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Relation Between the Fractional Domination Number of Some Graphs and Their Line Graphs

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Abstract:

Let G is the simple connected graph with order n and its line graph noted as L(G). Let the fractional dominating number denoted by γ_(f) (G) of graph G and the upper fractional dominating number Γ_(f) (G). In this study we have obtained union and join on 〖γ〗_(f) (G) and γ_(f) (L(G)) exploring the graphs for upper fractional domination number including Cycle, complete, Star, Bi-Star graph, wheel graph, Cubic graph, Graphs of Cartesian product like (K_2 ×〖P〗_(n)), (K_3 ×〖P〗_(n)) and (Cm×Cn) with consideration of the computational complexity. We have taken parameters related to fractional domination in line graphs towards generalization. The goal of this paper is to provide a generalized results of sum 〖γ〗_(f) (G)+γ_(f) (L(G)), Γ_(f) (G)+Γ_(f) L(G) and product 〖γ〗_(f) (G)*γ_(f) (L(G)), Γ_(f) (G)*Γ_(f) L(G) for some specific graph classes.

Keywords: Domination set, fractional dominating number, line graph, union and join.

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1. Introduction

Initially, the idea of graph domination was presented by author Ore in the year 1960, and after this it is very important and vital area of research in theory of graph. The research regarding fractional domination number presented by some researchers in the late 1980s as a generalized concept of domination. A graph's fractional dominating number is described as minimum total weights assigned to all the vertices of graph from fractional dominating set, where a fractional dominating set is a method that gives each vertex a weight such that the addition of the weights of all vertices is at least one. At the 18th south-eastern international conference based on Combinatorics, Graph theory and Computing S.T. Hedetniemi initially defined fractional dominating as a function $f: V \to [0,1]$ is said to be dominating function of the graph $G = (V, E)$ if $f(N[v]) \geq 1$ for all $v \in V$. The fractional dominating number of G is given by $\gamma_f\ (G)$.

M. Farber examined the issue of figuring out whether to use the linear programming formulation of $\gamma_f\ (G)$ could provide an integer solution to LPP.

$\gamma_f(G) = MIN \sum_{i=1}^{n} x_i$,

Subject to $N.X \geq 1_n$ *, where* $x_i > 0$.

The constraint system for domination is $N.X \geq 1_n$ and using a closed neighborhood matrix N we aim to reduce the weight of the function that accomplishes this with n-by-n binary matrix $N = A + I_n$, where I_n be the n-by-n identity matrix and A is adjacency matrix.

The authors [12] have discuss topological spaces generated by simple graphs using adjacency relation and non adjaceny relation on vertices. They establish important results showing relations between complete graph and discrete topological space. Also discuss the topological spaces related to complete graphs, isomorphic graphs and study their properties.

The author [8] have taken the simple connected graph G of n vertices and m edges. Let $L(G)$ is its line graph, In this paper they have obtained lower bound and upper bound for addition of graph and its line graph's domination number. They continue to work on the block graph and further domination number of graph. In the paper [9] they characterize regular graphs and unicyclic graphs of odd order for which $\gamma(G) + \gamma(L(G)) = n - 2$. The authors [1] have studied another fractional version of k-domination in the graphs and related parameters. Let G is the connected graph and k is the positive number with $k \leq rad(G)$.

A subset $D \subseteq V$ is known as a distance k-domination set of graph G iff every $v \in V - D$, there exists a vertex $u \in D$ such that $d(u, v) \leq k$.

The line graph of a graph represents the adjacencies between the line connections of the original graph. This study investigates relationship between the fractional domination number of a graph and their line graph for various graph classes. Our results show that there is a correlation between the sum of fractional domination number of a graph and its line graph, and this sum is NP-hard in general, however the correlation between the sum of the fractional domination number of a graph G and its line graph $L(G)$ is not straightforward. Generally, depends on the specific graphs involved or some graph classes, it can be solved in polynomial time. These findings contribute to our understanding of the properties of graphs.

In some graph classes, the fractional domination number of $L(G)$ can be related to the fractional domination number of G. If G has a high fractional domination number, this might imply that $L(G)$ also has a relatively high fractional domination number due to the structure and relationship between the two graphs.

Positive Correlation**:** In some cases, the fractional domination number of $L(G)$ can be related to the fractional domination number of G. If G has a high fractional domination number, this might imply that $L(G)$ also has a relatively high fractional domination number due to the structure and relationship between the two graphs.

No Simple Correlation**:** However, there is no universal rule stating that the sum of the fractional domination numbers of G and $L(G)$ must be correlated in a simple manner. The relationship is influenced by various factors such as the size of G, its connectivity, and how vertices in $L(G)$ are dominated by sets of vertices in G. The goal of this paper is to investigate a generalization of $\gamma_f(G)$,

$\gamma_f\ (L(G))$ and $\Gamma_f\ (G), \Gamma_f\ L(G)$ for some specific graphs. In this section we are going to give definitions which are useful in further sections.

The definitions are given [5, 7] like the following:

Definition 1.1. *Domination set and domination number:*

Let $G = (V, E)$ is a graph and subset D of vertex set V is known as a dominating set of graph if every vertex in $V - D$ is adjacent to at least one vertex in subset D. A dominating set D is known as a minimal dominating set if there isn't a dominating set of G that is a proper subset of D. The size of smallest dominating set of graph G is called the domination number of graph G and it is denoted by $\gamma(G)$. The maximum number of elements of minimal dominating set of graph G is known as upper domination number of graph and it is denoted by $\Gamma(G)$ The domination number noted as $\gamma\ (G)$ and upper domination number noted as $\Gamma(G)$ are described as:

$\gamma(G) = min\ \{|D|: D\ is\ minimal\ dominating\ set\ of\ G\}$, and

$\Gamma(G) = max\ \{|D|: D\ is\ minimal\ dominating\ set\ of\ G\}$.

Definition 1.2. *Fractional dominating function:*

A dominating function f to be any $f: V(G) \to [0, 1]$ is a function of graph G which allocate values for every vertex $v \in V(G)$ in unit interval [0,1]. The function f is known as fractional dominating function if for every vertex v $\in V(G)$, $f(N[v]) = \sum_{v \in N[v]} f(v) \geq 1$. It means the addition of the values allocated to the vertices in closed neighborhood of $v \in V(G)$ *such that* $N[v]$ is at least one, i.e. $(N[v]) \geq 1$. (Since then any vertex $v \in V(G)$ is in the closed neighborhood of at least one vertex in D, where D is subset of vertex set V).

Definition 1.3. *Fractional domination number:*

The dominating function f is known as a minimal fractional domination function if there doesn't exist another domination function $g \neq f$ *for which* $g(v) \leq f(v)$ *for all* $v \in V(G)$ equivalently f as an minimal fractional dominating function if for each vertex v with $f(v) > 0$, *there exist a vertex* $w \in N[v]$ *such that* $\sum_{v \in N[w]} f(v) = 1$ By lemma [8] Let f be a dominating function for a graph $G = (V, E)$ then f is minimal dominating if and only if for any $v \in V(G)$ whenever $f\ (v) > 0$ there exists some $u \in N[v]$ such that $f(N[v]) = 1$. (If there is a vertex v for which given condition is not true, means if every vertex in the closed neighborhood of v carry out $(N[v]) > 1$, then we are in a position to decrease $f\ (v)$ to obtain a smaller fractional dominating function and so f is not a minimal fractional dominating function. The fractional domination number of G, $\gamma_f\ (G)$ and upper fractional domination number of G, $\Gamma_f\ (G)$ are described as,

$\gamma_f\ (G) = min\ \{|f|: f\ is\ an\ minimal\ fractional\ dominating\ function\ of\ G\}$,

$\Gamma_f\ (G) = max\ \{|f|: f\ is\ an\ minimal\ fractional\ dominating\ function\ of\ G\}$.

where $|f| = \sum_{v \in V} f(v)$.

Definition 1.4. *Line graph:*

Line graph of G is $L\ (G)$ having its vertices are converted as the edges of original graph G with two vertices of graph $L\ (G)$ are adjacent for the condition if corresponding edges share a common vertex in graph G. Another definition that the vertices of $L\ (G)$ are the edges of graph G with two vertices are adjacent whenever the corresponding edges of G are adjacent.

In any minimal dominating set S is subset of $V\ (G)$, such that $N[S] = V(G)$ and the closed neighborhood of each $v \in S$ contains at least one vertex and which is not in the closed neighborhood of any other member of S . (If this is not true for some $v \in S$ then $S - \{v\}$ is a smaller dominating set, contradicting the minimality of S). Consequently, every maximal irredundant set is likewise a minimal dominant set. Hence $ir\ (G) \leq \gamma(G)$ earlier this result was obtained by [3] every minimal dominating set induces a minimum dominating function, and every minimal dominating function is said to be a minimal fractional dominating function. This implies that $\gamma_f\ (G) \leq \gamma(G) \leq \Gamma(\mathrm{G}) \leq \Gamma_f\ (G)$.

Some preliminary known results:

Theorem 1.1. ([5]). *For every graph G we have* $\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \leq \gamma\,(G) \leq n - \Delta(G)$.

Theorem 1.2. ([5]). *For every graph G we have* $\frac{n}{1+\Delta(G)} \leq \gamma_f\ (G) \leq \frac{n}{1+\delta(G)}$,

Where $\Delta(G)$ *is* noted as *maximum degree* and $\delta(G)$ *is noted as minimum degree of graph G.*

Theorem 1.3. ([5]). *For any graph G fractional domination number* $\gamma_f\ (G)\ =\ 1$ *if and only if* $\Delta(G) =\ n - 1$.

Corollary 1.1. ([5]). *Let* G *is any simple connected graph with* $\Delta(G) =\ n - 1$ *and let* f *be any fractional domination function with* $(G) = |f|$. *Then for only those vertices* $v \in V(G)$ *with* $deg\ (v)$ = $n - 1$ *satisfy* $f(G) > 0$.

Theorem 1.4. ([3]). *If the graph G is r-regular then* $\gamma_f\ (G) = \frac{n}{r+1}$.

In section 2 we determine the fractional and upper fractional dominating number of graph G and $L(G)$. We also determine the $\gamma_f\ (G) + \gamma_f\ (L(G))$, $\gamma_f\ (G) * \gamma_f\ (L(G))$ and $\Gamma_f\ (G) + \Gamma_f\ (L(G))$, $\Gamma_f\ (G) * \Gamma_f\ (L(G))$. We have taken the standard graphs like Cycle, Complete, Star, Bi-Star graph, Wheel graph, Cubic graph, Cartesian product graphs. In section 3 we introduce applications in terms of union and join of the considered graphs.

2. Relation between the fractional domination number of graph and their line graph

2.1 Cycle graph (C_n)

Simple graph G of order n where $(n \geq 3)$ and number of edges n is known as a cycle graph. Every edge in the graph creates a cycle with length n and each vertex has degree two. The line graph of cycle graph is $L(G)$ isomorphic to its cycle graph G with same vertex numbers and are adjacent in the provided graph if the corresponding edges share a common vertex.

Theorem 2.1.1. If G is the simple connected cycle graph (C_n) on n vertices and $L(G)$ is line graph, then

i) $2 \leq \gamma_f(G) + \gamma_f(L(G)) \leq \frac{2n}{3}$

ii) $1 \leq \gamma_f(G) * \gamma_f(L(G)) \leq n^2/9$

iii) $\Gamma_f(G) + \Gamma_f(L(G)) = n$

iv) $\Gamma_f(G) * \Gamma_f(L(G)) = n^2/4$

Proof: i) and ii) with the reference of previous chapter for cycle graph we have fractional domination number is $\gamma_f(C_n) = n/3, where\ n = 3,4,5, \dots n$. The line graph of cycle graph is $L(G)$ isomorphic to its cycle graph G with same vertex numbers and are adjacent in the provided graph if the corresponding edges share a common vertex. Therefore, for line graph $L(G)$ the fractional dominating number is $\gamma_f(L(C_n)) = n/3$. For the sum $\gamma_f(G) + \gamma_f(L(G))$ lower bound is 2 and upper bound is $\frac{2n}{3}$. Therefore we have $2 \leq \gamma_f(G) + \gamma_f(L(G)) \leq \frac{2n}{3}$ and $1 \leq \gamma_f(G) * \gamma_f(L(G)) \leq n^2/9,\ n = 3,4,5, \dots n$.

iii) and iv) The upper fractional domination number of cycle graph C_n is $\Gamma_f(C_n) = n/2$ is observed by assigning weight (½) to every vertex and the condition where vertex $w \in N[v]$ such that $\sum_{v \in N[w]} f(v) = 1$ condition is satisfied. Therefore we found that $\Gamma_f(G) + \Gamma_f(L(G)) = n$ and $\Gamma_f(G) * \Gamma_f(L(G)) = n^2/4$.
□

Theorem 2.1.2 *Let graph G is r-regular of order n then its upper fractional domination number is* $\Gamma_f(G) \leq n/r$

*Proof***:** Theorem 1.4. gives if graph $G\ be\ r$-regular of order n then $\gamma_f(G) = \frac{n}{r+1}$. The fractional dominating function f that allocates the weight $1/(r+1)$ to each vertex of G so that function is the minimal weight fractional dominating function, and its generalized form is $\gamma_f(G) = \frac{n}{r+1}$. So An identical argument works for given upper fractional domination number where function f allocating weights $(1/r)$ to each vertex of G so that $\Gamma_f(G)$ is maximum cardinality of function f where f is minimal fractional dominating function of G. Its generalized form is given by $\Gamma_f(G) \leq n/r$. Let $v \in V(G)$ such that $f(N[V]) = \sum_{v \in N[v]} f(v) \geq 1$. $\Gamma_f(G) =$ $max\ \{|f|: f\ is\ minimal\ fractional\ dominating\ function\ of\ G\}$.

Where $|f| = \sum_{v \in V} f(v)$. So if graph $G\ is\ r$-regular of order n then its upper fractional domination number is $\Gamma_f(G) \leq n/r$. □

2.2 Complete graph (K_n)

A simple connected graph G is considered to complete if every pair of vertices are adjacent. A complete graph G is simple undirected graph in which pair of different vertices is connected by a unique edge. In the line graph of complete graph number of vertices in $L(G)$ are equal in number of edges in graph G.

Theorem 2.2.1. Let graph G is a complete on n vertices and $L(G)$ is its line graph on $\frac{n(n-1)}{2}$ vertices then

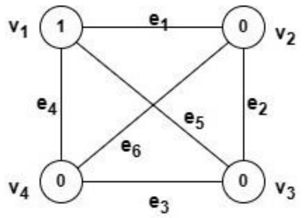
i) $\gamma_f\,(G) + \gamma_f\,(L(G)) = 1\; + \dfrac{n(n-1)/2}{(2n-4)+1}$

ii) $\gamma_f\,(G) * \gamma_f\,(L(G)) = \dfrac{\frac{n(n-1)}{2}}{(2n-4)+1}$

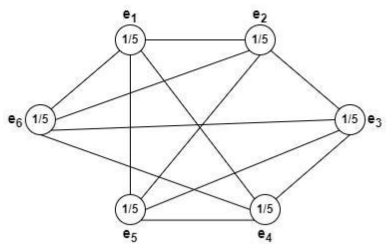
iii) $\Gamma_f\,(G) + \Gamma_f\,(L(G)) = 1 + \dfrac{n(n-1)/2}{2n-4}$

iv) $\Gamma_f\,(G) * \Gamma_f\,(L(G)) = \frac{n(n-1)/2}{2n-4}$

*Proof***:** i) and ii) If graph G is the complete then all pair of other vertices are connected by a unique edge and the fractional dominating number $\gamma_f\,(K_n\,) = 1$. Any set containing a single vertex is a dominating set in K_n. So $\gamma(G) = 1$. For any graph G we have $\gamma_f\,(G) \leq \gamma(G)$. The graph G is a complete on n vertices as $n-1$ regular graph we have $\Gamma_f\,(K_n) \leq \frac{n}{n-1}$. The line graph $L(G)$ is having its vertices are nothing but edges of G with two vertices of $L(G)$ are adjacent if common vertex shared by corresponding edges in G. In the line graph of complete graph number of vertices in $L(G)$ are equal in number of edges in graph G. (See the Figure 2.2.1A and 2.2.1B) where K_4 is complete graph with 4 vertices and 6 edges so in its line graph of K_4 we have 6 vertices with each vertex of degree 4. So $L(K_4)$ is 4-regular graph. In line graph of Complete Graph K_4 we have every two nonadjacent vertices are mutually adjacent to exactly four vertices.



(A) *Complete graph K_4*



(B) *Line graph of Complete graph K_4*

Figure 2.2.1

Similarly, K_5 is complete with 5 vertices and 10 edges so in the line graph of K_5 we get 10 vertices with each vertex of degree 6. For the generalized result we have the sequence of complete graph K_n with set of vertices as $\{1,2,3,4,5,6,7, \ldots n\}$. So for complete graph on set of vertices $n = \{2,3,4,5, \ldots, n\}$ the sequence formed for graphs with number of edges $\left\{1,3,6,10,15,21,28, \ldots \frac{n(n-1)}{2}\right\}$, is triangular number sequence. For G is complete graph *with n vertices as* $n-1$ regular connected graph where $n = \{2,3,4,5,6, \ldots\}$, the line graph of complete graph $L(G)$ *on* $\frac{n(n-1)}{2}$ vertices are $(2n-4)$ regular connected graph. Line graph having each vertex with degree sequence is $\{0,2,4,6,8,10, \ldots 2n-4\}$. So we have $\gamma_f\,(L(G)) = \frac{\frac{n(n-1)}{2}}{(2n-4)+1}$ hence for the sum $\gamma_f\,(G) + \gamma_f\,(L(G)) = 1\ + \frac{\frac{n(n-1)}{2}}{(2n-4)+1}$ and $\gamma_f\,(G) * \gamma_f\,(L(G)) = \frac{\frac{n(n-1)}{2}}{(2n-4)+1}$.

iii) and iv) The condition satisfies for the complete graph on n vertices as $(n-1)$ regulargraph where each vertex $w \in N[V]$ *such that* $\sum_{v\in N[w]} f(v) = 1$ and it is known that $\Gamma_f\,(K_n) = 1$. So if graph G is complete and $L(G)$ is its line graph then upper fractional dominating number of line graph is $\Gamma_f\,(L(G)) = \frac{n(n-1)/2}{2n-4}$ and for the sum we get $\Gamma_f\,(G) + \Gamma_f\,(L(G)) = 1 + \frac{n(n-1)/2}{2n-4}$ and $\Gamma_f\,(G) * \Gamma_f\,(L(G)) = \frac{n(n-1)/2}{(2n-4)}$. □

Number of Vertices	Number of Edges	Graph with each vertex of Degree		Line Graph with each vertex of Degree	
2	1	1	$\gamma_f\,(G) = 1$	0	$\gamma_f\,(L(G)) = 1$
3	3	2	$\gamma_f\,(G) = 1$	2	$\gamma_f\,(L(G)) = 1$
4	6	3	$\gamma_f\,(G) = 1$	4	$\gamma_f\,(L(G)) = 1.2$
5	10	4	$\gamma_f\,(G) = 1$	6	$\gamma_f\,(L(G)) = 1.42$
6	15	5	$\gamma_f\,(G) = 1$	8	$\gamma_f\,(L(G)) = 1.66$
…	…	…	…	…	…
n	$\frac{n(n-1)}{2}$	$n-1$	$\gamma_f\,(G) = 1$	$(2n-4)$	$\frac{\frac{n(n-1)}{2}}{(2n-4)+1}$

Table 2.2.1

2.3 Star graph (S_n)

Star graph S_n is noticed as complete bipartite graph like $(K_{1,n})$, An interior node serves as the central vertex of a tree containg n leaves.

Theorem 2.3.1 Let G is the star graph of order n and $L(G)$ is its line graph then

i) $\gamma_f\,(G) + \gamma_f\,(L(G)) = 2$

ii) $\gamma_f\,(G) * \gamma_f\,(L(G)) = 1$

iii) $\Gamma_f\,(G) + \Gamma_f\,(L(G)) \le n + \frac{n-1}{n-2}$

iv) $\Gamma_f\,(G) * \Gamma_f\,(L(G)) \le \frac{n^2-n}{n-2}$

Proof: i) and ii) The term star graph refers to be complete bipartite graph in which each vertex in the graph belongs to one set and other vertices to the other set. One central vertex is adjacent to every other vertex. So, if G be the Star Graph $(K_{1,5})$ then $\gamma_f\,(G) = 1$ see the Figure 2.3.1A). The Line graph of $(K_{1,5})$ is complete graph see the Figure 2.3.1B) and hence the fractional domination number is one i.e. $\gamma_f\,(G) = 1$. Therefore, star graph S_n is observed as complete bipartite graph like $(K_{1,n})$ where $n = \{3,4,5,\dots\}$ and line graphs are all complete graph with n vertices where $n = \{3,4,5,6,\dots\}$ so the general result for the sum $\gamma_f(G) + \gamma_f(L(G)) = 2$ and $\gamma_f(G) * \gamma_f(L(G)) = 1$.

(A) *Star Graph* $K_{1,5}$ (B) *Line graph of Star Graph* $K_{1,5}$

Figure 2.3.1

iii) and iv) As per observations the upper fractional dominating number of Star Graph $(K_{1,5})$ is 5 with considered closed neighborhood of every vertex satisfying the condition where the *vertex* $w \in N(v)$ such that $\sum_{v\in N[w]} f(v) = 1$. By assigning weight zero to the central vertex and weight one to remaining vertices. Upper fractional dominating number of Line graph of $K_{1,5}$ is one. A Star graph Sn is the complete bipartite graph like $(K_{1,n})$ where $n =$ 3,4,5, ... we have complete graph on n vertices where $n = \{3,4,5,6,\dots\}$ so the generalized result *for* G be the star graph Sn on n vertices and $L(G)$ be its line graph on $n-1$ vertices with $n-2$ regular graph then the sum $\Gamma_f(G) + \Gamma_f(L(G)) \le n + \frac{n-1}{n-2}$ and product $\Gamma_f(G) * \Gamma_f(L(G)) \le \frac{n^2-n}{n-2}$.

□

2.4 Bi-Star graph

The Bi-star graph is two copies of the star graph or two copies of $(K_{1,n})$ with a tree's core vertex being an internal node and n leaves.

Theorem 2.4.1. If simple graph G is the Bi-Star of n vertices and graph $L(G)$ be its line graph then

i) $\gamma_f(G) + \gamma_f(L(G)) = 3$

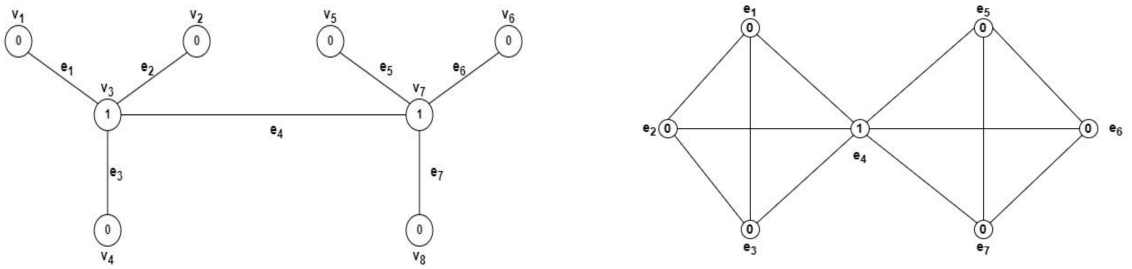
ii) $\gamma_f(G) * \gamma_f(L(G)) = 2$

iii) $\Gamma_f(G) + \Gamma_f(L(G)) = 2n + 2$

iv) $\Gamma_f(G) * \Gamma_f(L(G)) = 4n$

Proof i) and ii): ([11]) A bi-star graph is created by combining the apex vertices with two copies of a star graph. The set of vertices of $B_{n,n}$, $V(B_{n,n}) = \{u, v, ui, vi \mid 1 \le i \le n\}$, where u, v are center vertices and ui, vi are all pendent vertices. The set of edges of $B_{n,n}$ is $E(B_{n,n}) = \{uv, uui, vvi \mid 1 \le i \le n\}$. So, $|V(B_{n,n})| = 2n + 2$ and $|E(B_{n,n})| = 2n + 1$. The fractional dominating number of *Bi-star* graph will be $\gamma_f(G) = 2$ by assigning weight one to central vertices and weight zero to remaining vertices (see the Figure 2.4.1A). The given line graph of *Bi-star* $(K_{1,3})$ is connected graph with two copies of complete graph connected by one point so this graph is one point connectivity graph (see the Figure 2.4.1B). A vertex connected graph is one with a minimal number of vertices whose removal which results in a disconnected graph. We get line graph of *Bi-star* graph is one connected graph therefore $\gamma_f\,(L(G)) = 1$.

In general for *Bi-star* graph or two copies of $(K_{1,n})$ where $n = \{3,4,5,6,7,...\}$ we get $\gamma_f\left(K_{1,n}\right) = 2$ and $\gamma_f\left(L(K_{1,n})\right) = 1$. For the sum we have $\gamma_f\,(G) + \gamma_f\,(L(G)) = 3$ and $\gamma_f\,(G) * \gamma_f\,(L(G)) = 2$.



(A) *Bi-Star graph* $K(1,3)$ (B) *Line graph of Bi-Star graph* $K(1,3)$

Figure 2.4.1

iii) and iv) The upper fractional dominating number of $(K_{1,3})$ *is* $\Gamma_f\left((K_{1,3})\right) = 6$ with consideration of each vertex's closed neighborhood that satisfies the requirement where the *vertex* $w \in N[V]$ such that $\sum_{v\in N[w]} f(v) = 1$ by assigning weight zero to the central vertices and weight one to remaining vertices. The upper fractional dominating number of *Line graph of* $(K_{1,3})$ *is* $\Gamma_f\left(L(K_{1,3})\right) = 2$ by assigning weight one to vertex e_1 *and* e_5 for remaining vertices weight zero so that adjacency relation will satisfied. In general for Bi-star graph or two copies of $(K_{1,n})$ where $n = \{3,4,5,6,7\ ...\}$ we get result for G *be the Bi-star graph* $(K_{1,n})$ *on* n vertices its $\Gamma_f\left((K_{1,n})\right) = 2n$ and line graph of Bi-star graph $(K_{1,n})$ *on* $n-1$ vertices its $\Gamma_f\left(L(K_{1,n})\right) = 2$ then for sum $\Gamma_f\,(G) + \Gamma_f\,(L(G)) = 2n + 2$ and $\Gamma_f\,(G) * \Gamma_f\,(L(G)) = 4n$. □

2.5 Wheel Graph (W_n)

To create a wheel graph a cycle graph C_{n-1} ,is used, by adding one vertex or $W_n = C_{n-1} + K_1$. Every vertex is linked to every other vertex, forming a hub of C_{n-1}. The line graph of wheel graph $L(W_n)$ contains a complete graph K_{n-1} with cycle of $n-1$ vertices where each vertex of the cycle is connected to two vertices of K_{n-1} .

Theorem 2.5.1 *If G is wheel graph on n vertices with size m and $L(G)$ be its line graph of m vertices then $i)$ $\gamma_f\,(G) + \gamma_f\,(L(G)) = 1 + \frac{m}{\delta(G)+1}$, Where $\delta\,(G)$ is minimum degree of $L\,(G)$ and $ii)$ $\Gamma_f\,(G) + \Gamma_f\,(L(G)) \leq 2 + \frac{n-1}{2}$*

Proof i): [11] A wheel graph of n vertices contains a cycle graph of length $n-1$ and all the vertices of the cycle are connected to a single central vertex. Thus, wheel graph on n vertices denoted by W_n for $n > 3$ is acquired by connecting a vertex to every vertex of cycle graph. The fractional dominating

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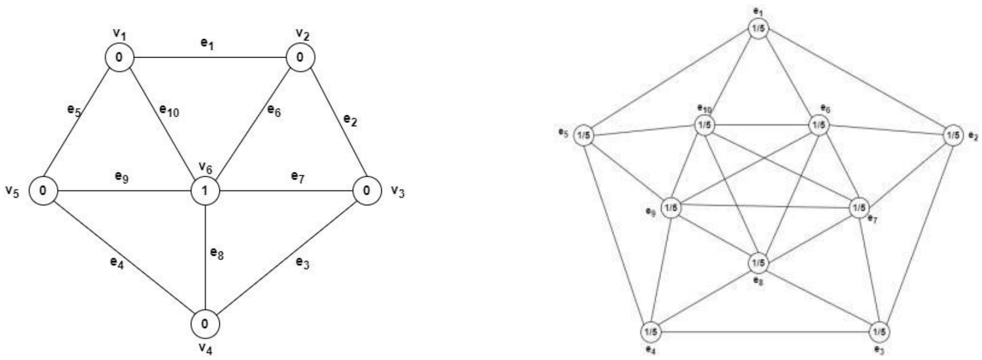
number of wheel graph will be $\gamma_f\ (W_n) = 1$. Wheel graph of n vertices contains m edges so the line graph of wheel graph $L\ (W_n)$ is a graph in which vertices corresponds to the edges of the original wheel graph so it contains m vertices with a complete graph K_{n-1} and the cycle of $n-1$ vertices where every vertex of the cycle is adjacent to other two vertices of K_{n-1}. The max degree of line graph $\Delta\ (G) = n$ and min degree of line graph $\delta\ (G) = 4$. For example if wheel graph $G = W_4$ with number of vertices $n = 4$ and number of edges $m = 6$ then line graph $L(G)$ contains $m = 6$ vertices with a complete graph $K_{4\text{-}1}$ and the cycle of $n - 1 = 3$ vertices where every vertex of the cycle is adjacent to two vertices of $K_{4\text{-}1}$. We know that $\gamma_f\ (W_4) = 1$ and the line graph $L(G)$ where $G = W_4$ is of order $m = 6$ and $\delta(G) = 4$ *we get* $\gamma_f\ (L(W_4)) = \frac{m}{\delta(G)+1} \leq 6/5$ by assigning weight (1/5) for each vertex of line graph. If wheel graph $G = W_5$ is of order $n = 5$ and size $m = 8$ then line graph $L(G)$ contains $m = 8$ vertices with a complete graph $K_{5\text{-}1}$ and cycle of $n - 1 = 4$ vertices where each vertex of cycle is connected to two vertices of $K_{5\text{-}1}$. We found that if $\gamma_f\ (W_5) = 1$ and its line graph of order $m = 8$ and $\delta(G) = 4$ then $\gamma_f\ (L(W_5)) = \frac{m}{\delta(G)+1} \leq 8/5$ by assigning weight (1/5) for each vertex of line graph. If wheel graph $G = W_6$ is of order $n = 6$ and size $m = 10$ (See the Figure 2.5.1A) then line graph $L(G)$ contains $m = 10$ vertices with complete graph $K_{6\text{-}1}$ and the cycle of $n - 1 = 5$ vertices where each vertex of cycle is connected to two vertices of $K_{6\text{-}1}$ (See the Figure 2.5.1B). We found if $\gamma_f\ (W_6) = 1$ its line graph of order $m = 10$ and $\delta(G) = 4$ then *we get* $\gamma_f\ (L(W_6)) \leq \frac{m}{\delta(G)+1} = 2$ Hence, we get the generalized result for the sum $\gamma_f\ (G) + \gamma_f\ (L(G)) \leq 1 + \frac{m}{\delta(G)+1}$.



(A) *Wheel graph* W_6 (B) *Line graph of Wheel graph* W_6

Figure 2.5.1

ii) Upper fractional dominating number of wheel Graph W_n *is* $\Gamma_f\ (W_n) = 2$ with consideration of the closed neighborhood of each vertex satisfying the condition where the vertex $w \in N[V]$ such that $\sum_{v \in N[w]} f(v) = 1$. For upper fractional dominating number of line graph of wheel Graph W_n *is* $\Gamma_f(L(W_n)) \leq \frac{n-1}{2}$ as we have $W_n = C_{n-1} + K_1$ and the line graph of wheel graph $L(W_n)$ contains a complete graph K_{n-1} with the cycle of $n-1$ vertices where each vertex of the cycle is adjacent to two vertices of K_{n-1} and each vertex in cycle having degree four. If we assign the weights (½) for each vertex of complete graph K_{n-1} and each vertex in cycle having weight zero will get the result $\Gamma_f\ (L(W_n)) \leq \frac{n-1}{2}$ and hence result for the sum $\Gamma_f\ (G) + \Gamma_f\ (L(G)) \leq 2 + \frac{n-1}{2}$.

2.6 Cubic graph

The regular graph with degree three is known as cubic graph. Trivalent graphs are another name for cubic graphs. The Cubic graphs with n vertices exist only for even n whereas the condition for a graph is to be cubic is given by $m/n = 3/2$, with m as size of graph and n be the order of graph.

Theorem 2.6.1 Let graph G is the connected cubic graph or trivalent graph with n vertices and m edges and $L(G)$ be its line graph with order m then

i) $\gamma_f(G) + \gamma_f(L(G)) = \frac{5n+4m}{20}$

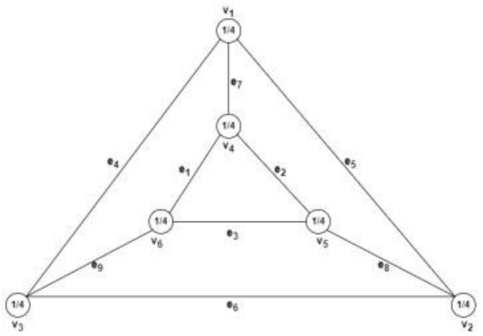
ii) $\gamma_f(G) * \gamma_f(L(G)) = \frac{mn}{20}$

iii) $\Gamma_f(G) + \Gamma_f(L(G)) \leq \frac{4n+3m}{12}$

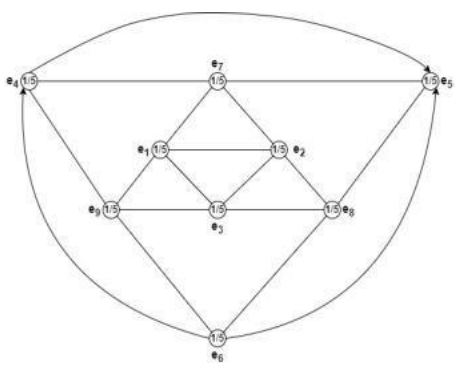
iv) $\Gamma_f(G) * \Gamma_f(L(G)) \leq \frac{mn}{12}$

Proof i) and ii): A graph with degree three at every vertex is called a cubic graph or 3-regular graph. Cubic graph are also called trivalent graph. Cubic graphs on n vertices exist only for even n. For example if G be the cubic graph with $n = 6$ vertices or 3-regular graph with size $m = 9$ then $\gamma_f(G) = \frac{n}{3+1} = 6/4$ by the definition of fractional domination number

(see the Figure 2.6.1A). Its line graph $L(G)$ has $m = 9$ vertices as 4-regular graph so $\gamma_f(L(G)) = \frac{m}{4+1} = 9/5$ (see the Figure 2.6.1B). The cubic graph with $n = 4$ called tetrahedral graph, The cubic graph with $n = 8$ called cubical graph, for all such connected cubic graphs which are 3-regular graphs of order $n = \{4,6,8,...\}$ we get the generalized result as $\gamma_f(G) = \frac{n}{r+1}$ where $r = 3$. For its line graph $L(G)$ of order m with $(r+1)$ regular graph where $r = 3$, we have $\gamma_f(L(G)) = \frac{m}{r+2}$ so for the sum we get $\gamma_f(G) + \gamma_f(L(G)) = \frac{n}{r+1} + \frac{m}{r+2}$ if $r = 3$ then $\gamma_f(G) + \gamma_f(L(G)) = \frac{n}{4} + \frac{m}{5} = \frac{5n+4m}{20}$ and $\gamma_f(G) * \gamma_f(L(G)) = \frac{n}{4} * \frac{m}{5} = \frac{mn}{20}$.



(A) *Cubic graph of 6 vertices*



(B) *Line graph of 3-prism graph*

Figure 2.6.1

iii) and *iv)* For upper fractional domination number we used result 2.1.2 Let G *is* r-regular graph of n vertices and m edges then $\Gamma_f\,(G) \le n/r$. So The generalized result for connected cubic graphs with degree three at every vertex of order $n = \{4,6,8,\ldots\}$ and size m for $r = 3$, we get $\Gamma_f\,(G) \le \frac{n}{3}$ and $L(G)$ is $(r+1)$ regular graph so $\Gamma_f\,(L(G)) \le \frac{m}{r+1}$ we get $\Gamma_f\,(L(G)) \le \frac{m}{4}$. So sum $\Gamma_f\,(G) + \Gamma_f\,(L(G)) \le \frac{n}{3} + \frac{m}{4} = \frac{4n+3m}{12}$ and $\Gamma_f\,(G) * \Gamma_f\,(L(G)) \le \frac{n}{3} * \frac{m}{4} = \frac{mn}{12}$. □

2.7 Cartesian product of graph $(K_2 \times P_n)$

If we take Path graph P_n and Complete graph K_2 then Cartesian product graph by $K_2 \times P_n$. For the Complete graph like K_2 we have the set of vertices $V(K_2) = \{k_1, k_2\}$ and the set of vertices of path graph P_n is $\{p_1, p_{2,\ldots,} p_n\}$ respectively.

The set of verticesvfor Cartesian product of $K_2 \times P_n$ is $V\,(K_2 \times P_n) = V\,(K_2) \times V(\,P_n)$ and $V(K_2) \times V(\,P_n) = \{(k_i, p_j) \mid k_i \in V(K_2), p_j \in V(\,P_n)\}$ and 'e' be the edge for Cartesian product $K_2 \times P_n$ iff $e = <(k_i, p_j), (k_r, p_s)>$ where the condition holds as

$i)\; i = r \text{ and } p_j\, p_s \in \text{Edge set of } (\,P_n),$

$ii)\; j = s \text{ and } p_j\, p_s \in \text{Edge set of } (K_2).$

We need the following theorems on $\gamma_f\,(K_2 \times P_n)$.

Theorem 2.7.1 ([10,11]) *If* G *is the Cartesian product graph* $(K_2 \times P_n)$ *for* $n > 1$ *then*

$$\gamma_f\,(K_2 \times P_n) = \begin{cases} \frac{n+1}{2}, & if\ n\ is\ odd \\ \frac{n^2+2n}{2(n+1)}, & if\ n\ is\ even \end{cases}$$

Theorem 2.7.2 *If* G *is the Cartesian product* $(K_2 \times P_n)$ *of order* $2n$ *for* $n > 1$ *and* $L(G)$ *be its line graph of order* $3n-2$ *then* $\gamma_f\left(L(K_2 \times P_n)\right) = \frac{2n}{3}$.

*Proof***:** In Cartesian product graph $(K_2 \times P_n)$ for $n > 1$, we noted that K_2 and P_n both are one distance graph and the Cartesian product of the two unit distance graphs is also unit distance graph. Let $G_1 = (K_2 \times P_2)$ is the cartesian product of K_2 *and* P_2 we can see that it is cycle graph of order 4 so its fractional domination number is $\gamma_f\,(K_2 \times P_2) = \frac{(2)^2+2(2)}{2(2+1)} = 4/3$ and its line graph $L(G_1)$ is isomorphic to original graph $G_1 = (K_2 \times P_2)$ of order $3(2)-2 = 4$ so $\gamma_f\left(L(K_2 \times P_2)\right) = \frac{2(2)}{3} = 4/3$. Let $G_2 = (K_2 \times P_3)$ be the cartesian product of K_2 *and* P_3 we can see that $\gamma_f\,(K_2 \times P_3) = \frac{3+1}{2} = 2$. Its line graph $L(G_2)$ is of order $3(3)-2 = 7$ so $\gamma_f\left(L(K_2 \times P_3)\right) = \frac{2(3)}{3} = 2$ by assigning each vertex with weight (1/3) except central vertex. For central (hub) vertex we can assign weight zero so that adjacency relation will satisfied.

Similarly if $G_3 = K_2 \times P_4$ be the Cartesian product of K_2 *and* P_4 is graph of 8 vertices with $\{v2, v3, v6, v7\}$ having weights (1/5) and $\{v1, v4, v5, v8\}$ having weights (2/5) so $\gamma_f\,(K_2 \times P_4) =$

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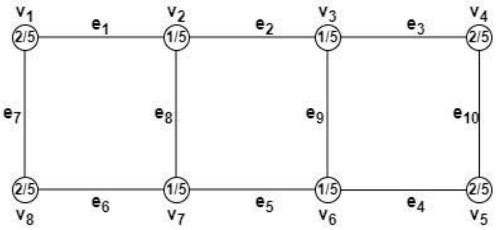
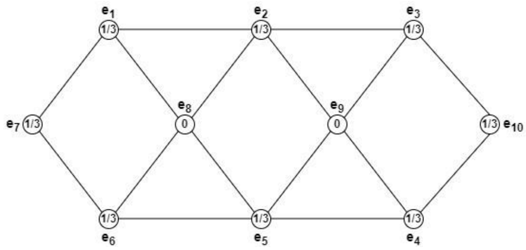
Vol 31 No. 6s (2024)

$\frac{4^2+2(4)}{2(4+1)} = \frac{12}{5} = 2.4$ (See the Figure 2.7.2A).



(A) *Cartesian product of* $(K_2 \times P_4)$



(B) *Its Line graph* $(K_2 \times P_4)$

Figure 2.7.2

Its line graph $L(G_3)$ is of order $3(4) - 2 = 10$ so $\gamma_f\left(L(K_2 \times P_4)\right) = \frac{2(4)}{3} = 8/3$ with vertices $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_{10},\}$ having weights (1/3) and vertices $\{e_8, e_9\}$ having weights zero. (See the Figure 2.7.2B). These are minimal fractional domination numbers by considering the closed neighborhood of each vertex satisfying the condition where the vertex $w \in N[V]$ such that $\sum_{v\in N[w]} f(v) = 1$. Hence in general form by theorem 2.7.1 we get $\gamma_f (K_2 \times P_n) = \frac{n+1}{2}, if\ n\ is\ odd$ and $\gamma_f (K_2 \times P_n) = \frac{n^2+2n}{2(n+1)}, if\ n\ is\ even$. Therefore in its line graph $L(K_2 \times P_n)$ with $3n - 2$ vertices for $n > 1$ we have $\gamma_f\left(L(K_2 \times P_n)\right) = \frac{2n}{3}$. □

2.8 Cartesian product graph ($K_3 \times P_n$)

If we take the path graph P_n and Complete graph K_3 then Cartesian product graph represented by $K_3 \times P_n$. For the three vertex Complete graph K_3, The vertex set $V(K_3) = \{k_1, k_2, k_3\}$ and the vertex set for path graph P_n is $\{p_1, p_2, ... p_n\}$ resp. The vertex set for Cartesian product of $(K_3 \times P_n)$ is $V(K_3 \times P_n) = V(K_3) \times V(P_n)$ and

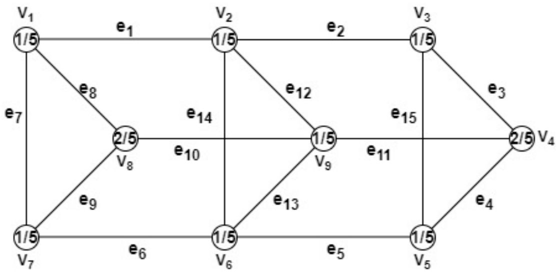
$V(K_3) \times V(P_n) = \{(k_i, p_j) \mid k_i \in V(K_3), p_j \in V(P_n)\}$ and 'e' be the edge for Cartesian product $(K_3 \times P_n)$ iff $e = < (k_i, p_j), (k_r, p_s) >$ where the condition holds as

$i)\ i = r\ and\ p_j\ p_s \in Edge\ set\ of\ (P_n),\ ii)\ j = s\ and\ p_j\ p_s \in Edge\ set\ of\ (K_3).$

Theorem 2.8.1 *Let G is the Cartesian product graph* $(K_3 \times P_n)$ *of order* $3n$ *for* $n \geq 3$ *and* $L(G)$ *be its line graph of order* $(6n - 3)$. *then* $\gamma_f (K_3 \times P_n) = \frac{3n+2}{5}$ and $\gamma_f (L(K_3 \times P_n)) = \frac{3(2n+1)}{7}$.

Proof: For the Complete graph K_3 and path graph P_n, we have graph of Cartesian product denoted by $(K_3 \times P_n)$ of order $3n$ *for* $n \geq 3$. Let $G_1 = (K_3 \times P_3)$ be the Cartesian product of K_3 *and* P_3 we can see that it is graph of order 9 so its fractional domination number is $\gamma_f (K_3 \times P_3) = \frac{3(3)+2}{5} = 11/5$ by assigning weight (1/5) to every vertex of the set $\{v1, v2, v3, v5, v6, v7, v9,\}$ and weight (2/5) to every vertex of the set $\{v4, v8,\}$. For *G is Cartesian product graph* $(K_3 \times P_n)$ *of order* $3n$ *for* $n \geq 3$ then for fractional domination numbers we found sequence of numbers as {11/5, 14/5, 17/5, 20/5, …$\frac{3n+2}{5}$} so that in general $\gamma_f (K_3 \times P_n) = \frac{3n+2}{5}$ see the Figure 2.8.1A).

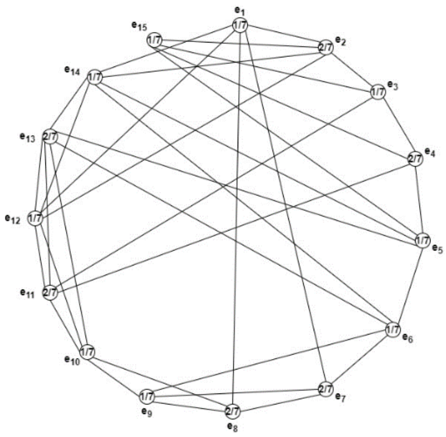
In contradiction if we have $\gamma_f\,(K_3 \times\, P_n\,) = k$ and $k < \frac{3n+2}{5}$ then adjacency relation is not satisfied with $\sum_{v \in N[w]} f(v) = 1$ therefore there does not exist such k where $k < \frac{3n+2}{5}$. Hence with minimal weights we have $\gamma_f\,(K_3 \times\, P_n\,) = \frac{3n+2}{5}$.



(A) *Cartesian product of* ($\boldsymbol{K_3} \times\, \boldsymbol{P_3}$)

Figure 2.8.1

In its line graph $L(G_1) = L(K_3 \times\, P_3)$ of order $6(3) - 3 = 15$ we get $\gamma_f\left(L(K_3 \times\, P_3\,)\right) = \frac{3[2(3)+1]}{7} = \frac{21}{7} = 3$. By assigning weight (1/7) to every vertex of the set $\{e1, e3, e5, e6, e9, e10, , e12, e14, e15\}$ and weight (2/7) to every vertex of the set $\{e2, e4, e7, e8\; e11, e13\}$ see the Figure 2.8.1B).



(B) *Line graph of Cartesian product of* ($\boldsymbol{K_3} \times\, \boldsymbol{P_3}$)

Figure 2.8.1

Let $G_2 = (K_3 \times\, P_4)$ be the Cartesian product of K_3 *and* P_4 we can see that it is graph of order 12 so its fractional domination number is $\gamma_f\,(K_3 \times\, P_4\,) = \frac{3(4)+2}{5} = 14/5$ by assigning weight (1/5) to 10 number of vertices and weight (2/5) to other two vertices. In its line graph $L(G_2) = L(K_3 \times\, P_4)$ of order $[6(4) - 3] = 21$ so $\gamma_f\left(L(K_3 \times\, P_4\,)\right) = \frac{3[2(4)+1]}{7} = \frac{27}{7}$ by assigning weight (1/7) to 15 number of vertices and weight (2/7) to six number of vertices. These are minimal fractional domination numbers by definition satisfying the condition where the vertex $w \in N[V]$ such that $\sum_{v \in N[w]} f(v) = 1$. Hence if graph G *is the Cartesian product* $(K_3\; \times\; P_n\,)$ *of order* $3n$ *for* $n\; \geq 3$ then for the fractional domination numbers we found sequence of numbers as {11/5, 14/5, 17/5, 20/5, …$\frac{3n+2}{5}$} and $L(G)$ *be its line graph of order* $(6n-3)$ *then* fractional domination numbers formed by sequence {21/7, 27/7, 33/7, 39/7, …$\frac{3(2n+1)}{7}$} therefore $\gamma_f\,(K_3 \times P_n) = \frac{3n+2}{5}$ and $\gamma_f\left(L(K_3 \times P_n)\right) = \frac{3(2n+1)}{7}$.

In contradiction if we have $\gamma_f\left(L(K_3 \times P_n)\right) = k$ and $k < \frac{3(2n+1)}{7}$ then adjacency relation is not satisfied with $\sum_{v\in N[w]} f(v) = 1$ therefore there does not exist such k where $k < \frac{3(2n+1)}{7}$. Hence with minimal weights we have $\gamma_f\left(L(K_3 \times P_n)\right) = \frac{3(2n+1)}{7}$. □

2.9 The Cartesian product of two cycle graphs $C_m \times C_n$

Cycle graph C_m and C_n taken to form Cartesian product and is denoted by $(C_m \times C_n)$ whose vertex set is $V\,(C_m) \times V\,(C_n)$ where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are connected if $u_1 = v_1$ in C_m and u_2 is adjacent to v_2 in C_n or u_1 is adjacent to v_1 in C_m and $u_2 = v_2$ in C_n. It is also known as torus grid graph $T_{m,n}$ formed by Cartesian product of two cycle graphs C_m and C_n.

Theorem 2.9.1 *Let cycle graphs C_m and C_n are two graphs with m and n vertices respectively and let $G = C_m \times C_n$ is Cartesian product of two cycle graphs. Let $L(G)$ be its line graph then*

i) $\gamma_f\,(G) + \gamma_f\,(L(G)) = \frac{mn}{5} + \frac{2mn}{7}$

ii) $\gamma_f\,(G) * \gamma_f\,(L(G)) = \frac{mn}{5} * \frac{2mn}{7}$,

iii) $\Gamma_f\,(G) + \Gamma_f\,(L(G)) \le \frac{mn}{4} + \frac{mn}{3}$

iv) $\Gamma_f\,(G) * \Gamma_f\,(L(G)) \le \frac{mn}{4} * \frac{mn}{3}$.

Proof i) and ii): We have taken reference to [7], in graphs of Cartesian product $C_m \times C_n)$ the set of vertices of C_m will be $V = \{v1, v2, v3, \dots, vm\}$ and the set of vertices of C_n will be $U = \{u1, u2, u3, \dots, um\}$ The graph G has order mn and size $2mn$ with each vertex has degree 4 or 4-regular graph. The line graph $L(G)$ has order $2mn$ with each vertex has degree 6 or 6-regular graph. For fractional domination number we used the theorem 1.9 for following cases:

Case 1: When m and n are even and distinct if $m = 4, n = 6$ then the graph $G = (C_m \times C_n)$ has order 24 and size 48 with each vertex has degree 4. Hence we have $\gamma_f\,(C_4 \times C_6) = 24/5$. The line graph $L(G)$ has 48 number of vertices with each vertex has degree 6. Therefore we get $\gamma_f\,(L(G)) = \gamma_f\,(C_4 \times C_6) = 48/7$.

Case 2: *When m is even and n is odd* if $m = 4, n = 3$ then the graph $G = (C4 \times C3)$ for example circulant graph has order 12 and size 24 with every vertex has degree 4. Hence $\gamma_f\,(C_4 \times C_3)$= 12/5. The line graph G has order 24 with each vertex has degree 6. Therefore we get $\gamma_f\,(L(G)) = \gamma_f\,(C_4 \times C_3) = 24/7$.

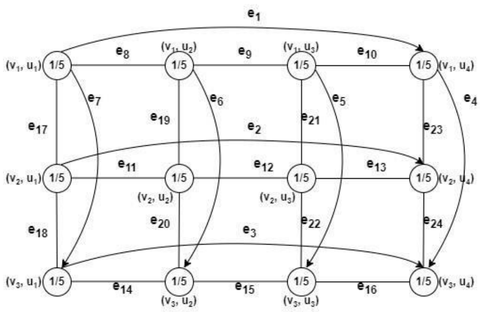
Case 3: *When m and n are odd and equal* if $m = 3, n = 3$ then the graph $G = C_3 \times C_3$ for example generalized quadrangle graph has order 9 and size 18 with each vertex has degree 4. Therefore we get $\gamma_f\,(C_3 \times C_3) = 9/5$. The line graph (G) has order 18 with every vertex has degree

6. Therefore $\gamma_f\,(L(G)) = \gamma_f\,(C_4 \times C_3) = 18/7$.

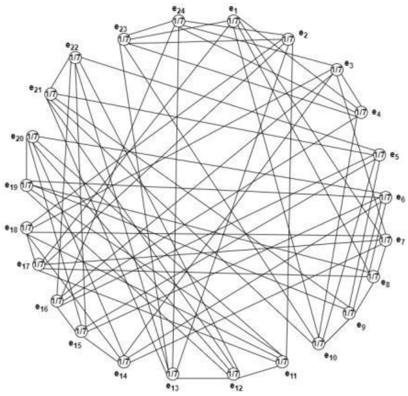
Case 4: *When* m *and* n *are even and equal* if $m = 4$, $n = 4$ then the graph $G = C_4 \times C_4$ for example tesseract graph has order 16 and size 32 with each vertex has degree 4. Hence we get $\gamma_f\,(C_4 \times C_4) = \frac{16}{5}$. The line graph (G) has order 32 with every vertex has degree 6. Therefore $\gamma_f\,(L(G)) = \gamma_f\,(C_4 \times C_4) = 32/7$.

Case 5: When m and n *are odd and* distinct if $m = 3, n = 5$ *then the graph* $G = C_3 \times C_5$ *has* 15 *number* of vertices and size 30 with every vertex has degree 4. Hence $\gamma_f\,(C_3 \times C_5) = \frac{15}{5} = 3$. The line graph (G) has 30 number of vertices with each vertex has degree 6. Therefore $\gamma_f\,(L(G)) = \gamma_f\,(C_3 \times C_5) = 30/7$. Hence in generalized if $G = C_m \times C_n$ *be* 4-regular graph with order mn *and graph* $L(G)$ *be* 6-regular graph of order $2mn$ *then* $\gamma_f\,(G) + \gamma_f\,(L(G)) = \frac{mn}{5} + \frac{2mn}{7}$ *and* $\gamma_f\,(G) * \gamma_f\,(L(G)) = \frac{mn}{5} * \frac{2mn}{7}$.

Example 2.9.1 When $m = 3, n = 4$ the Cartesian product graph $G = C_3 \times C_4$ given in Figure 2.9.1A). So $\gamma_f\,(C_3 \times C_4) = 12/5$.



(A) *Cartesian product graph* $G = C_3 \times C_4$



(A) *Line graph of Cartesian product* $G = C_3 \times C_4$

Figure 2.9.1

The line graph of Cartesian product $G = C_3 \times C_4$ given in Figure 2.9.1B) The line graph $L(G)$ has 24 vertices with each vertex has degree 6. Therefore $\gamma_f\,(L(G)) = \gamma_f\,(C_3 \times C_4) = 24/7$.

iii) and iv) The theorem 2.1.2 give $G = C_m \times C_n$ *be* 4-regular graph of order mn then its upper fractional domination number is $\Gamma_f\,(C_m \times C_n) \leq mn/r$ and $L(G)$ *be* 6-regular graph of order $2mn$ then $\Gamma_f\,(L(C_m \times C_n)) \leq \frac{2mn}{r+2}$ hence $\Gamma_f\,(G) + \Gamma_f\,(L(G)) \leq \frac{mn}{r} + \frac{2mn}{r+2}$ and $\Gamma_f\,(G) * \Gamma_f\,(L(G)) \leq \frac{mn}{4} * \frac{mn}{3}$. □

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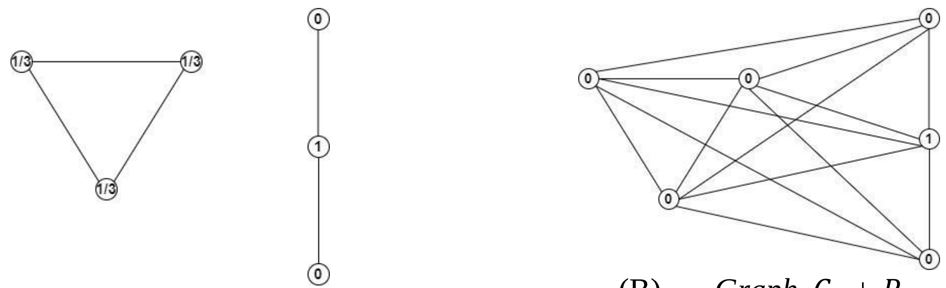
3 Applications

Join and Union

The first graph G and second graph H have two set of vertices as V_1 and V_2 which lack coherence or disjoint whereas the edge sets are X_1 *and* X_2 respectively. The union $(G \cup H)$ represented as graphs such that $V = (V_1 \cup V_2)$ *and* $X = (X_1 \cup X_2)$ If graph G has order n_1 and size m_1, graph H has order n_2 and size m_2 then for union $G \cup H$ we have $n_1 + n_2$ vertices and $m_1 + m_2$ edges. So It is recognized that $\gamma_f(G \cup H) = \gamma_f(G) + \gamma_f(H)$ *and* $\Gamma_f(G \cup H) = \Gamma_f(G) + \Gamma_f(H)$ here we have taken graph $(C_3 \cup P_3)$ so immediately we have $\gamma_f(C_3 \cup P_3) = \gamma_f(C_3) + \gamma_f(P_3)$ *and* $\Gamma_f(C_3 \cup P_3) = \Gamma_f(C_3) + \Gamma_f(P_3)$. (see Figure 3A).

The Join for graph G and H denoted by $(G + H)$ has vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{(gh) | g \in V(G), h \in V(H)\}$. If graph G has order n_1 and size m_1, graph H has order n_2 and m_2 number of edges then for join $G + H$ we have $n_1 + n_2$ vertices and $(m_1 + m_2 + n_1 n_2)$ number of edges. Here we consider the graph $(C_3 + P_3)$ (see Figure 3B).



(A) *Graph* $C_3 \cup P_3$ (B) *Graph* $C_3 + P_3$

Figure 3

We will now state several results involving fractional domination of graphs. Clearly,

$$\gamma_f(G + H) \le \min\{\gamma_f(G), \gamma_f(H)\} \text{ and } \Gamma_f(G + H) = \max\{\Gamma_f(G), \Gamma_f(H)\}.$$

We have the following theorems on $\gamma_f(G + H)$ *and* $\Gamma_f(G + H)$.

Theorem 3.1 ([4]). Given graphs G and H, we have

$$\gamma_f(G + H) = \begin{cases} 1 & \text{if } \gamma_f(G) = 1 \text{ or } \gamma_f(H) = 1, \\ 2 - \dfrac{\gamma_f(G) + \gamma_f(H) - 2}{\gamma_f(G) * \gamma_f(H) - 1} & \text{Otherwise}. \end{cases}$$

And $\Gamma_f(G + H) = 2 - \dfrac{\Gamma_f(G) + \Gamma_f(H) - 2}{\Gamma_f(G) * \Gamma_f(H) - 1}$.

Theorem 3.2 ([4]). For graph G and graph H we have using $1/\infty = 0$,

$$\gamma_f(G + H) = 2 - \begin{cases} 1/\gamma_f(H), & \text{if } \gamma_f(G) = \infty \\ 1/\gamma_f(G), & \text{if } \gamma_f(H) = \infty \\ \dfrac{\Gamma_f(G) + \Gamma_f(H) - 2}{\Gamma_f(G) * \Gamma_f(H) - 1}, & \text{Otherwise} \end{cases}$$

(Figure 3A) as union for 3 vertex cycle and 3 vertex path graph and (Figure 3B) as join for the same. So $\gamma_f(C_3) = 1$ whereas $\Gamma_f(C_3) = 3/2$, $\gamma_f(P_3) = 1$ and $\Gamma_f(P_3) = 3/2$, Same concept for 5 vertex cycle graph and 4 vertex path graph, as per the join and union we get $\gamma_f(C_5) = 5/3$ and $\Gamma_f(C_5) = 5/2$ whereas $\gamma_f(P_4) = 2$ and $\Gamma_f(P_4) = 4/2$. Therefore theorem on union gives $\gamma_f(C_5\ U\ P_4) = 5/3 + 2 = 11/3$ and $\Gamma_f(C_5\ U\ P_4) = 5/2 + 2 = 9/2$. Theorem on join gives $\gamma_f(C_5 + P_4) = 2 - \frac{(5/3)+2-2}{(5/3)*2-1} = 9/7$ and $\Gamma_f(C_5 + P_4) = 2 - \frac{(5/2)+2-2}{(5/2)*2-1} = 11/8$.

Theorem 3.3 *Let G is cycle graph C_n and $L(G)$ is line graph, we have*

$$\gamma_f(G + L(G)) = \begin{cases} 1 & if \ \ \gamma_f(G) = 1\ or\ \ \gamma_f(L(G)) = 1, \\ 2 - \frac{\frac{2n}{3} - 2}{\frac{n^2}{9} - 1} & Otherwise \end{cases}$$

and $\Gamma_f(G + L(G)) = 2 - \frac{n-2}{\frac{n^2}{4} - 1}$.

Proof. Theorem 2.1.1 and Theorem 3.1 gives $\gamma_f(C_n) = n/3$ and $\gamma_f(L(C_n)) = n/3$ so $\gamma_f(G) + \gamma_f(L(G)) = \frac{2n}{3}$ and $\gamma_f(G) * \gamma_f(L(G)) = \frac{n^2}{9}$ hence $\gamma_f(C_n + L(C_n)) = 1\ if\ \ \gamma_f(G) = 1\ or\ \ \gamma_f(L(G)) = 1$ otherwise $\gamma_f(C_n + L(C_n)) = 2 - \frac{\frac{2n}{3} - 2}{\frac{n^2}{9} - 1} = 2 - \frac{6n-18}{n^2-9}$. Similarly $\Gamma_f(C_n) = n/2$ and $\Gamma_f(L(C_n)) = n/2$ so $\Gamma_f(C_n) + \Gamma_f(L(C_n)) = n$ and $\Gamma_f(C_n) * \Gamma_f(L(C_n)) = \frac{n^2}{4}$ hence $\Gamma_f(C_n + L(C_n)) = 2 - \frac{n-2}{\frac{n^2}{4} - 1} = 2 - \frac{4n-8}{n^2-4}$. □

Theorem 3.4 *For the graph G as complete graph K_n and $L(G)$ be its line graph, we have*

$$\gamma_f(G + L(G)) = \begin{cases} 1 & if \ \ \gamma_f(G) = 1\ or\ \ \gamma_f(L(G)) = 1, \\ 2 - \frac{1 + \frac{n(n-1)/2}{(2n-4)+1} - 2}{\frac{n(n-1)/2}{(2n-4)+1} - 1} & Otherwise \end{cases}$$

and $\Gamma_f(G + L(G)) = 2 - \frac{1 + \frac{n(n-1)/2}{2n-4} - 2}{\frac{n(n-1)/2}{2n-4} - 1}$.

Proof: Theorem 2.2.1 and Theorem 3.1 gives $\gamma_f(K_n) = 1$ and $\gamma_f(L(K_n)) = \frac{n(n-1)/2}{(2n-4)+1}$ so $\gamma_f(G) + \gamma_f(L(G)) = 1 + \frac{n(n-1)/2}{(2n-4)+1}$ and $\gamma_f(G) * \gamma_f(L(G)) = \frac{n(n-1)/2}{(2n-4)+1}$.

Hence $\gamma_f(K_n + L(K_n)) = 1,\ if\ \ \gamma_f(G) = 1\ or\ \ \gamma_f(L(G)) = 1$, otherwise

$\gamma_f\left(K_n + L(K_n)\right) = 2 - \frac{1+\frac{n(n-1)/2}{(2n-4)+1}-2}{\frac{n(n-1)/2}{(2n-4)+1}-1}$. Similarly $\Gamma_f(K_n) = 1$ and $\Gamma_f(L(K_n)) = \frac{n(n-1)/2}{2n-4}$ so $\Gamma_f(G) + \Gamma_f(L(G)) = 1 + \frac{n(n-1)/2}{2n-4}$ and $\Gamma_f(G) * \Gamma_f(L(G)) = \frac{n(n-1)/2}{2n-4}$ hence $\Gamma_f\left(K_n + L(K_n)\right) = 2 - \frac{1+\frac{\frac{n(n-1)}{2}}{2n-4}-2}{\frac{\frac{n(n-1)}{2}}{2n-4}-1}$. □

Theorem 3.5 *Let G is the S_n as star graph, $L(G)$ be its line graph, we have*

$$\gamma_f(G + L(G)) = \begin{cases} 1 & if \quad \gamma_f(G) = 1 \; or \; \gamma_f\left(L(G)\right) = 1, \\ 2 & Otherwise \end{cases}$$

and $\Gamma_f(G + L(G)) \leq 2 - \frac{n+\frac{n-1}{n-2}-2}{\frac{n^2-n}{n-2}-1}$.

Proof: Theorem 2.3.1 and Theorem 3.1 gives $\gamma_f(S_n) = 1$ and $\gamma_f(L(S_n)) = 1$ so $\gamma_f(G) + \gamma_f(L(G)) = 2$ and $\gamma_f(G) * \gamma_f(L(G)) = 1$ hence $\gamma_f\left(S_n + L(S_n)\right) = 1$ if $\gamma_f(G) = 1 \; or \; \gamma_f\left(L(G)\right) = 1$ otherwise $\gamma_f\left(S_n + L(S_n)\right) = 2$. Similarly $\Gamma_f(S_n) = n$ and $\Gamma_f\left(L(S_n)\right) = \frac{n-1}{n-2}$ so $\Gamma_f(G) + \Gamma_f\left(L(G)\right) \leq n + \frac{n-1}{n-2}$ and $\Gamma_f(C_n) * \Gamma_f(L(C_n)) \leq \frac{n^2-n}{n-2}$ hence $\Gamma_f\left(S_n + L(S_n)\right) \leq 2 - \frac{n+\frac{n-1}{n-2}-2}{\frac{n^2-n}{n-2}-1}$. □

Theorem 3.6 *Let G is the graph $(K_{1,n})$ a as bistar graph and $L(G)$ be its line graph, we have*
$\gamma_f(G + L(G)) = 1$ if $\gamma_f(G) = 1 \; or \; \gamma_f\left(L(G)\right) = 1$

and $\Gamma_f(G + L(G)) = 2 - \frac{2n}{4n-1}$.

Proof: Theorem 2.4.1 and Theorem 3.1 gives $\gamma_f(G) = 2$ and $\gamma_f(L(G)) = 1$ so $\gamma_f(G) + \gamma_f(L(G)) = 3$ and $\gamma_f(G) * \gamma_f(L(G)) = 2$ hence $\gamma_f\left(G + L(G)\right) = 1$ if $\gamma_f(G) = 1 \; or \; \gamma_f\left(L(G)\right) = 1$. Similarly $\Gamma_f(G) = 2n$ and $\Gamma_f\left(L(G)\right) = 2$ so $\Gamma_f(G) + \Gamma_f\left(L(G)\right) = 2n + 2$and $\Gamma_f(G) * \Gamma_f\left(L(G)\right) = 4n$, $\Gamma_f\left(G + L(G)\right) \leq 2 - \frac{2n}{4n-1}$. □

Theorem 3.7 *For the graph G be the wheel graph on n vertices with size m and $L(G)$ be its line graph, where $\delta(G)$ is minimum degree of $L(G)$ we have*

$$\gamma_f(G + L(G)) = \begin{cases} 1 & if \quad \gamma_f(G) = 1 \; or \; \gamma_f\left(L(G)\right) = 1, \\ 2 - \frac{1+\frac{m}{\delta(G)+1}-2}{\frac{m}{\delta(G)+1}-1} & Otherwise \end{cases}$$

and $\Gamma_f(G + L(G)) \leq 2 - \frac{2+\frac{n-1}{2}-2}{n-2}$.

Proof: Theorem 2.5.1 and Theorem 3.1 give $\gamma_f(W_n) = 1$ and $\gamma_f(L(W_n)) = \frac{m}{\delta(G)+1}$ so $\gamma_f(G) + \gamma_f(L(G)) = 1 + \frac{m}{\delta(G)+1}$ and $\gamma_f(G) * \gamma_f(L(G)) = \frac{m}{\delta(G)+1}$ hence $\gamma_f(W_n + L(W_n)) = 1$ if $\gamma_f(G) = 1$ *or* $\gamma_f(L(G)) = 1$,

otherwise $\gamma_f(W_n + L(W_n)) = 2 - \frac{1 + \frac{m}{\delta(G)+1} - 2}{\frac{m}{\delta(G)+1} - 1}$ Similarly $\Gamma_f(W_n) = 2$ and $\Gamma_f(L(W_n)) = n - 1/2$ so $\Gamma_f(G) + \Gamma_f(L(G)) \leq 2 + \frac{n-1}{2}$ and $\Gamma_f(G) * \Gamma_f(L(G)) \leq n - 1$ hence $\Gamma_f(G + L(G)) \leq 2 - \frac{2 + \frac{n-1}{2} - 2}{n-2}$. □

Theorem 3.8 *If graph G be the connected cubic graph or trivalent graph with order n, size m and $L(G)$ be its line graph of order m we have*

$$\gamma_f(G + L(G)) = \begin{cases} 1 & if \ \gamma_f(G) = 1 \ or \ \gamma_f(L(G)) = 1, \\ 2 - \frac{\frac{5n+4m}{20} - 2}{\frac{mn}{20} - 1} & Otherwise \end{cases}$$

and $\Gamma_f(G + L(G)) \leq 2 - \frac{\frac{4n+3m}{12} - 2}{\frac{mn}{12} - 1}$.

Proof: Theorem 2.6.1 and Theorem 3.1 gives $\gamma_f(G) = n/4$ and $\gamma_f(L(G)) = m/5$ so $\gamma_f(G) + \gamma_f(L(G)) = \frac{n}{4} + \frac{m}{5} = \frac{5n+4m}{20}$ and $\gamma_f(G) * \gamma_f(L(G)) = \frac{n}{4} * \frac{m}{5} = mn/20$ hence $\gamma_f(G + L(G)) = 1$ if $\gamma_f(G) = 1$ *or* $\gamma_f(L(G)) = 1$ otherwise $\gamma_f(G + L(G)) = 2 - \frac{\frac{5n+4m}{20} - 2}{\frac{mn}{20} - 1}$. Similarly $\Gamma_f(G) = n/3$ and $\Gamma_f(L(G)) = m/4$. So $\Gamma_f(G) + \Gamma_f(L(G)) \leq \frac{n}{3} + \frac{m}{4} = \frac{4n+3m}{12}$ and $\Gamma_f(G) * \Gamma_f(L(G)) \leq mn/12$ hence $\Gamma_f(G + L(G)) \leq 2 - \frac{\frac{4n+3m}{12} - 2}{\frac{mn}{12} - 1}$. □

Theorem 3.9 *Let G is the Cartesian product graph $(K_2 \times P_n)$ for $n > 1$ and $L(G)$ be line graph of order $3n - 2$* we have

$$\gamma_f(G + L(G)) = \begin{cases} 2 - \frac{\frac{n+1}{2} + \frac{2n}{3} - 2}{\frac{n+1}{2} * \frac{2n}{3} - 1} & if \ n \ is \ odd \\ 2 - \frac{\frac{n^2+2n}{2(n+1)} + \frac{2n}{3} - 2}{\frac{n^2+2n}{2(n+1)} * \frac{2n}{3} - 1} & if \ n \ is \ even. \end{cases}$$

Proof: Theorem 2.7.1 and Theorem 3.1 gives $\gamma_f(K_2 \times P_n) = \frac{n+1}{2}, if \ n \ is \ odd$ and $\gamma_f(K_2 \times P_n) = \frac{n^2+2n}{2(n+1)}, if \ n \ is \ even$. The theorem 2.7.2 gives $\gamma_f(L(K_2 \times P_n)) = \frac{2n}{3}$. so $\gamma_f(G) + \gamma_f(L(G)) = \frac{n+1}{2} + \frac{2n}{3}$ if n is odd and $\gamma_f(G) + \gamma_f(L(G)) = \frac{n^2+2n}{2(n+1)} + \frac{2n}{3}$ *if n is even*. Hence

$\gamma_f\left(G+L(G)\right)=2-\frac{\frac{n+1}{2}+\frac{2n}{3}-2}{\frac{n+1}{2}\frac{2n}{3}-1}$ $if\ n\ is\ odd$ and $\gamma_f\left(G+L(G)\right)=2-\frac{\frac{n^2+2n}{2(n+1)}+\frac{2n}{3}-2}{\frac{n^2+2n}{2(n+1)}*\frac{2n}{3}-1}$ $if\ n\ is\ even$. □

Theorem 3.10 *Let G is the Cartesian product graph* $(K_3 \times P_n)$ *of order* $3n$ *for* $n \geq 3$ *and* $L(G)$ *is its line graph of order* $(6n-3)$ *then*

$$\gamma_f\left(G+L(G)\right)=\begin{cases}1 & if \quad \gamma_f(G)=1\ or\ \ \gamma_f\left(L(G)\right)=1,\\ 2-\dfrac{\frac{3n+2}{5}+\frac{3(2n+1)}{7}-2}{\frac{3n+2}{5}*\frac{3(2n+1)}{7}-1} & Otherwise\end{cases}$$

Proof: Theorem 2.8.1 and Theorem 3.1 gives $\gamma_f(K_3 \times P_n)=\frac{3n+2}{5}$ and $\gamma_f\left(L(K_2\times P_n)\right)=\frac{3(2n+1)}{7}$ so $\gamma_f(G)+\gamma_f(L(G))=\frac{3n+2}{5}+\frac{3(2n+1)}{7}$ and $\gamma_f(G)*\gamma_f(L(G))=\frac{3n+2}{5}*\frac{3(2n+1)}{7}$ hence $\gamma_f\left(G+L(G)\right)=1\ if\ \ \gamma_f(G)=1\ or\ \ \gamma_f\left(L(G)\right)=1$ and $\gamma_f\left(G+L(G)\right)=2-\frac{\frac{3n+2}{5}+\frac{3(2n+1)}{7}-2}{\frac{3n+2}{5}*\frac{3(2n+1)}{7}-1}$ otherwise. □

Theorem 3.11 *Let* C_m *and* C_n *are the two cycle graphs with* m *and* n *verticesrespectively and let* $G=C_m\times C_n$ *be the Cartesian product. Let* $L(G)$ *be its line graph we have*

$$\gamma_f\left(G+L(G)\right)=\begin{cases}1 & if \quad \gamma_f(G)=1\ or\ \ \gamma_f\left(L(G)\right)=1,\\ 2-\dfrac{\frac{mn}{5}+\frac{2mn}{7}-2}{\frac{mn}{5}*\frac{2mn}{7}-1} & Otherwise\end{cases}$$

and $\Gamma_f(G+L(G))\leq 2-\frac{\frac{mn}{4}+\frac{mn}{3}-2}{\frac{mn}{4}*\frac{mn}{3}-1}$

Proof: Theorem 2.9.1 and Theorem 3.1 gives $\gamma_f(G)=\frac{mn}{5}$ and $\gamma_f(L(G))=\frac{2mn}{7}$ where $G=C_m\times C_n$ observed as Cartesian product cycle graphs so $\gamma_f(G)+\gamma_f(L(G))=\frac{mn}{5}+\frac{2mn}{7}$ and $\gamma_f(G)*\gamma_f(L(G))=\frac{mn}{5}+\frac{2mn}{7}$ hence $\gamma_f\left(G+L(G)\right)=1, if\ \ \gamma_f(G)=1\ or\ \ \gamma_f\left(L(G)\right)=1$, otherwise $\gamma_f\left(G+L(G)\right)=2-\frac{\frac{mn}{5}+\frac{2mn}{7}-2}{\frac{mn}{5}*\frac{2mn}{7}-1}$. Similarly $\Gamma_f(G)\leq\frac{mn}{4}$ and $\Gamma_f(L(G))\leq\frac{mn}{3}$ so $\Gamma_f(G)+\Gamma_f\left(L(G)\right)\leq\frac{mn}{4}+\frac{mn}{3}$ and $\Gamma_f(G)*\Gamma_f\left(L(G)\right)\leq\frac{mn}{4}*\frac{mn}{3}$ hence $\Gamma_f(G+L(G))\leq 2-\frac{\frac{mn}{4}+\frac{mn}{3}-2}{\frac{mn}{4}*\frac{mn}{3}-1}$.

4. Applications

- Transportation Networks: Graph theory helps in optimizing transportation networks, such as road networks or flight routes. Line graphs can represent these networks where nodes represent intersections or airports, and edges represent roads or flight paths.

- Communication networks**:** Graph theory is used in designing and optimizing communication networks, such as telephone networks or internet routing. Line graphs can represent these networks where nodes are switching points or routers, and edges represent communication links.
- Computational Biology: [13] Here we have used these concepts in computational biological systems such as gene regulatory networks, protein-protein interaction networks and healthcare network optimization. Line graphs can represent these networks where nodes are biological entities and edges represent interactions or biochemical reactions.

5. Conclusion

we have obtained bounds like lower and upper for sum of $\gamma_f(G) + \gamma_f(L(G))$ and exploring the graphs for upper fractional domination number including cycle, complete, star, Bi-star, wheel, cubic graph, graph of Cartesian product like $(K_2 \times P_n)$, $(K_3 \times P_n)$ and $(C_m \times C_n)$. For these graph classes, the fractional domination number of $L(G)$ can be related to the fractional domination number of G. Our results show that there is a correlation between the sum of fractional domination number of a graph and its line graph. Parameters related to fractional domination in line graphs towards generalization.

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