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Lukasiewicz Fuzzy BM-Algebra and BM-Ideal

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Abstract:

Introduction: $\mathscr{F}uzzy$ Sets is a mathematical framework that expands the traditional concept of sets by enabling elements to have degrees of membership. This enables partial membership based on degree of likeness. In classical set theory, an element can be represented as a crisp set, indicated by x, which either belongs to or does not belong to the set. In contrast, an $\mathscr{F}uzzy$ Sets allows for various levels of membership. The level of membership has a value somewhere between 0 and 1, with 0 representing non-participation and 1 representing full participation. The shape of the member function varies according to the application and intended behaviour. Jan Lukasiewicz was a logical thinker and philosopher. He contributed to the advancement of proportional logic. Lukasiewicz or Lukasz logic is an uncommon and highly appreciated logic that follows the Lukasz t-norm and t-conorm operations to compute the intersection and union of $\mathscr{F}uzzy$ Sets. This logic enables reasoning with unclear or incomplete knowledge, making it appropriate for a variety of applications including ambiguity and imprecision.

Objectives: Incorporation of Lukasz logic theory to $\mathscr{F}uzzy$ set in BM-algebra for the betterment of algorithms to address a variety of real-world issues, including risk management, decision making, managing public transit, diagnosing medical conditions and more.

Methods: Applying BM-algebra to $\mathscr{F}uzzy$ set theory and incorporating Lukasz logic theory with the inclusion of certain attributes, in order to facilitate the production of Lukasz $\mathscr{F}uzzy$ BM-algebra and BM-ideal, wherein the characteristics and attributes of the Lukasz $\mathscr{F}uzzy$ BM-algebra and BM-ideal are examined, and the relationships between them are demonstrated by a few examples.

Results:

Theorem 3.5. Every Lukasz $\mathscr{F}uzzy$ set L_U^ε is a Lukasz $\mathscr{F}uzzy$ $BM\text{-}algebra$ of $\mathfrak{G}$ iff it satisfies: $L_U^\varepsilon(\dot{p} * \dot{q}) \geq min\{L_U^\varepsilon(\dot{p}), L_U^\varepsilon(\dot{q})\}, \forall\, \dot{p}, \dot{q} \in \mathfrak{G}$.

Theorem 3.6. Show that ε-Lukasz $\mathscr{F}uzzy$ set L_U^ε in $\mathfrak{G}$ is an ε-Lukasz $\mathscr{F}uzzy$ $BM\text{-}algebra$ of $\mathfrak{G}$, if U is a $\mathscr{F}uzzy$ sub algebra of $\mathfrak{G}$. An example has been provided to show that the converse is not true.

Theorem 4.3. Every Lukasz $\mathscr{F}uzzy$ set L_U^ε of a $\mathscr{F}uzzy$ set U in $\mathfrak{G}$ is a Lukasz $\mathscr{F}uzzy$ $BM\text{-}ideal$ of $\mathfrak{G}$ if and only if it satisfies

(i) $\forall\, \dot{p} \in \mathfrak{G}, \forall\, u_a \in (0,1], [\dot{p}/u_a] \in L_U^\varepsilon \Rightarrow [0/u_a] \in L_U^\varepsilon$

(ii) $\forall\, \dot{p}, \dot{q} \in \mathfrak{G}, L_U^\varepsilon(\dot{p}) \geq min\{L_U^\varepsilon(\dot{p} * \dot{q}), L_U^\varepsilon(\dot{q})\}$

Conclusions: The application of BM-algebra within Lukasiewicz $\mathscr{F}uzzy$ logic operation can optimize public transportation system by scheduling time and routing based on passengers need. It also improves service reliability using operational constraints taken from the field.

This study give rise to the notion of Lukasz $\mathscr{Fuzzy}$ BM-algebra and Lukasz $\mathscr{Fuzzy}$ BM-ideal along with some of their properties are investigated. In addition to the characterization of both Lukasz $\mathscr{Fuzzy}$ BM-algebra and BM-ideal, the relations of $\mathscr{Fuzzy}$ subalgebra, $\mathscr{Fuzzy}$ ideal, Lukasz $\mathscr{Fuzzy}$ set, Lukasz $\mathscr{Fuzzy}$ BM-algebra and Lukasz $\mathscr{Fuzzy}$ BM-ideal are discussed. Some examples are provided based on those relations. In the future, we will construct an algorithm for the advancement of transportation, making use of the ideas and results of this study.

Keywords: BM-Algebra, $\mathscr{Fuzzy}$ subalgebra, Lukasz $\mathscr{Fuzzy}$ set, Lukasz $\mathscr{Fuzzy}$ BM-Algebra, Lukasz $\mathscr{Fuzzy}$ BM-Ideal.

1. Introduction

In 1966, BCK/BCI-algebra are developed by Y. Imai, K. Iseki and S. Tanaka [4]. There were other algebraic structures besides BCI and BCK algebras. These structures belong to universal algebra that describes fragments of proportional calculus. Such algebraic structures are $BCC/BCH/B/BE$-algebras, etc. These algebras can be explored both theoretically and practically in Mathematics and Computer science. In 2006, a specialized B-algebra, called BM-algebras was delivered by [2]. The concept of Lukasz $\mathscr{Fuzzy}$ subalgebra in BCK/BCI-algebras was built by Jun using the thoughts of Lukasz t-norm [8]. Later, he extended it to Lukasz $\mathscr{Fuzzy}$ ideal in BCK/BCI-algebras in 2023 [7]. In 2002, Jun and Ahn designed the concept of Lukasz $\mathscr{Fuzzy}$ set in $BE\text{-}algebras$ to be Lukasz $\mathscr{Fuzzy}$ $BE\text{-}algebras$ and BE-filters [10] along with the discussion of relationship between Lukasz $\mathscr{Fuzzy}$ BE-algebra and Lukasz $\mathscr{Fuzzy}$ $BE\text{-}filters$. Their capacity to handle partial truth values, include fuzzy principles and integrate them into the decision-making process leads to a variety of applications. To build an advanced algorithm for the solution of real-life problems, we can explore various algebraic structures.

This study led to explore the concept of Lukasz $\mathscr{Fuzzy}$ BM-algebra and BM-ideal using the notion of Lukasz $\mathscr{Fuzzy}$ set to the given $\mathscr{Fuzzy}$ set in BM-algebra and investigated some of their properties, characterizations and relations with some examples.

2. Objectives

Definition 2.1

The set $\mathfrak{G}$ be a non-empty set.The $\boldsymbol{BM\text{-}algebra}$ satisfies the given axioms:

(BM_1) $\dot{p} * 0 = \dot{p}$

$(BM_2)(\dot{r} * \dot{p}) * (\dot{r} * \dot{q}) = \dot{q} * \dot{r}$, for all $\dot{p}, \dot{q}, \dot{r} \in \mathfrak{G}$

under binary operation " $*$ " with a constant element "0"

Remark 2.2

Every $BM\text{-}algebra$ satisfies

(i) $(\dot{p} * \dot{p}) = 0$

(ii) $(0 * (0 * \dot{p})) = \dot{p}$
(iii) $(0 * (\dot{p} * \dot{q})) = \dot{q} * \dot{p}$
(iv) $(\dot{p} * \dot{r}) * (\dot{q} * \dot{r}) = \dot{p} * \dot{q}$
(v) $(\dot{p} * \dot{q}) = 0 \Leftrightarrow (\dot{q} * \dot{p}) = 0$ for all $\dot{p}, \dot{q}, \dot{r} \in \mathfrak{G}$.

Definition 2.3

A BM-algebra $\mathfrak{G}$ is called **subalgebra** of $\mathfrak{G}$ if $\mathfrak{S}$ be a subset of $\mathfrak{G}$ then $\dot{p} * \dot{q} \in \mathfrak{S}, \forall \dot{p}, \dot{q} \in \mathfrak{S}$

Definition 2.4

A BM-algebra $\mathfrak{G}$ is called **ideal** of $\mathfrak{G}$ if $\mathfrak{S}$ be a subset of $\mathfrak{G}$ then $0 \in \mathfrak{G}$ and $\dot{p} * \dot{q} \in \mathfrak{S}, \dot{q} \in \mathfrak{S} \Rightarrow \dot{p} \in \mathfrak{S}, \forall \dot{p}, \dot{q} \in \mathfrak{G}$

Definition 2.5

Zadeh's $\mathscr{F}uzzy$ set U in $\mathfrak{G}$ takes the form

$$U(\dot{q}) = \begin{cases} u \in (0,1] & if\ \dot{q} = \dot{p} \\ 0 & if\ \dot{q} \neq \dot{p} \end{cases}$$

is regarded as ***$\mathscr{F}$uzzy* point** with support $\dot{p}$ and value u. It is viewed by $[\dot{p}/u]$.

Definition 2.6

A $\mathscr{F}uzzy$ point $[\dot{p}/u]$ in every $\mathscr{F}uzzy$ set U in $\mathfrak{G}$ is

(i) **contained** in U, noted by $[\dot{p}/u] \in U$ if $U(\dot{p}) \geq u$.

(ii) **quasi-coincident** with U, noted by $[\dot{p}/u]\mathrm{q}U$ if $U(\dot{p}) + u > 1$

Definition 2.7

A $\mathscr{F}uzzy$ set U is called ***$\mathscr{F}$uzzy* subalgebra** of a BM-algebra $\mathfrak{G}$ if it satisfies

(FA_1) $U(\dot{p} * \dot{q}) \geq min\{U(\dot{p}), U(\dot{q})\}$, $\forall\ \dot{p}, \dot{q} \in \mathfrak{G}$

Definition 2.8

A $\mathscr{F}uzzy$ set U is called ***$\mathscr{F}$uzzy* ideal** of a BM-algebra $\mathfrak{G}$ if it satisfies

(FI_1) $U(0) \geq U(\dot{p})$

(FI_2) $U(\dot{p}) \geq min\{U(\dot{p} * \dot{q}), U(\dot{q})\}, \forall\ \dot{p}, \dot{q} \in \mathfrak{G}$

3. Methods

Definition 3.1

An $\boldsymbol{\varepsilon - Lukasz\ \mathcal{F}}$**uzzy** $\boldsymbol{Set}$ of $\mathscr{F}uzzy$ set U in $\mathfrak{G}$ is a function from the BM-algebra $\mathfrak{G}$ to $[0,1]$ and $\varepsilon \in [0,1]$.

$$L_U^{\varepsilon}: \mathfrak{G} \to [0,1], \dot{p} \mapsto max\,\{0, U(\dot{p}) + \varepsilon - 1\} \qquad (3.1)$$

Remark 3.2

If U is a $\mathscr{F}uzzy$ set in $\mathfrak{G}$, then its ε-Lukasz $\mathscr{F}uzzy$ set L_U^{ε} satisfies

$U(\dot{p}) \geq U(\dot{q}) \Rightarrow L_U^{\varepsilon}(\dot{p}) \geq L_U^{\varepsilon}(\dot{q}),\ \forall\, \dot{p}, \dot{q} \in \mathfrak{G}$ (3.2)

Proof

Suppose U be a $\mathcal{Fuzzy}$ set in $\mathfrak{G}$ and $U(\dot{p}) \geq U(\dot{q})$ then

$$L_U^{\varepsilon}(\dot{p}) = max\{0, U(\dot{p}) + \varepsilon - 1\}$$

$$\geq max\{0, U(\dot{q}) + \varepsilon - 1\} = L_U^{\varepsilon}(\dot{q}).$$

Thus $L_U^{\varepsilon}(\dot{p}) \geq L_U^{\varepsilon}(\dot{q})$.

Definition 3.3

An ε-Lukasz $\mathcal{Fuzzy}$ set L_U^{ε} in $\mathfrak{G}$ is called an **ε- Lukasz $\mathcal{Fuzzy}$ subalgebra of $\mathfrak{G}$** or **ε- Lukasz $\mathcal{Fuzzy}$ $BM - algebra$ of $\mathfrak{G}$** if it satisfies

(LFA_1) $[\dot{p}/u_a], [\dot{q}/u_b] \in L_U^{\varepsilon} \Rightarrow [(\dot{p} * \dot{q})/min\{u_a, u_b\}] \in L_U^{\varepsilon}$ (3.3)

for all $\dot{p}, \dot{q} \in G, \varepsilon \in (0,1)$ and $u_a, u_b \in (0,1]$.

Example 3.4

Let $\mathfrak{G} = \{0, \dot{p}_1, \dot{p}_2, \dot{p}_3\}$ be a set and Table 3.1 shows the Cayley table of $\mathfrak{G}$ under " $*$ "

$*$	0	$\dot{p}_1$	$\dot{p}_2$	$\dot{p}_3$
0	0	$\dot{p}_1$	$\dot{p}_2$	$\dot{p}_3$
$\dot{p}_1$	$\dot{p}_1$	0	0	$\dot{p}_2$
$\dot{p}_2$	$\dot{p}_2$	0	0	$\dot{p}_1$
$\dot{p}_3$	$\dot{p}_3$	$\dot{p}_2$	$\dot{p}_1$	0

TABLE 3.1 Cayley table with respect to " $*$ "

Then $\mathfrak{G}$ is a $BM - algebra$.

Defining a $\mathcal{Fuzzy}$ set U in $\mathfrak{G}$ as follows:

$$U: \mathfrak{G} \to [0,1], \dot{p} \mapsto \begin{cases} 0.88\ if\ \dot{p} = 0 \\ 0.69\ if\ \dot{p} = \{\dot{p}_1, \dot{p}_2\}. \\ 0.77\ if\ \dot{p} = \dot{p}_3 \end{cases}$$

If it is taken that $\varepsilon = 0.61$, then the Lukasz $\mathcal{Fuzzy}$ set L_U^{ε} of U in $\mathfrak{G}$ is provided as follows:

$$L_U^{\varepsilon}: \mathfrak{G} \to [0,1], \dot{p} \mapsto \begin{cases} 0.49\ if\ \dot{p} = 0 \\ 0.3\ \ if\ \dot{p} = \{\dot{p}_1, \dot{p}_2\} \\ 0.36\ if\ p = \dot{p}_3 \end{cases}$$

Typically, it is verified that L_U^{ε} is a Lukasz $\mathcal{Fuzzy}$ $BM\text{-}algebra$ of $\mathfrak{G}$.

Definition 3.5

A Lukasz $\mathcal{Fuzzy}$ set L_U^{ε} in $\mathfrak{G}$ is called **Lukasz $\mathcal{Fuzzy}$ $BM\text{-}ideal$** of $\mathfrak{G}$ if it satisfies

(LFI_1) L_U^{ε} (0) is an upper bound of $\{L_U^{\varepsilon}\ (\dot{p}) | \dot{p} \in \mathfrak{G}\}$ (3.4)

(LFI_2) $[(\dot{p} * \dot{q})/u_a] \in L_U^\varepsilon$, $[\dot{q}/u_b] \in L_U^\varepsilon \Rightarrow [\dot{p}/min\{u_a, u_b\}] \in L_U^\varepsilon$ (3.5)

for all $\dot{p}, \dot{q} \in \mathfrak{G}$ and $u_a, u_b \in (0,1]$.

Example 3.6

Suppose the set $\mathfrak{G} = \{0, \dot{p}_1, \dot{p}_2, \dot{p}_3\}$ be a $BM\text{-}algebra$ with respect to a binary operation " $*$ "given by Table 3.2

$*$	0	$\dot{p}_1$	$\dot{p}_2$	$\dot{p}_3$
0	0	$\dot{p}_1$	$\dot{p}_2$	$\dot{p}_3$
$\dot{p}_1$	$\dot{p}_1$	0	$\dot{p}_3$	$\dot{p}_2$
$\dot{p}_2$	$\dot{p}_2$	$\dot{p}_3$	0	$\dot{p}_1$
$\dot{p}_3$	$\dot{p}_3$	$\dot{p}_2$	$\dot{p}_1$	0

TABLE 3.2 Cayley table with respect to " $*$ "

Then $\mathfrak{G}$ is a $BM\text{-}algebra$. $\mathscr{F}uzzy$ set U in $\mathfrak{G}$ is defined as follows:

$$U: \mathfrak{G} \to [0,1], \dot{p} \mapsto \begin{cases} 0.91 \; if \; \dot{p} = 0 \\ 0.78 \; if \; \dot{p} = \{\dot{p}_1, \dot{p}_2\}. \\ 0.83 \; if \; \dot{p} = \dot{p}_3 \end{cases}$$

If it is taken that $\varepsilon = 0.54$, then the Lukasz $\mathscr{F}uzzy$ set L_U^ε of U in $\mathfrak{G}$ is provided as below

$$L_U^\varepsilon: \mathfrak{G} \to [0,1], \dot{p} \mapsto \begin{cases} 0.45 \; if \; \dot{p} = 0 \\ 0.32 \; if \; \dot{p} = \{\dot{p}_1, \dot{p}_2\} \\ 0.37 \; if \; \dot{p} = \dot{p}_3 \end{cases}$$

Typically, it is verified that L_U^ε is a Lukasz $\mathscr{F}uzzy$ $BM\text{-}ideal$ of $\mathfrak{G}$.

4. Results

Theorem 4.1

Every Lukasz $\mathscr{F}uzzy$ set L_U^ε is a Lukasz $\mathscr{F}uzzy$ $BM\text{-}algebra$ of $\mathfrak{G}$ iff it satisfies:

$L_U^\varepsilon(\dot{p} * \dot{q}) \geq min\{L_U^\varepsilon(\dot{p}), L_U^\varepsilon(\dot{q})\}, \forall \dot{p}, \dot{q} \in \mathfrak{G}$ (4.1)

Proof

Suppose U be a $\mathscr{F}uzzy$ set in $\mathfrak{G}$.

For instance, L_U^ε is a Lukasz $\mathscr{F}uzzy$ $BM\text{-}algebra$ of $\mathfrak{G}$.

Let $\dot{p}, \dot{q} \in \mathfrak{G}$ and it is clear that $[\dot{p}/L_U^\varepsilon(\dot{p})] \in L_U^\varepsilon$ and $[\dot{q}/L_U^\varepsilon(\dot{q})] \in L_U^\varepsilon$.

From (3.3), it is evident that

$[(\dot{p} * \dot{q})/min\{L_U^\varepsilon(\dot{p}), L_U^\varepsilon(\dot{q})\}] \in L_U^\varepsilon$,

and hence $L_U^\varepsilon(\dot{p} * \dot{q}) \geq min\{L_U^\varepsilon(\dot{p}), L_U^\varepsilon(\dot{q})\}$ for all $\dot{p}, \dot{q} \in \mathfrak{G}$.

Conversely, suppose that L_U^ε satisfies (4.1).

Also let $\dot{p}, \dot{q} \in \mathfrak{G}$ and $u_a, u_b \in (0,1]$ be such that $[\dot{p}/u_a] \in L_U^\varepsilon$, $[\dot{q}/u_b] \in L_U^\varepsilon$.

Then $L_U^\varepsilon(\dot{p}) \geq u_a$ and $L_U^\varepsilon(\dot{q}) \geq u_b$,

which imply from (4.1) that

$L_U^\varepsilon(\dot{p} * \dot{q}) \geq min\{L_U^\varepsilon(\dot{p}), L_U^\varepsilon(\dot{q})\} \geq min\{u_a, u_b\}$.

Thus $[(\dot{p} * \dot{q})/min\{u_a, u_b\}] \in L_U^\varepsilon$.

Therefore L_U^ε is a Lukasz $\mathscr{F}uzzy\ BM\text{-}algebra$ of $\mathfrak{G}$.

Theorem 4.2

Show that ε-Lukasz $\mathscr{F}uzzy$ set L_U^ε in $\mathfrak{G}$ is an ε-Lukasz $\mathscr{F}uzzy\ BM\text{-}algebra$ of $\mathfrak{G}$, if U is a $\mathscr{F}uzzy$ sub algebra of $\mathfrak{G}$.

Proof

For instance, U is a $\mathscr{F}uzzy$ subalgebra of $\mathfrak{G}$.

Let $\dot{p}, \dot{q} \in \mathfrak{G}$ and $u_a, u_b \in (0,1]$ be such that $[\dot{p}/u_a] \in L_U^\varepsilon$, $[\dot{q}/u_b] \in L_U^\varepsilon$.

Then $L_U^\varepsilon(\dot{p}) \geq u_a$ and $L_U^\varepsilon(\dot{q}) \geq u_b$.

Thus

$$\begin{aligned} L_U^\varepsilon(\dot{p} * \dot{q}) &= max\{0, U(\dot{p} * \dot{q}) + \varepsilon - 1\} && [\because (3.1)] \\ &\geq max\{0, min\{U(\dot{p}), U(\dot{q})\} + \varepsilon - 1\} && [\because (4.1)] \\ &= max\big\{0, min\{U(\dot{p}) + \varepsilon - 1, U(\dot{q}) + \varepsilon - 1\}\big\} \\ &= min\big\{max\{0, U(\dot{p}) + \varepsilon - 1\}, max\{0, U(\dot{q}) + \varepsilon - 1\}\big\} \\ &= min\{L_U^\varepsilon(\dot{p}), L_U^\varepsilon(\dot{q})\} && [\because (3.1)] \\ &\geq min\{u_a, u_b\}. \end{aligned}$$

So, $[(\dot{p} * \dot{q})/min\{u_a, u_b\}] \in L_U^\varepsilon$.

Hence L_U^ε is a ε-Lukasz $\mathscr{F}uzzy\ BM\text{-}algebra$ of $\mathfrak{G}$.

The subsequent example demonstrates why the reverse portion of Theorem 4.2 is false.

Example 4.3

Suppose the set $\mathfrak{G} = \{0, \dot{p}_1, \dot{p}_2\}$ be a $BM\text{-}algebra$ and Table 4.1 shows binary operation " $*$ " in $\mathfrak{G}$

$*$	0	$\dot{p}_1$	$\dot{p}_2$
0	0	$\dot{p}_2$	$\dot{p}_1$
$\dot{p}_1$	$\dot{p}_1$	0	$\dot{p}_2$
$\dot{p}_2$	$\dot{p}_2$	$\dot{p}_1$	0

TABLE 4.1 Cayley table with respect to " $*$ "

Defining a $\mathscr{F}uzzy$ set U in $\mathfrak{G}$ as follows:

$$U: \mathfrak{G} \to [0,1], \dot{p} \mapsto \begin{cases} 0.72 \; if \; \dot{p} = 0 \\ 0.51 \; if \; \dot{p} = \dot{p}_1 \\ 0.43 \; if \; \dot{p} = \dot{p}_2 \end{cases}.$$

Provided that $\varepsilon = 0.49$, then the ε-Lukasz $\mathscr{F}uzzy$ set L_U^ε of U in $\mathfrak{G}$ is formed as follows:

$$L_U^\varepsilon: \mathfrak{G} \to [0,1], \dot{p} \mapsto \begin{cases} 0.21 \; if \; \dot{p} = 0 \\ 0 \quad if \; \dot{p} = \dot{p}_1 \\ 0 \quad if \; \dot{p} = \dot{p}_2 \end{cases}$$

Typically, it is verified that L_U^ε is an ε-Lukasz $\mathscr{F}uzzy$ $BM\text{-}algebra$ of $\mathfrak{G}$. But U is not a $\mathscr{F}uzzy$ subalgebra of $\mathfrak{G}$ because of

$U(0 * \dot{p}_1) = U(\dot{p}_2) = 0.43 \ngeq 0.51 = min\{U(0), U(\dot{p}_1)\}$.

Theorem 4.4

Every Lukasz $\mathscr{F}uzzy$ set L_U^ε of a $\mathscr{F}uzzy$ set U in $\mathfrak{G}$ is a Lukasz $\mathscr{F}uzzy$ $BM\text{-}ideal$ of $\mathfrak{G}$ if and only if it satisfies

(i) $\forall\, \dot{p} \in \mathfrak{G}, \forall\, u_a \in (0,1], [\dot{p}/u_a] \in L_U^\varepsilon \Rightarrow [0/u_a] \in L_U^\varepsilon$ (4.2)

(ii) $\forall\, \dot{p}, \dot{q} \in \mathfrak{G}, L_U^\varepsilon(\dot{p}) \geq min\{L_U^\varepsilon(\dot{p} * \dot{q}), L_U^\varepsilon(\dot{q})\}$ (4.3)

Proof

For instance, L_U^ε is a Lukasz $\mathscr{F}uzzy$ $BM\text{-}ideal$ of $\mathfrak{G}$. Let $\dot{p} \in \mathfrak{G}$ and $u_a \in (0,1]$ be such that $[\dot{p}/u_a] \in L_U^\varepsilon$.

Utilising (3.4), leads to $L_U^\varepsilon(0) \geq L_U^\varepsilon(\dot{p}) \geq u_a$, and so $[0/u_a] \in L_U^\varepsilon$.

Note that $[(\dot{p} * \dot{q})/L_U^\varepsilon(\dot{p} * \dot{q})] \in L_U^\varepsilon$, $[\dot{q}/L_U^\varepsilon(\dot{q})] \in L_U^\varepsilon$ for all $\dot{p}, \dot{q} \in \mathfrak{G}$.

From (3.5), it is evident that

$[\dot{p}/min\{L_U^\varepsilon(\dot{p} * \dot{q}), L_U^\varepsilon(\dot{q})\}] \in L_U^\varepsilon$,

and hence $L_U^\varepsilon(\dot{p}) \geq min\{L_U^\varepsilon(\dot{p} * \dot{q}), L_U^\varepsilon(\dot{q})\}$ for all $\dot{p}, \dot{q} \in \mathfrak{G}$.

Conversely, let us consider L_U^ε satisfies (4.2) and (4.3).

Since $[\dot{p}/L_U^\varepsilon(\dot{p})] \in L_U^\varepsilon$ for all $\dot{p} \in \mathfrak{G}$,

we have $[0/L_U^\varepsilon(\dot{p})] \in L_U^\varepsilon$ and so $L_U^\varepsilon(0) \geq L_U^\varepsilon(\dot{p})$ for all $\dot{p} \in \mathfrak{G}$ by (4.2).

Hence, $L_U^\varepsilon(0)$ is an upper bound of $\{L_U^\varepsilon(\dot{p}) | \dot{p} \in \mathfrak{G}\}$.

Also let $\dot{p}, \dot{q} \in \mathfrak{G}$ and $u_a, u_b \in (0,1]$ be such that $[(\dot{p} * \dot{q})/u_a] \in L_U^\varepsilon$, $[\dot{q}/u_b] \in L_U^\varepsilon$.

Then $L_U^\varepsilon(\dot{p} * \dot{q}) \geq u_a$ and $L_U^\varepsilon(\dot{q}) \geq u_b$,

which imply from (4.3) that

$L_U^{\varepsilon}(\dot{p}) \geq min\{L_U^{\varepsilon}(\dot{p} * \dot{q}), L_U^{\varepsilon}(\dot{q})\} \geq min\{u_a, u_b\}$.

Thus $[\dot{p}/min\{u_a, u_b\}] \in L_U^{\varepsilon}$.

Therefore L_U^{ε} is a Lukasz $\mathscr{F}uzzy\ BM\text{-}ideal$ of $\mathfrak{G}$.

5. Discussion

Remark 5.1

Prove that ε -Lukasz $\mathscr{F}uzzy$ set L_U^{ε} satisfies

$L_U^{\varepsilon}(0) \geq L_U^{\varepsilon}(\dot{p}), \forall\ \dot{p} \in \mathfrak{G}$**,** if U is a $\mathscr{F}uzzy$ sub algebra of $\mathfrak{G}$, then it

Proof

Let U be a $\mathscr{F}uzzy$ subalgebra of $\mathfrak{G}$.

Then, $U(0) = U(\dot{p} * \dot{p}) \geq min\{U(\dot{p}), U(\dot{p})\} = U(\dot{p})$.

Therefore $U(0) \geq U(\dot{p})$ for all $\dot{p} \in \mathfrak{G}$.

From (3.2), it is evident that

$L_U^{\varepsilon}(0) \geq L_U^{\varepsilon}(\dot{p})$ for all $\dot{p} \in \mathfrak{G}$.

Remark 5.2

Every $\mathscr{F}uzzy$ subalgebra U of $\mathfrak{G}$ is said to be ε-Lukasz $\mathscr{F}uzzy$ set L_U^{ε} if

$[\forall\ \dot{p}, \dot{q} \in \mathfrak{G}]\ [L_U^{\varepsilon}(\dot{p}) = L_U^{\varepsilon}(0) \Leftrightarrow L_U^{\varepsilon}(\dot{p} * \dot{q}) \geq L_U^{\varepsilon}(\dot{q})]$.

Proof

Let U be a $\mathscr{F}uzzy$ subalgebra of $\mathfrak{G}$.

For instance, $L_U^{\varepsilon}(\dot{p}) = L_U^{\varepsilon}(0)$ for all $\dot{p} \in \mathfrak{G}$.

Then

$$L_U^{\varepsilon}(\dot{p} * \dot{q}) \geq min\{L_U^{\varepsilon}(\dot{p}), L_U^{\varepsilon}(\dot{q})\} = min\{L_U^{\varepsilon}(0), L_U^{\varepsilon}(\dot{q})\} = L_U^{\varepsilon}(\dot{q}).$$

Combining the results of Theorem 4.2 and Remark 3.2 leads to

$L_U^{\varepsilon}(\dot{p} * \dot{q}) \geq L_U^{\varepsilon}(\dot{q})$ for all $\dot{p}, \dot{q} \in \mathfrak{G}$.

Conversely, suppose that $L_U^{\varepsilon}(\dot{p} * \dot{q}) \geq L_U^{\varepsilon}(\dot{q})$ for all $\dot{p}, \dot{q} \in \mathfrak{G}$.

Utilising (BM_1) leads to

$$L_U^{\varepsilon}(\dot{p}) = L_U^{\varepsilon}(\dot{p} * 0) \geq L_U^{\varepsilon}(0).$$

Combining the results of above inequality and Remark 5.1 leads to

$L_U^{\varepsilon}(\dot{p}) = L_U^{\varepsilon}(0)$ for all $\dot{p} \in \mathfrak{G}$.

Remark 5.3

Every $\mathscr{F}uzzy$ subalgebra U of $BM\text{-}algebra$ $\mathfrak{G}$ is said to be an ε-Lukasz $\mathscr{F}uzzy$ set L_U^{ε} if

$[\forall\, \dot{p} \in \mathfrak{G}]\ [L_U^{\varepsilon}(0 * \dot{p}) \geq L_U^{\varepsilon}(\dot{p})]$.

Proof

Let U be a $\mathcal{F}uzzy$ subalgebra of BM-algebra $\mathfrak{G}$. Then,

$U(0 * \dot{p}) \geq min\{U(0), U(\dot{p})\} = U(\dot{p})$ for all $\dot{p} \in \mathfrak{G}$.

From (3.2), it is evident that

$$L_U^{\varepsilon}(0 * \dot{p}) \geq L_U^{\varepsilon}(\dot{p}) \text{ for all } \dot{p} \in \mathfrak{G}.$$

Remark 5.4

If U is a $\mathcal{F}uzzy$ sub algebra of $\mathfrak{G}$, then prove that ε-Lukasz $\mathcal{F}uzzy$ set L_U^{ε} satisfies

$[\dot{p}/u_a] \in L_U^{\varepsilon}, [\dot{q}/u_b] \in L_U^{\varepsilon} \Rightarrow \left[\left(0 * (\dot{p} * \dot{q})\right)/min\{u_a, u_b\}\right] \in L_U^{\varepsilon}, \forall\, \dot{p}, \dot{q} \in \mathfrak{G}$ and $u_a, u_b \in (0,1]$.

Proof

Let U be a $\mathcal{F}uzzy$ subalgebra of $BM\text{-}algebra$ $\mathfrak{G}$.

It is given that $\dot{p}, \dot{q} \in \mathfrak{G}$ and $u_a, u_b \in (0,1]$ which implies $[\dot{p}/u_a] \in L_U^{\varepsilon}, [\dot{q}/u_b] \in L_U^{\varepsilon}$.

Then $L_U^{\varepsilon}(\dot{p}) \geq u_a$ and $L_U^{\varepsilon}(\dot{q}) \geq u_b$.

Thus

$$\begin{aligned}
L_U^{\varepsilon}\left(0 * (\dot{p} * \dot{q})\right) &= max\left\{0, U\left(0 * (\dot{p} * \dot{q})\right) + \varepsilon - 1\right\} \\
&= max\{0, U(\dot{q} * \dot{p}) + \varepsilon - 1\} \\
&\geq max\{0, min\{U(\dot{q}), U(\dot{p})\} + \varepsilon - 1\} \\
&\geq max\left\{0, min\{U(\dot{q}) + \varepsilon - 1, U(\dot{p}) + \varepsilon - 1\}\right\} \\
&\geq min\left\{max\{0, U(\dot{q}) + \varepsilon - 1\}, max\{0, U(\dot{p}) + \varepsilon - 1\}\right\} \\
&\geq min\{L_U^{\varepsilon}(\dot{q}), L_U^{\varepsilon}(\dot{p})\} \geq min\{u_b, u_a\}.
\end{aligned}$$

So $\left[\left(0 * (\dot{p} * \dot{q})\right)/min\{u_a, u_b\}\right] \in L_U^{\varepsilon}$.

Lemma 5.5

Every Lukasz $\mathcal{F}uzzy\ BM\text{-}ideal$ L_U^{ε} of $\mathfrak{G}$ satisfies the condition if $\dot{p} \leq \dot{q}$ and $\dot{q} * \dot{p} = 0$, then

$$[\dot{q}/u_a] \in L_U^{\varepsilon} \Rightarrow [\dot{p}/u_a] \in L_U^{\varepsilon}, \forall\, \dot{p}, \dot{q} \in \mathfrak{G}, \forall\, u_a \in (0,1] \qquad (5.1)$$

Proof

Let $\dot{p}, \dot{q} \in \mathfrak{G}$ and $u_a \in (0,1]$ be such that $\dot{p} \leq \dot{q}$, $\dot{q} * \dot{p} = 0$ and $[\dot{q}/u_a] \in L_U^{\varepsilon}$

then $(\dot{p} * \dot{q}) = 0$ and $L_U^{\varepsilon}(\dot{q}) \geq u_a$ so,

$L_U^{\varepsilon}(\dot{p}) \geq \min\{L_U^{\varepsilon}(\dot{p} * \dot{q}), L_U^{\varepsilon}(\dot{q})\} = min\{L_U^{\varepsilon}(0), L_U^{\varepsilon}(\dot{q})\} = L_U^{\varepsilon}(\dot{q}) \geq u_a$.

Hence $[\dot{p}/u_a] \in L_U^{\varepsilon}$.

Lemma 5.6

Every Lukasz $\mathscr{F}uzzy\ BM\text{-}ideal\ L_U^{\varepsilon}$ of $\mathfrak{G}$ fulfils the condition if $\dot{p} * \dot{q} \leq \dot{r}$ and $\dot{r} * (\dot{p} * \dot{q}) = 0$, then

$$[\dot{q}/u_a] \in L_U^{\varepsilon}, [\dot{r}/u_b] \in L_U^{\varepsilon} \Rightarrow [\dot{p}/min\{u_a, u_b\}] \in L_U^{\varepsilon} \quad (5.2)$$

for all $\dot{p}, \dot{q}, \dot{r} \in \mathfrak{G}$, and $u_a, u_b \in (0,1]$

Proof

Let $\dot{p}, \dot{q}, \dot{r} \in \mathfrak{G}$ and $u_a, u_b \in (0,1]$ be such that $\dot{p} * \dot{q} \leq \dot{r}$, $\dot{r} * (\dot{p} * \dot{q}) = 0$, $[\dot{q}/u_a] \in L_U^{\varepsilon}$ and $[\dot{r}/u_b] \in L_U^{\varepsilon}$.

Then, $(\dot{p} * \dot{q}) * \dot{r} = 0$, $L_U^{\varepsilon}(\dot{q}) \geq u_a$ and $L_U^{\varepsilon}(\dot{r}) \geq u_b$.

Hence,

$$L_U^{\varepsilon}(\dot{p}) \geq \min\{L_U^{\varepsilon}(\dot{p} * \dot{q}), L_U^{\varepsilon}(\dot{q})\}$$
$$\geq min\{min\{L_U^{\varepsilon}((\dot{p} * \dot{q}) * \dot{r}), L_U^{\varepsilon}(\dot{r})\}, L_U^{\varepsilon}(\dot{q})\}$$
$$= min\{min\{L_U^{\varepsilon}(0), L_U^{\varepsilon}(\dot{r})\}, L_U^{\varepsilon}(\dot{q})\}$$
$$= \min\{L_U^{\varepsilon}(\dot{r}), L_U^{\varepsilon}(\dot{q})\} \geq min\{u_b, u_a\}$$

and so $[\dot{p}/min\{u_a, u_b\}] \in L_U^{\varepsilon}$.

Remark 5.7

If L_U^{ε} is a Lukasz $\mathscr{F}uzzy$ ideal of $\mathfrak{G}$, then it satisfies the following inequalities

(i) $[\dot{p} \leq \dot{q}\ and\ \dot{q} * \dot{p} = 0 \Rightarrow L_U^{\varepsilon}(\dot{p}) \geq L_U^{\varepsilon}(\dot{q})]$ (5.3)

(ii) $[\dot{p} * \dot{q} \leq \dot{r}\ and\ \dot{r} * (\dot{p} * \dot{q}) = 0 \Rightarrow L_U^{\varepsilon}(\dot{p}) \geq min\{L_U^{\varepsilon}(\dot{q}), L_U^{\varepsilon}(\dot{r})\}]$ (5.4)

for all $\dot{p}, \dot{q}, \dot{r} \in \mathfrak{G}$

Theorem 5.8

Every Lukasz $\mathscr{F}uzzy$ set L_U^{ε} in $\mathfrak{G}$ is a Lukasz $\mathscr{F}uzzy\ BM\text{-}ideal$ of $\mathfrak{G}$ if U is a $\mathscr{F}uzzy$ ideal of BM-algebra $\mathfrak{G}$.

Proof

For instance, L_U^{ε} is a Lukasz $\mathscr{F}uzzy$ set of a $\mathscr{F}uzzy$ ideal U in $\mathfrak{G}$.

Let $\dot{p}, \dot{q} \in \mathfrak{G}$ and $u_a, u_b \in (0,1]$ be such that $[(\dot{p} * \dot{q})/u_a] \in L_U^{\varepsilon}$, $[\dot{q}/u_b] \in L_U^{\varepsilon}$.

Then $L_U^{\varepsilon}(\dot{p} * \dot{q}) \geq u_a$ and $L_U^{\varepsilon}(\dot{q}) \geq u_b$.

Thus

$$L_U^{\varepsilon}(\dot{p}) = max\{0, U(\dot{p}) + \varepsilon - 1\} \quad [\because (3.1)]$$
$$\geq max\{0, min\{U(\dot{p} * \dot{q}), U(\dot{q})\} + \varepsilon - 1\} \quad [\because (4.3)]$$

$$= max\{0, min\{U(\dot{p} * \dot{q}) + \varepsilon - 1, U(\dot{q}) + \varepsilon - 1\}\}$$
$$= min\{max\{0, U(\dot{p} * \dot{q}) + \varepsilon - 1\}, max\{0, U(\dot{q}) + \varepsilon - 1\}\}$$
$$= min\{L_U^{\varepsilon}(\dot{p} * \dot{q}), L_U^{\varepsilon}(\dot{q})\} \qquad [\because (3.1)]$$
$$\geq min\{u_a, u_b\}.$$

So, $[\dot{p}/min\{u_a, u_b\}] \in L_U^{\varepsilon}$.

Hence L_U^{ε} is a Lukasz $\mathscr{F}uzzy\ BM\text{-}ideal$ of $\mathfrak{G}$.

The subsequent example demonstrates why the reverse portion of Theorem 5.8 is false.

Example 5.9

Consider the BM-algebra set $\mathfrak{G}$ in Example 3.4 and $\mathscr{F}uzzy$ set U in $\mathfrak{G}$ defined by

$$U: \mathfrak{G} \to [0,1], \dot{p} \mapsto \begin{cases} 0.81\ if\ \dot{p} = 0 \\ 0.42\ if\ \dot{p} = \dot{p}_1 \\ 0.57\ if\ \dot{p} = \dot{p}_2 \\ 0.31\ if\ \dot{p} = \dot{p}_3 \end{cases}.$$

Then U is not a $\mathscr{F}uzzy$ ideal of $\mathfrak{G}$.

Since $U(\dot{p}_1) = 0.42 \ngeq 0.57 = min\{U(\dot{p}_1 * \dot{p}_2), U(\dot{p}_2)\}$

Given that $\varepsilon = 0.55$, then the Lukasz $\mathscr{F}uzzy$ set L_U^{ε} of U in $\mathfrak{G}$ is provided as below:

$$L_U^{\varepsilon}: \mathfrak{G} \to [0,1], \dot{p} \mapsto \begin{cases} 0.36\ if\ \dot{p} = 0 \\ 0 \quad\ \ if\ \dot{p} = \dot{p}_1 \\ 0.12\ if\ \dot{p} = \dot{p}_2 \\ 0 \quad\ \ if\ \dot{p} = \dot{p}_3 \end{cases}.$$

and it is a Lukasz $\mathscr{F}uzzy\ BM\text{-}ideal$ of $\mathfrak{G}$.

Theorem 5.10

Every Lukasz $\mathscr{F}uzzy\ BM\text{-}ideal$ of $\mathfrak{G}$ is a Lukasz $\mathscr{F}uzzy\ BM\text{-}algebra$ of $\mathfrak{G}$.

Proof

For instance, L_U^{ε} is a Lukasz $\mathscr{F}uzzy\ BM\text{-}ideal$ of $\mathfrak{G}$. Let $\dot{p}, \dot{q} \in \mathfrak{G}$ and

$u_a, u_b \in (0,1]$ be such that $[\dot{p}/u_a] \in L_U^{\varepsilon}$, $[\dot{q}/u_b] \in L_U^{\varepsilon}$.

Since $\dot{p} * \dot{q} \leq \dot{p}$ and $\dot{p} * (\dot{p} * \dot{q}) = 0$

we have $[(\dot{p} * \dot{q})/u_a] \in L_U^{\varepsilon}$ by (5.1).

Hence $[\dot{p}/min\{u_a, u_b\}] \in L_U^{\varepsilon}$by (3.5), and so $[(\dot{p} * \dot{q})/min\{u_a, u_b\}] \in L_U^{\varepsilon}$by (5.1).

Therefore L_U^{ε} is a Lukasz $\mathscr{F}uzzy\ BM\text{-}algebra$ of $\mathfrak{G}$.

The subsequent example demonstrates why the reverse portion of Theorem 5.10 is false.

Example 5.11

A set in $BM\text{-}algebra$ $\mathfrak{G} = \{0, \dot{p}_1, \dot{p}_2\}$ be considered and the Table 5.1 is built under the "$*$" operation

$*$	0	$\dot{p}_1$	$\dot{p}_2$
0	0	$\dot{p}_1$	$\dot{p}_2$
$\dot{p}_1$	$\dot{p}_1$	0	$\dot{p}_1$
$\dot{p}_2$	$\dot{p}_2$	$\dot{p}_1$	0

TABLE 5.1 Cayley table with respect to "$*$"

Defining a $\mathcal{F}uzzy$ set U in $\mathfrak{G}$ as follows

$$U: \mathfrak{G} \to [0,1], \dot{p} \mapsto \begin{cases} 0.84 \; if \; \dot{p} = 0 \\ 0.72 \; if \; \dot{p} = \dot{p}_1. \\ 0.51 \; if \; \dot{p} = \dot{p}_2 \end{cases}$$

Given that $\varepsilon = 0.58$, the ε-Lukasz $\mathcal{F}uzzy$ set L_U^{ε} of U in $\mathfrak{G}$ is provided as below

$$L_U^{\varepsilon}: \mathfrak{G} \to [0,1], \dot{p} \mapsto \begin{cases} 0.42 \; if \; \dot{p} = 0 \\ 0.3 \;\; if \; \dot{p} = \dot{p}_1 \\ 0.09 \; if \; \dot{p} = \dot{p}_2 \end{cases}$$

Typically, it is verified that L_U^{ε} is an ε-Lukasz $\mathcal{F}uzzy\ BM\text{-}algebra$ of $\mathfrak{G}$. But L_U^{ε} is not a Lukasz $\mathcal{F}uzzy\ BM\text{-}ideal$ of $\mathfrak{G}$ because of

$U(\dot{p}_2) = 0.09 \ngeq 0.3 = min\{U(\dot{p}_2 * \dot{p}_1), U(\dot{p}_1)\}$.

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