{\omega}), Д^{+}(\xi)\}\} = \text{rmin}\{Ћ^{+}(\varkappa, \zeta), Ћ^{+}(\tilde{\omega}, \xi)\}, \forall\, (\varkappa, \zeta), (\tilde{\omega}, \xi) \in \mathfrak{C}_1 \times \mathfrak{C}_1$. Also $Ћ^{-}[(\varkappa, \zeta)-(\tilde{\omega}, \xi)] = Ћ^{-}(\varkappa-\tilde{\omega}, \zeta-\xi) = \text{rmax}\{Д^{-}(\varkappa-\tilde{\omega}), Д^{-}(\zeta-\xi)\} \leq \text{rmax}\{\text{rmax}\{Д^{-}(\varkappa), Д^{-}(\tilde{\omega})\}, \text{rmax}\{Д^{-}(\zeta), Д^{-}(\xi)\}\} = \text{rmax}\{\text{rmax}\{Д^{-}(\varkappa), Д^{-}(\zeta)\}, \text{rmax}\{Д^{-}(\tilde{\omega}), Д^{-}(\xi)\}\} = \text{rmax}\{Ћ^{-}(\varkappa, \zeta), Ћ^{-}(\tilde{\omega}, \xi)\}, \forall\, (\varkappa, \zeta), (\tilde{\omega}, \xi) \in \mathfrak{C}_1 \times \mathfrak{C}_1$. And $Ћ^{-}[(\varkappa, \zeta)(\tilde{\omega}, \xi)^{-1}] = Ћ^{-}(\varkappa\tilde{\omega}^{-1}, \zeta\xi^{-1}) = \text{rmax}\{Д^{-}(\varkappa\tilde{\omega}^{-1}), Д^{-}(\zeta\xi^{-1})\} \leq \text{rmax}\{\text{rmax}\{Д^{-}(\varkappa), Д^{-}(\tilde{\omega})\}, \text{rmax}\{Д^{-}(\zeta), Д^{-}(\xi)\}\} = \text{rmax}\{\text{rmax}\{Д^{-}(\varkappa), Д^{-}(\zeta)\}, \text{rmax}\{Д^{-}(\tilde{\omega}), Д^{-}(\xi)\}\} = \text{rmax}\{Ћ^{-}(\varkappa, \zeta), Ћ^{-}(\tilde{\omega}, \xi)\}$, for all $(\varkappa, \zeta), (\tilde{\omega}, \xi)$ in $\mathfrak{C}_1 \times \mathfrak{C}_1$. Hence Ћ is a $B_{VV}SF$ of $\mathfrak{C}_1 \times \mathfrak{C}_1$.

Conversely, assume Ћ is a $B_{VV}SF$ of $\mathfrak{C}_1 \times \mathfrak{C}_1$.

$\text{rmin}\{Д^{+}(\varkappa-\tilde{\omega}), Д^{+}(\zeta-\xi)\} = Ћ^{+}(\varkappa-\tilde{\omega}, \zeta-\xi) = Ћ^{+}[(\varkappa, \zeta)-(\tilde{\omega}, \xi)] \geq \text{rmin}\{Ћ^{+}(\varkappa, \zeta), Ћ^{+}(\tilde{\omega}, \xi)\} = \text{rmin}\{\text{rmin}\{Д^{+}(\varkappa), Д^{+}(\zeta)\}, \text{rmin}\{Д^{+}(\tilde{\omega}), Д^{+}(\xi)\}\}$, put $\zeta = \mathfrak{o}$ and $\xi = \mathfrak{o}$, where $\mathfrak{o}$ is an first operation identity element of $\mathfrak{C}_1$, then $Д^{+}(\varkappa-\tilde{\omega}) \geq \text{rmin}\{Д^{+}(\varkappa), Д^{+}(\tilde{\omega})\}, \forall\, \varkappa, \tilde{\omega} \in \mathfrak{C}_1$.

And $\text{rmin}\{Д^{+}(\varkappa\tilde{\omega}^{-1}), Д^{+}(\zeta\xi^{-1})\} = Ћ^{+}(\varkappa\tilde{\omega}^{-1}, \zeta\xi^{-1}) = Ћ^{+}[(\varkappa, \zeta)(\tilde{\omega}, \xi)^{-1}] \geq \text{rmin}\{Ћ^{+}(\varkappa, \zeta), Ћ^{+}(\tilde{\omega}, \xi)\} = \text{rmin}\{\text{rmin}\{Д^{+}(\varkappa), Д^{+}(\zeta)\}, \text{rmin}\{Д^{+}(\tilde{\omega}), Д^{+}(\xi)\}\}$, put $\zeta = \mathfrak{o}$ and $\xi = \mathfrak{o}$, where $\mathfrak{o}$ is an first operation identity element of $\mathfrak{C}_1$, then $Д^{+}(\varkappa\tilde{\omega}^{-1}) \geq \text{rmin}\{Д^{+}(\varkappa), Д^{+}(\tilde{\omega})\}, \forall\, \varkappa, \tilde{\omega} \in \mathfrak{C}_1$. Also $\text{rmax}\{Д^{-}(\varkappa-\tilde{\omega}), Д^{-}(\zeta-\xi)\} = Ћ^{-}(\varkappa-\tilde{\omega}, \zeta-\xi) = Ћ^{-}[(\varkappa, \zeta)-(\tilde{\omega}, \xi)] \leq \text{rmax}\{Ћ^{-}(\varkappa, \zeta), Ћ^{-}(\tilde{\omega}, \xi)\} = \text{rmax}\{\text{rmax}\{Д^{-}(\varkappa), Д^{-}(\zeta)\}, \text{rmax}\{Д^{-}(\tilde{\omega}), Д^{-}(\xi)\}\}$, put $\zeta = \mathfrak{o}$ and $\xi = \mathfrak{o}$, where $\mathfrak{o}$ is an first operation identity element of $\mathfrak{C}_1$, then $Д^{-}(\varkappa-\tilde{\omega}) \leq \max\{Д^{-}(\varkappa), Д^{-}(\tilde{\omega})\}, \forall\, \varkappa, \tilde{\omega} \in \mathfrak{C}_1$. And $\text{rmax}\{Д^{-}(\varkappa\tilde{\omega}^{-1}), Д^{-}(\zeta\xi^{-1})\} = Ћ^{-}(\varkappa\tilde{\omega}^{-1}, \zeta\xi^{-1}) = Ћ^{-}[(\varkappa, \zeta)(\tilde{\omega}, \xi)^{-1}] \leq \text{rmax}\{Ћ^{-}(\varkappa, \zeta), Ћ^{-}(\tilde{\omega}, \xi)\} = \text{rmax}\{\text{rmax}\{Д^{-}(\varkappa), Д^{-}(\zeta)\}, \text{rmax}\{Д^{-}(\tilde{\omega}), Д^{-}(\xi)\}\}$, put $\zeta = \mathfrak{o}$ and $\xi = \mathfrak{o}$, where $\mathfrak{o}$ is an first operation identity element of $\mathfrak{C}_1$, then $Д^{-}(\varkappa\tilde{\omega}^{-1}) \leq \text{rmax}\{Д^{-}(\varkappa), Д^{-}(\tilde{\omega})\}, \forall\, \varkappa, \tilde{\omega} \in \mathfrak{C}_1$.

CONCLUSION

Using the above theorems, we can find more results. It can be extended into different types of BVV algebra.

REFERENCES

[1] Anandh.B and Giri.R, "Notes on Intuitionistic (T, S)-Fuzzy Subfields of A Field", *IOSR Journal of Mathematics (IOSR-JM)*, Volume 12, Issue 5 Ver. III (2016), 30–34.

[2] Anitha.M.S., Murugenantha Prasad & K.Arjunan, Notes on bipolar valued fuzzy subgroups of a group, *Bulletin of Society for Mathematical Services and Standards,* Vol. 2 No. 3 (2013), pp. 52-59.

[3] Azriel Rosenfeld, fuzzy groups, *Journal of mathematical analysis and applications* 35(1971), 512-517.

[4] Balasubramanian.A, K.L.Muruganantha Prasad & K.Arjunan, “Properties of Bipolar interval valued fuzzy subgroups of a group”, *International Journal of Scientific Research,* Vol. 4, Iss. 4 (2015), 262 - 268.

[5] Cicily Flora. S and Arockiarani.I, A new class of generalized bipolar vague sets, *International Journal of Information Research and review,*3(11), (2016), 3058– 3065.

[6] Gau W.L and Buehrer D.J, Vague sets, *IEEE Transactions on systems, Man and Cybernetics,* 23(1993), 610 – 614.

[7] Grattan-Guiness, “Fuzzy membership mapped onto interval and many valued quantities”, *Z.Math.Logik. Grundladen Math.* 22 (1975), 149 – 160.

[8] K.M.Lee, bipolar valued fuzzy sets and their operations. *Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand* (2000), 307-312.

[9] K.M.Lee, Comparison of interval valued fuzzy sets, intuitionistic fuzzy sets and bipolar valued fuzzy sets. J. fuzzy Logic Intelligent Systems, 14 (2) (2004), 125-129.

[10] Muthusamy. M, Palaniappan. N and Arjunan. K, “Homomorphism and anti homomorphism of level subfield of intuitionistic fuzzy subfield of a field”, *International journal of Computationaland Applied Mathematics*, Vol. 4, number 3 (2009), 299 – 306.

[11] RanjitBiswas, Vague groups, *International Journal of Computational Coginition,* 4(2), (2006), 20 – 23.

[12] Yamini. C, Arjunan. K, and Ananth.B., “Bipolar valued multi fuzzy subfield of a field”, *International Journal of Management, Technology And Engineering,* volume 8, issue XI(2018), 1706 - 1711.

[13] Yasodara.B and KE.Sathappan, “Bipolar-valued multi fuzzy subsemifields of a semifield”, *International Journal of Mathematical Archive*, 6(9) (2015), 75 – 80.

[14] L.A.Zadeh, fuzzy sets, *Inform. And Control,* 8(1965), 338-353.

[15] W.R.Zhang, Bipolar Fuzzy sets and Relations, *a computational Frame work for cognitive modeling and multiple decision Analysis, proceedings of Fuzzy IEEE conferences*, (1994), 305– 309.