Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 6s (2024)

Some Properties of The Spectrum of The Power Digraph $\Gamma(n,k)$

Sanjay Kumar Thakur [1]*, Gautam Chandra Ray [2], Pinkimani Goswami [3]

[1]Department of Mathematics, Science College, Kokrajhar, India, e-mail: sanj26sc@yahoo.com

[2]Department of Mathematics, CIT, Kokrajhar, India, e-mail: gc.roy@cit.ac.in

[3]Department of Mathematics, University of Science and Technology, Baridua,
e-mail: pinkimanigoswami@yahoo.com *corresponding author

Article History:

Received: *23-05-2024*

Revised: *10-07-2024*

Accepted: *22-07-2024*

Abstract:

For every positive integer n and k , a power digraph modulo n, denoted by $\Gamma(n,k)$ is constructed with the vertex set $\mathbb{Z}_n = \{0,1,2,\cdots,n-1\}$, and a directed edge from a vertex x to a vertex y exists if and only if $x^k \equiv y(mod\ n)$, where $x,y \in \mathbb{Z}_n$. In this work, we define the out-adjacency (A_Γ^+) and the in-adjacency (A_Γ^-) matrices of the digraph $\Gamma(n,k)$ and some results on A_Γ^+ and A_Γ^- are discussed. It is proved that the matrices A_Γ^+ and A_Γ^- are singular if $k|\phi(n)$ or $p^2|n$, for some prime p. Some spectral properties of $\Gamma(n,k)$ are also presented. Moreover, it is proved that the algebraic multiplicity of 1 as an eigenvalue of A_Γ^+ is the number of components of the digraph $\Gamma(n,k)$.

Keywords: Digraph, Adjacency matrices of power digraph (mod n), eigenvalues.

AMS Subject classification: 11A07, 05C50

1. Introduction

In recent years, exploring the interconnections between Graph theory, Group theory, and Number theory has emerged as an attractive and effective study area, for example, [3, 4, 6, 8, 10, 11, 13, 14, 17, 18, 20]. In this article, for each positive integers n and k, we consider a power digraph modulo n denoted by $\Gamma(n,k)$ whose vertex set is $\mathbb{Z}_n = \{0,1,2,\cdots,n-1\}$ and the ordered pair (x,y) is a directed arc (or directed edge) of $\Gamma(n,k)$ from x to y iff $x^k \equiv y(mod\ n)$, where $x,y \in \mathbb{Z}_n$. In [3, 6, 9, 12, 14, 17, 18, 21] some properties of the power digraph $\Gamma(n,k)$ were studied.

The adjacency matrix is a commonly used matrix representation for graphs, and numerous researchers have investigated the connection between the eigenvalues of the adjacency matrix and the graph's structures in the past, for example, [1, 2, 7]. In the case of a multidigraph G with n vertices, the adjacency matrix of G defined in [1] as the $n \times n$ matrix $A(G) = [a_{ij}]$, where a_{ij} represents the number of directed edges that start at the vertex i and ends at the vertex j. It is important to note that based on this definition, the adjacency matrix of a multidigraph is not symmetric in general. So, it may have complex eigenvalues. Furthermore, a graph is completely determined by its adjacency eigenvalues and corresponding eigenvectors. This is evident from the fact that a graph G can be uniquely determined by $A(G)$. In the case of an undirected simple graph G, $A(G)$ is symmetric.

It is important to mention that the study of adjacency matrices of $\Gamma(n,k)$, the power digraph modulo n is still open. In this paper, we aim to define the adjacency matrices of the digraph $\Gamma(n,k)$ and try to explore some properties associated with them.

We organize the rest of the paper as follows: In Section 2, we provide some definitions and results from Graph Theory and Matrix Theory. In Section 3, we define the out-adjacency (A_Γ^+) and the in-adjacency (A_Γ^-) matrices of the digraph $\Gamma(n,k)$ and some results on A_Γ^+ and A_Γ^- are discussed. It is proved that the matrices A_Γ^+ and A_Γ^- are singular if $k|\phi(n)$ or $p^2|n$, for some prime p. In Section 4, some spectral properties of $\Gamma(n,k)$ are presented. It is also proved that the algebraic multiplicity of 1 as an eigenvalue of A_Γ^+ is the number of components of the digraph $\Gamma(n,k)$.

2. Preliminaries

For each positive integers n and k, we consider a power digraph modulo n denoted by $\Gamma(n,k)$ (in short, directed graph $\Gamma(n,k)$ or digraph $\Gamma(n,k)$) whose vertex set is $\mathbb{Z}_n$ and any two vertices $x,y \in \mathbb{Z}_n$ are connected by a directed arc from x to y if and only if $x^k \equiv y(mod\ n)$.

We denote the vertex set of the digraph $\Gamma(n,k)$ by $V\big(\Gamma(n,k)\big)$ or by $V(\Gamma)$ (= $\mathbb{Z}_n$) and the arc set by $A\big(\Gamma(n,k)\big)$ or by $A(\Gamma)$. The distinct vertices $v_1, v_2, v_3, \dots, v_t$ in $V(\Gamma)$ will form a *cycle* of length t if

$$v_1{}^k \equiv v_2(mod\ n)$$
$$v_2{}^k \equiv v_3(mod\ n)$$
$$v_3{}^k \equiv v_4(mod\ n)$$
$$\vdots$$
$$v_t{}^k \equiv v_1(mod\ n)$$

We call a cycle of length t as a t- cycle and a cycle of length 1 is named as a *fixed point* (or a *self-loop*). A vertex is *isolated* if it is not connected to any other vertex in $\Gamma(n,k)$. Some researchers have developed theorems to find the number of fixed points of the digraph $\Gamma(n,k)$, denoted by $L(n)$ for some values of k see [5, 15, 16, 19, 20]. From these theorems, it is clear that 0 is always a fixed point of $\Gamma(n,k)$ and so the number of fixed points, $L(n) > 0$.

The *in-degree* of a vertex $v \in V(\Gamma)$, denoted by $d_\Gamma^-(v)$ is the number of directed arcs incident into the vertex v and the *out-degree* of a vertex v, denoted by $d_\Gamma^+(v)$ is the number of directed arcs incident out of the vertex v. Since the residue of a number modulo n is unique, so $d_\Gamma^+(v) = 1$ and $d_\Gamma^-(v) \geq 0$ for each vertex $v \in V(\Gamma)$. Also, for an isolated fixed point $v \in V(\Gamma)$, $d_\Gamma^+(v) = d_\Gamma^-(v) = 1$. The *total degree* (or simply *degree*) of a vertex $v \in V(\Gamma)$, denoted by $d_\Gamma(v)$ is the sum of out-degree and in-degree of v i.e. $d_\Gamma(v) = d_\Gamma^+(v) + d_\Gamma^-(v)$.

A *component* of a digraph is a subdigraph which is a maximal connected subgraph of the associated nondirected graph.

As the out-degree of each vertex of the digraph $\Gamma(n,k)$ is equal to 1, the number of components of $\Gamma(n,k)$ equals the number of all cycles. The cycles may or may not be isolated.

We call a digraph *regular* if the in-degree of each vertex is equal to 1. Every component of such a digraph is a cycle. A digraph is *semi-regular* if there exists a positive integer d such that each vertex either has in-degree 0 or d.

For $n > 1$, let us divide the digraph $\Gamma(n,k)$ into two subdigraphs $\Gamma_1(n,k)$ and $\Gamma_2(n,k)$, where $\Gamma_1(n,k)$ is the subdigraph induced on the set of the vertices $v \in \mathbb{Z}_n$ such that $\gcd(v,n) = 1$ and $\Gamma_2(n,k)$ is the subdigraph induced on the set of the vertices $v \in \mathbb{Z}_n$ such that $\gcd(v,n) \neq 1$. Clearly, the vertex set of $\Gamma_1(n,k)$ is the unit group $\mathbb{Z}_n^*$ with order $\phi(n)$, where $\phi(n)$ denotes the Euler's totient function. Also, 1 and $(n-1)$ are vertices of $\Gamma_1(n,k)$ and 0 is always a vertex of $\Gamma_2(n,k)$. One can easily observe that $\Gamma_1(n,k) \cup \Gamma_2(n,k) = \Gamma(n,k)$ and $\Gamma_1(n,k) \cap \Gamma_2(n,k) = \phi$.

From definition of $\Gamma(n,k)$, it is clear that $|A(\Gamma)| = n$. Since the number of arcs in a directed graph equals the number of their tails (or their heads), we have the following theorem.

Theorem 2.1. [22] (Handshaking theorem) In the digraph $\Gamma(n,k)$,

$$\sum_{v\in V(\Gamma)} d_\Gamma^+(v) = \sum_{v\in V(\Gamma)} d_\Gamma^-(v) = |A(\Gamma)|$$

A directed walk in a digraph D is an alternating sequence $v_1, e_1, v_2, e_2, v_3, \dots, e_{n-1}, v_n$ of vertices and arcs in which each arc e_i is $v_i v_{i+1}$. A directed path is a walk in which all vertices are distinct. If there is a directed path from a vertex u to a vertex v, then v is said to be reachable from u.

In a digraph D, a semi-walk is an alternating sequence $v_1, e_1, v_2, e_2, v_3, \dots, e_{n-1}, v_n$ of vertices and arcs in which each arc e_i may be either $v_i v_{i+1}$ or $v_{i+1} v_i$. A semi-path is a semi-walk in which all vertices are distinct.

A digraph is strongly connected (or strong) if every two vertices are mutually reachable. A digraph is unilaterally connected (or unilateral) if for any two vertices at least one is reachable from the other. A digraph is weakly connected (or weak) if every two vertices are joined by a semi-path.

Every strongly connected (or strong) digraph is a unilateral digraph and every unilateral digraph is weak. But the converse statements are not true.

A digraph is disconnected if it is not even weak.

Note 2.1. From the definition of the digraph $\Gamma(n,k)$, it is clear that $\Gamma(n,k)$ is a disconnected graph, and the components of $\Gamma(n,k)$ are weakly connected.

A tree is a connected acyclic graph. A tree in which one vertex has been designated as the root is a rooted tree. The edges of a rooted tree can be assigned a natural orientation, either away from or towards the root, in which case the structure becomes a directed rooted tree. When a directed rooted tree has an orientation away from the root, it is called an arborescence or out-tree and when it has an orientation towards the root, it is called an anti-arborescence or in-tree. A vertex in a rooted tree is called a leaf if $d_\Gamma^-(v) = 0$.

A block diagonal matrix is a square matrix of the form

$$B = \begin{bmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ 0 & A_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{mm} \end{bmatrix}$$

Where $A_{11}, A_{22}, \cdots, A_{mm}$ are square matrices lying along the diagonal and all other entries of the matrix is 0 (zero matrices). Determinant of B is given by

$$\det(B) = \det(A_{11}) \times \det(A_{22}) \times \cdots \times \det(A_{mm}).$$

3. Adjacency matrices of the digraph $\Gamma(n,k)$

In this section, we try to define adjacency matrices of the digraph $\Gamma(n,k)$ and try to study some properties associated with them.

Definition 3.1. We define the *out-adjacency* matrix of the digraph $\Gamma(n,k)$ as an $n \times n$ matrix $[a_{ij}]$ such that

$$a_{ij} = \begin{cases} 1, & \text{if } (\mathrm{v_i}, \mathrm{v_j}) \in \mathrm{A}(\Gamma) \\ 0, & \text{otherwise} \end{cases}$$

We denote this matrix by $A^{+}(\Gamma(n,k))$ or by A_Γ^{+}.

Definition 3.2. We define the in-adjacency matrix of the digraph $\Gamma(n,k)$ as an $n \times n$ matrix $[a_{ij}]$ such that

$$a_{ij} = \begin{cases} 1, & \text{if } (\mathrm{v_j}, \mathrm{v_i}) \in \mathrm{A}(\Gamma) \\ 0, & \text{otherwise} \end{cases}$$

We denote this matrix by $A^{-}(\Gamma(n,k))$ or by A_Γ^{-}.

From definition of $\Gamma(n,k)$ it is clear that $\Gamma(n,k)$ is a disconnected graph, so the out-adjacency matrix A_Γ^{+} can also be defined as a block diagonal matrix $[A_{ij}]_{m\times m}$ i.e. $A_\Gamma^{+} = [A_{ij}]_{m\times m}$, such that

$$A_{ij} = \begin{cases} [a_{uv}]_{q\times q}, & \text{for } \mathrm{i = j;\ q \le m \le n} \\ 0, & \text{for } \mathrm{i \ne j.} \end{cases}$$

where,

$$a_{uv} = \begin{cases} 1, & \text{if there is a directed arc from } \mathrm{u^{th}} \text{ vertex to } \mathrm{v^{th}} \text{ vertex.} \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, the in-adjacency matrix A_Γ^{-} can be defined as a block diagonal matrix $[A_{ij}]_{m\times m}$ i.e.

$A_\Gamma^{-} = [A_{ij}]_{m\times m}$, such that

$$A_{ij} = \begin{cases} [a_{uv}]_{q\times q}, & \text{for } \mathrm{i = j;\ q \le m \le n} \\ 0, & \text{for } \mathrm{i \ne j.} \end{cases}$$

where,

$$a_{uv} = \begin{cases} 1, & \text{if there is a directed arc from } \mathrm{v^{th}} \text{ vertex to } \mathrm{u^{th}} \text{ vertex.} \\ 0, & \text{otherwise.} \end{cases}$$

We have the following observations about A_Γ^{+} and A_Γ^{-} of a digraph $\Gamma(n,k)$:

i. Each non-zero element on the main diagonal of A_Γ^{+} and A_Γ^{-} represents a loop at the corresponding vertex.

ii. The number of non-zero entries of either A_Γ^{+} or A_Γ^{-} equals the number of directed arcs in $\Gamma(n,k)$.

iii. Permutations of any rows together with a permutation of the corresponding columns do not alter the power digraph $\Gamma(n,k)$; indicating that the permutation simply rearranges the vertices.

iv. The out-adjacency matrix A_Γ^+ (or in-adjacency matrix A_Γ^-) is not unique (follows from iii.).

v. The out-adjacency (or in-adjacency) matrix A_Γ^+ (or A_Γ^-) of the digraph $\Gamma(n,k)$ can be written as a block-diagonal matrix with diagonal elements as the out-adjacency (or in-adjacency) matrices of the component digraphs of the digraph $\Gamma(n,k)$.

Example 3.1. Let us consider the digraph $\Gamma(6,2)$.

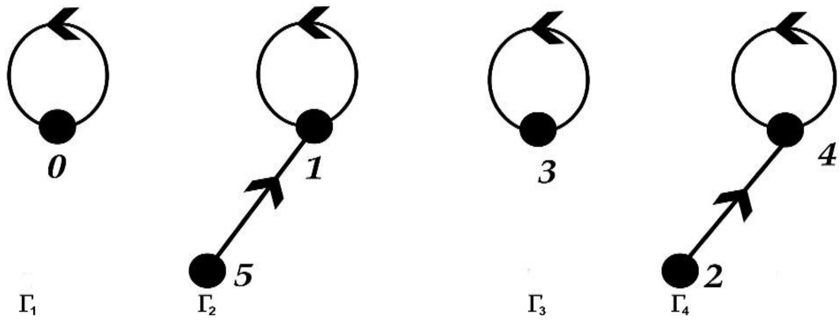

Figure 1: Digraph $\Gamma(6,2)$ with components $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$.

Here,

$$A_\Gamma^+ = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 & 0 \\ 5 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}, \quad A_\Gamma^- = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 & 1 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}, \text{ and } (A_\Gamma^+)^t = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 & 1 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Let us apply the elementary operations $R_2 \leftrightarrow R_6$ and then $C_2 \leftrightarrow C_6$; $R_3 \leftrightarrow R_6$ and then $C_3 \leftrightarrow C_6$ and finally $R_5 \leftrightarrow R_6$ and then $C_5 \leftrightarrow C_6$ to the matrix A_Γ^+, we get the following matrix

$$\begin{array}{c|cccccc} & 0 & 5 & 1 & 3 & 2 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$, which gives us the digraph $\Gamma(6,2)$.

Hence, permuting rows together with the corresponding columns, the matrix A_Γ^+ can be written as

$$A_\Gamma^+ = \begin{array}{c} \\ 0 \\ 5 \\ 1 \\ 3 \\ 2 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 5 & 1 & 3 & 2 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} A_{\Gamma_1}^+ & 0 & 0 & 0 \\ 0 & A_{\Gamma_2}^+ & 0 & 0 \\ 0 & 0 & A_{\Gamma_3}^+ & 0 \\ 0 & 0 & 0 & A_{\Gamma_4}^+ \end{bmatrix}$$

Where, $A_{\Gamma_1}^+ = [1]$, $A_{\Gamma_2}^+ = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $A_{\Gamma_3}^+ = [1]$, and $A_{\Gamma_4}^+ = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ are out-adjacency matrices of the component digraphs $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4 respectively. Thus, the out-adjacency matrix A_Γ^+ of the digraph $\Gamma(6,2)$ can be written as a block-diagonal matrix with diagonal elements as the out-adjacency matrices of the component digraphs of the digraph $\Gamma(6,2)$.

Similarly, the in-adjacency matrix A_Γ^- of the digraph $\Gamma(6,2)$ can be written as a block-diagonal matrix with diagonal elements as the in-adjacency matrices of the component digraphs of the digraph $\Gamma(6,2)$ $i.e.$

$$A_\Gamma^- = \begin{array}{c} \\ 0 \\ 5 \\ 1 \\ 3 \\ 2 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 5 & 1 & 3 & 2 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} A_{\Gamma_1}^- & 0 & 0 & 0 \\ 0 & A_{\Gamma_2}^- & 0 & 0 \\ 0 & 0 & A_{\Gamma_3}^- & 0 \\ 0 & 0 & 0 & A_{\Gamma_4}^- \end{bmatrix}$$

Where, $A_{\Gamma_1}^- = [1]$, $A_{\Gamma_2}^- = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $A_{\Gamma_3}^- = [1]$, and $A_{\Gamma_4}^- = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ are in-adjacency matrices of the component digraphs $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4 respectively.

Result 3.1. $(A_\Gamma^+)^t = A_\Gamma^-$ and $(A_\Gamma^-)^t = A_\Gamma^+$.

Proof. Clearly, the matrices A_Γ^+, A_Γ^-, and $(A_\Gamma^+)^t$ are of the same order $n \times n$.

Also, the $(i,j)^{th}$ element of $(A_\Gamma^+)^t =$ the $(j,i)^{th}$ element of A_Γ^+

$=$ the $(i,j)^{th}$ element of A_Γ^-. [By definition of A_Γ^-]

Hence, $(A_\Gamma^+)^t = A_\Gamma^-$.

Similarly, it can be shown that $(A_\Gamma^-)^t = A_\Gamma^+$. □

Result 3.2. Let A_Γ be the adjacency matrix of the underlying graph G of the digraph $\Gamma(n,k)$, then $A_\Gamma = A_\Gamma^+ + A_\Gamma^-$.

Proof. Let $A_\Gamma^+ = [a_{ij}]_{n\times n}$, where,

$$a_{ij} = \begin{cases} 1, & \text{if there is a directed arc from the vertex } v_i \text{ to the vertex } v_j. \\ 0, & \text{otherwise.} \end{cases}$$

and, $A_\Gamma^- = [b_{ij}]_{n\times n}$,

where,

$$b_{ij} = \begin{cases} 1, & \text{if there is a directed arc from the vertex } v_j \text{ to the vertex } v_i. \\ 0, & \text{otherwise.} \end{cases}$$

Also, let $A_\Gamma = [c_{ij}]_{n\times n}$,

where,

$$c_{ij} = \begin{cases} 2, & \text{if there is a loop at the vertex } v_i. \\ 1, & \text{if there is an edge between the vertices } v_i \text{ and } v_j. \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the matrices A_Γ and $A_\Gamma^+ + A_\Gamma^-$ are of the same order $n \times n$.

We have,

$$\begin{aligned} A_\Gamma^+ + A_\Gamma^- &= [a_{ij}]_{n\times n} + [b_{ij}]_{n\times n} \\ &= [a_{ij} + b_{ij}]_{n\times n} \\ &= [d_{ij}]_{n\times n}, \text{where, } d_{ij} = \begin{cases} 2, & \text{if there is a loop at the vertex } v_i. \\ 1, & \text{if there is an edge between the vertices } v_i \text{ and } v_j. \\ 0, & \text{otherwise.} \end{cases} \\ &= [c_{ij}]_{n\times n} \\ &= A_\Gamma \end{aligned}$$

Hence, $A_\Gamma = A_\Gamma^+ + A_\Gamma^-$. □

Remark 3.1.

i. A_Γ is symmetric i.e. $(A_\Gamma)^t = A_\Gamma$

ii. $A_\Gamma = A_\Gamma^+ + (A_\Gamma^+)^t$

iii. $A_\Gamma = A_\Gamma^- + (A_\Gamma^-)^t$

Result 3.3. The sum of entries in the i^{th} row of A_Γ^+ is 1.

Proof. Let $A_\Gamma^+ = [a_{ij}]$ be an out-adjacency matrix of the digraph $\Gamma(n,k)$ and let $R_i = [a_{i1}, a_{i2}, \cdots, a_{in}]$ be the i^{th} row of A_Γ^+ corresponding to the vertex $v_i \in V(\Gamma)$. As the residue of a number modulo n is unique, the number of directed arcs leaving the vertex v_i is exactly one. It contributes thereby 1 exactly in one of the entries of R_i and 0 in the remaining entries of R_i. Thus, $\sum_{j=1}^{n} a_{ij} = 1$. □

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 6s (2024)

Corollary 3.1. The sum of entries in the i^{th} row of A_Γ^+ is $d_\Gamma^+(v_i)$, where $d_\Gamma^+(v_i)$ is the out-degree of the i^{th} vertex v_i.

Proof. As the out-degree of each vertex $v_i \in \Gamma(n,k)$ is 1, so we have

$$d_\Gamma^+(v_i) = 1, \quad \forall\, v_i \in V(\Gamma).$$

$$\Rightarrow d_\Gamma^+(v_i) = 1 = \sum_{j=1}^{n} a_{ij} \quad \text{[By Result 3.3]}$$

$$i.e.\ \sum_{j=1}^{n} a_{ij} = d_\Gamma^+(v_i).$$ □

Result 3.4. The sum of entries in the i^{th} row of A_Γ^- is $d_\Gamma^-(v_i)$, where $d_\Gamma^-(v_i)$ is the in-degree of the i^{th} vertex v_i.

Proof. Let $A_\Gamma^- = [a_{ij}]$ be an in-adjacency matrix of the digraph $\Gamma(n,k)$ and let $R_i = [a_{i1}, a_{i2}, \cdots, a_{in}]$ be the i^{th} row of the matrix A_Γ^- corresponding to the vertex $v_i \in V(\Gamma)$. We now consider the sum $\sum_{j=1}^{n} a_{ij}$. Clearly, 1 is added to this sum $\sum_{j=1}^{n} a_{ij}$ exactly once for each directed arc coming to the vertex v_i and thereby using the definition of the in-degree of a vertex the result follows immediately i.e. $\sum_{j=1}^{n} a_{ij} = indeg(v_i) = d_\Gamma^-(v_i)$. □

Result 3.5. The sum of entries in the j^{th} column of A_Γ^+ is $d_\Gamma^-(v_j)$, where $d_\Gamma^-(v_j)$ is the in-degree of the j^{th} vertex v_j.

Proof. Let $A_\Gamma^+ = [a_{ij}]$ be an out-adjacency matrix of the digraph $\Gamma(n,k)$ and let $C_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$ be the j^{th}column of A_Γ^+ corresponding to the vertex $v_j \in V(\Gamma)$. We now consider the sum $\sum_{i=1}^{n} a_{ij}$. Clearly, 1 is added to this sum $\sum_{i=1}^{n} a_{ij}$ exactly once for each directed arc coming to the vertex v_j and thereby using the definition of the in-degree of a vertex the result follows immediately i.e. $\sum_{i=1}^{n} a_{ij} = indeg(v_j) = d_\Gamma^-(v_j)$. □

Result 3.6. The sum of entries in the j^{th} column of A_Γ^- is 1.

Proof. Let $A_\Gamma^- = [a_{ij}]$ be an in-adjacency matrix of the digraph $\Gamma(n,k)$ and let $C_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$ be the j^{th} column of A_Γ^- corresponding to the vertex $v_j \in V(\Gamma)$. As the residue of a number modulo n is unique, the number of directed arcs leaving the vertex v_j is exactly one. It contributes thereby 1 exactly in one of the entries of C_j and 0 in the remaining entries of C_j. Thus, $\sum_{i=1}^{n} a_{ij} = 1$. □

Corollary 3.2. The sum of entries in the j^{th} column of A_Γ^- is $d_\Gamma^-(v_j)$, where $d_\Gamma^-(v_j)$ is the in-degree of the j^{th} vertex v_j.

Result 3.7. The sum of all entries in the matrix A_Γ^+ is $\sum_{i=1}^{n} d_\Gamma^+(v_i)$.

Proof. Let $A_\Gamma^+ = [a_{ij}]_{n\times n}$ be an out-adjacency matrix of the digraph $\Gamma(n,k)$. Suppose $R_1, R_2, \cdots, R_n$ be the n-rows of the matrix A_Γ^+. By Corollary 3.1., the sum of entries in the i^{th} row (i.e. R_i) is $d_\Gamma^+(v_i)$, for all $i = 1,2,\cdots,n$ and consequently, the sum of entries in all these rows is $d_\Gamma^+(v_1) + d_\Gamma^+(v_2) + \cdots + d_\Gamma^+(v_n) = \sum_{i=1}^n d_\Gamma^+(v_i)$ i.e. $\sum_{i=1}^n \sum_{j=1}^n a_{ij} = \sum_{i=1}^n d_\Gamma^+(v_i)$. □

Result 3.8. The sum of all entries in the matrix A_Γ^+ is $\sum_{i=1}^n d_\Gamma^-(v_i)$.

Proof. The result can be easily established using Result 3.4. □

Remark 3.2. If $A_\Gamma^+ = [a_{ij}]_{n\times n}$ be an out-adjacency matrix of the digraph $\Gamma(n,k)$ then

i. $\sum_{i=1}^n \sum_{j=1}^n a_{ij} = \sum_{i=1}^n d_\Gamma^+(v_i) = \sum_{i=1}^n d_\Gamma^-(v_i) = n$

ii. $\sum_{i=1}^n \sum_{j=1}^n a_{ij} = |A(\Gamma)| = |V(\Gamma)| = n$.

Result 3.9. The sum of all entries in the matrix A_Γ^- is $\sum_{i=1}^n d_\Gamma^-(v_i)$.

Proof. The proof is left for the reader. □

Result 3.10. The sum of all entries in the matrix A_Γ^- is $\sum_{i=1}^n d_\Gamma^+(v_i)$.

Proof. The proof is left for the reader. □

Remark 3.3. If $A_\Gamma^- = [a_{ij}]_{n\times n}$ be an in-adjacency matrix of the digraph $\Gamma(n,k)$ then

i. $\sum_{i=1}^n \sum_{j=1}^n a_{ij} = \sum_{i=1}^n d_\Gamma^+(v_i) = \sum_{i=1}^n d_\Gamma^-(v_i) = n$

ii. $\sum_{i=1}^n \sum_{j=1}^n a_{ij} = |A(\Gamma)| = |V(\Gamma)| = n$.

Result 3.11. Let $A_\Gamma^+ = [a_{ij}]_{n\times n}$ be an out-adjacency matrix of the digraph $\Gamma(n,k)$, then the number of directed walks of length m from vertex v_i to vertex $v_j (i.e. v_i \to v_j$ directed walk) in $\Gamma(n,k)$ is the element in the $(i,j)^{th}$ position of the matrix $(A_\Gamma^+)^m$, where m is a non-negative integer.

Proof. We shall try to prove the result using mathematical induction on m.

If $m = 0$, then the number of directed walks of length 0 from vertex v_i to vertex v_j is 0 resulting $a_{ij} = 0$, for $i \neq j$. Also the number of directed walks of length 0 from a vertex v_i to itself is 1 resulting $a_{ij} = 1$, for $i = j$ which gives us the identity matrix I. So we get $(A_\Gamma^+)^0 = I$.

If $m = 1$, then the number of directed walks of length 1 from vertex v_i to vertex v_j is the number of directed arcs from the vertex v_i to vertex v_j which is equal to a_{ij} of the out-adjacency matrix A_Γ^+. So we get $(A_\Gamma^+)^1 = A_\Gamma^+$.

We now assume that the result is true for $m > 1$ and try to establish the result for $m + 1$. Let us denote the $(i,j)^{th}$ element of $(A_\Gamma^+)^m$ by b_{ij} i.e. $(A_\Gamma^+)^m = [b_{ij}]_{n\times n}$.

As,

$$
\begin{aligned}
(A_\Gamma^+)^{m+1} &= (A_\Gamma^+)^m \cdot (A_\Gamma^+) \\
&= [b_{ij}]_{n\times n} \cdot [a_{ij}]_{n\times n} \\
&= [c_{ij}]_{n\times n}
\end{aligned}
$$

where, $c_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj}$.

By assumption, b_{ik} is the number of $v_i \to v_k$ directed walks of length m. Also, $a_{kj} = 0$ or 1, so $b_{ik}a_{kj} = 0$ or b_{ik}. Then $b_{ik}a_{kj}$ is exactly the number of $v_i \to v_j$ directed walks of length $(m+1)$ with vertex v_k adjacent to vertex v_j. As the sum includes this for each of the vertices, we notice that $c_{ij} (= \sum_{k=1}^{n} b_{ik}a_{kj})$ is the number of $v_i \to v_j$ directed walks of length $(m+1)$ and hence the result holds for $(A_\Gamma^+)^{m+1}$. So by induction, the result is established. □

Result 3.12. Let $A_\Gamma^- = [a_{ij}]_{n\times n}$ be the in-adjacency matrix of the digraph $\Gamma(n,k)$, then the number of directed walks of length m from vertex v_j to vertex v_i $(i.e. v_i \leftarrow v_j$ directed walk) in $\Gamma(n,k)$ is the element in the $(i,j)^{th}$ position of the matrix $(A_\Gamma^-)^m$, where m is a non-negative integer.

Proof. It can be proven in the same way as Result 3.11, using the definition of A_Γ^-. □

Result 3.13. Let $A_\Gamma^+ = [a_{ij}]_{n\times n}$ be an out-adjacency matrix of the digraph $\Gamma(n,k)$. Then the matrix $B_\Gamma = [b_{ij}]$ has at least two entries which is zero, where $B_\Gamma = A_\Gamma^+ + (A_\Gamma^+)^2 + (A_\Gamma^+)^3 + \cdots + (A_\Gamma^+)^{n-1}$ and $n > 1$.

Proof. By definition of $\Gamma(n,k)$, it is clear that the digraph $\Gamma(n,k)$ is disconnected for $n > 1$. So there exists two or more than two disjoint components of $\Gamma(n,k)$ that have no directed arcs in between them. Let there be such s number of components namely $\Gamma_1, \Gamma_2, \cdots, \Gamma_s$. In this case, the out-adjacency matrix A_Γ^+ of $\Gamma(n,k)$ can be partitioned into block diagonal matrices as

$$A_\Gamma^+ = \begin{bmatrix} A_{\Gamma_1}^+ & 0 & 0 & \cdots & 0 \\ 0 & A_{\Gamma_2}^+ & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{\Gamma_s}^+ \end{bmatrix}$$

where $A_{\Gamma_1}^+, A_{\Gamma_2}^+, \cdots, A_{\Gamma_s}^+$ are out-adjacency matrices of the components $\Gamma_1, \Gamma_2, \cdots, \Gamma_s$ respectively.

Now, let us consider the matrix, $B_\Gamma = A_\Gamma^+ + (A_\Gamma^+)^2 + (A_\Gamma^+)^3 + \cdots + (A_\Gamma^+)^{n-1}$. Clearly, each entry in $(A_\Gamma^+)^m (1 \le m \le n-1)$ counts the number of directed walks of length m from vertex v_i to vertex v_j. As the digraph $\Gamma(n,k)$ is disconnected, so a directed walk from one component to another component is not possible, and hence the entry in the matrix $(A_\Gamma^+)^m$ corresponding to those directed walks will be zero. Thus the only non-zero entries in B_Γ will come from the individual components $\Gamma_1, \Gamma_2, \cdots, \Gamma_s$. Moreover, each out-adjacency matrix $A_{\Gamma_i}^+$ corresponding to components $\Gamma_i (1 \le i \le s)$ is a non-zero square matrix. So, the submatrices in the diagonal blocks of B_Γ will be non-zero matrices but non-diagonal blocks will be zero because there are no arcs between the components. Hence, at least two entries in the matrix B_Γ will be zero. □

Result 3.14. Let $A_\Gamma^- = [a_{ij}]_{n\times n}$ be the in-adjacency matrix of the digraph $\Gamma(n,k)$. Then the matrix $C_\Gamma = [c_{ij}]$ has at least two entries which is zero, where $C_\Gamma = A_\Gamma^- + (A_\Gamma^-)^2 + (A_\Gamma^-)^3 + \cdots + (A_\Gamma^-)^{n-1}$ and $n > 1$.

Proof. It can be proven in the same way as Result 3.13. □

Lemma 3.1. The digraph $\Gamma(n,k)$ has at least one vertex of in-degree 0 iff $k|\phi(n)$ or $p^2|n$, for some prime p.

Proof. Let $\Gamma(n,k)$ have at least one vertex of in-degree 0. To show $k|\phi(n)$ or $p^2|n$, for some prime p.

Let $p^2 \nmid n$, for any prime p. In this case, the digraph $\Gamma_1(n,k)$ is semi-regular and so $d_\Gamma^-(v) = 0$ or $k^{\omega(n)}$, for $v \in \Gamma_1(n,k)$, where

$$\omega(n) = \begin{cases} \omega_0(n) + 1, & \text{if } k^2|n \\ \omega_0(n), & \text{if } k^2 \nmid n \end{cases}$$

and $\omega_o(n)$ is the number of distinct primes dividing n which are congruent to $1(mod\ k)$.

As the set of residues which are co-prime to n, forms a group under multiplication modulo n of order $\phi(n)$, so the set of vertices of $\Gamma_1(n,k)$ forms a group under multiplication modulo n of order $\phi(n)$. Let $v \in \Gamma_1(n,k)$ such that $d_\Gamma^-(v) = k^{\omega(n)}$ and let H = $\{0 \le m \le n-1\ |(m,n) = 1, m^k \equiv 1(mod\ n)\}$. Then H is a subgroup of the group $\Gamma_1(n,k)$ of order $k^{\omega(n)}$ and hence $k^{\omega(n)}|\phi(n)$ which implies $k|\phi(n)$.

Now, let $k \nmid \phi(n)$. To show $p^2|n$, for some prime p.

If possible, let $p^2 \nmid n$ for any prime p, then n is a square-free integer. Now, n is square-free and $k \nmid \phi(n)$ so in this case the digraph $\Gamma(n,k)$ is cyclic. By definition, a digraph is cyclic if all of its components are cyclic. Moreover, if all the components of the digraph $\Gamma(n,k)$ are cycles, then the digraph $\Gamma(n,k)$ is regular and so $d_\Gamma^-(v) = 1, \forall v \in \Gamma(n,k)$, which contradicts the fact that there exists at least one vertex of in-degree 0. This contradiction implies that $p^2|n$, for some prime p.

Conversely, let $k|\phi(n)$ or $p^2|n$, for some prime p. To show the digraph $\Gamma(n,k)$ has at least one vertex of in-degree 0.

If $k|\phi(n)$, then the digraph $\Gamma_1(n,k)$ is a semi-regular digraph and hence $d_\Gamma^-(v) = 0$ or $k^{\omega(n)}$, for $v \in \Gamma_1(n,k)$. Thus, there exists at least one vertex v such that $d_\Gamma^-(v) = 0$.

If $p^2|n$, for some prime p, then some (or all) vertices of the digraph $\Gamma_2(n,k)$ forms a rooted in-tree with root 0 and therefore there exists at least one leaf v in this rooted in-tree such that $d_\Gamma^-(v) = 0$. □

Lemma 3.2. The out-adjacency matrix A_Γ^+ of $\Gamma(n,k)$ contains at least one block diagonal submatrix whose determinant is *zero* if $k|\phi(n)$ or $p^2|n$, for some prime p.

Proof. Let us consider the digraph $\Gamma(n,k)$ with $k|\phi(n)$ or $p^2|n$, for some prime p. Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_s$ be the s components of the digraph $\Gamma(n,k)$ and $n > 2$. Let A_Γ^+ be the out-adjacency matrix of the digraph $\Gamma(n,k)$ and

$$A_\Gamma^+ = \begin{bmatrix} A_{\Gamma_1}^+ & 0 & 0 & \cdots & 0 \\ 0 & A_{\Gamma_2}^+ & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{\Gamma_s}^+ \end{bmatrix}$$

where $A_{\Gamma_1}^+, A_{\Gamma_2}^+, \cdots, A_{\Gamma_s}^+$ are out-adjacency matrices of the components $\Gamma_1, \Gamma_2, \cdots, \Gamma_s$ respectively.

By Lemma 3.1., the digraph $\Gamma(n,k)$ has at least one vertex of in-degree 0, so let v_t be such a vertex of the digraph $\Gamma(n,k)$ such that indeg$(v_t) = 0$. Therefore, each entry of the column C_{v_t} (say) corresponding to the vertex v_t in A_Γ^+ will be zero. Now, some element(s) of C_{v_t} is (are) also column element(s) of one of the block diagonal submatrix $A_{\Gamma_i}^+(say)$,$1 \leq i \leq s$ and consequently one column of $A_{\Gamma_i}^+$ is a zero column resulting $det(A_{\Gamma_i}^+) = 0$. □

Result 3.15. If $k|\phi(n)$ or $p^2|n$, for some prime p then the out-adjacency matrix A_Γ^+ of $\Gamma(n,k)$ is a singular matrix.

Proof. Let $k|\phi(n)$ or $p^2|n$, for some prime p. Also, let,

$$A_\Gamma^+ = \begin{bmatrix} A_{\Gamma_1}^+ & 0 & 0 & \cdots & 0 \\ 0 & A_{\Gamma_2}^+ & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{\Gamma_s}^+ \end{bmatrix}$$

where $A_{\Gamma_1}^+, A_{\Gamma_2}^+, \cdots, A_{\Gamma_s}^+$ are respectively out-adjacency matrices of the components $\Gamma_1, \Gamma_2, \cdots, \Gamma_s$ of the digraph $\Gamma(n,k)$.

We have,

$$det(A_\Gamma^+) = det(A_{\Gamma_1}^+) \times det(A_{\Gamma_2}^+) \times \cdots \times det(A_{\Gamma_s}^+). \qquad (1)$$

By Lemma 3.2., A_Γ^+ contains at least one block submatrix $A_{\Gamma_m}^+$ (say), $1 \leq m \leq s$ such that $det(A_{\Gamma_m}^+) = 0$ and hence from (1) we get, $det(A_\Gamma^+) = 0$. This shows that the matrix A_Γ^+ is a singular matrix. □

Result 3.16. If $k|\phi(n)$ or $p^2|n$, for some prime p then the in-adjacency matrix A_Γ^- of $\Gamma(n,k)$ is a singular matrix.

Proof. Let $k|\phi(n)$ or $p^2|n$, for some prime p.

We have,

$$\begin{aligned} det(A_\Gamma^-) &= det((A_\Gamma^+)^t) \quad \text{[By Result 3.1.]} \\ &= det(A_\Gamma^+) \\ &= 0 \qquad \text{[By Result 3.15.]} \end{aligned}$$

This shows that the matrix A_Γ^- is a singular matrix. □

4. Spectrum of the digraph $\Gamma(n,k)$

The characteristic polynomial of a matrix A is the polynomial $det(A - \lambda I)$. The roots of the characteristic polynomial are the eigenvalues of A. A non-zero vector v is an eigenvector of A with eigenvalue λ if the equation $Av = \lambda v$ is satisfied.

The eigenvalue(s) of a graph G is (are) defined as the eigenvalue(s) of its adjacency matrix. The spectrum of a graph G is the set of eigenvalues of G together with their algebraic multiplicities. If a graph G has t distinct eigenvalues $\lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_t$ with multiplicities $m(\lambda_1), m(\lambda_2), m(\lambda_3), \cdots, m(\lambda_t)$ then the spectrum of G is

$$Spec(G)=\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_t \\ m(\lambda_1) & m(\lambda_2) & m(\lambda_3) & \cdots & m(\lambda_t) \end{pmatrix}$$

Also, we have,

$det((A_\Gamma^+)^t - \lambda I) = det((A_\Gamma^+)^t - \lambda I^t)$, where I is an Identity matrix of order n.

$\Rightarrow det(A_\Gamma^- - \lambda I) = det(A_\Gamma^+ - \lambda I)^t$ [By Result 3.1., $(A_\Gamma^+)^t = A_\Gamma^-$]

$\Rightarrow det(A_\Gamma^- - \lambda I) = det(A_\Gamma^+ - \lambda I)$ [$\because det(X^t) = det(X)$, where X is a square matrix.]

So, the characteristic polynomial of A_Γ^+ = The characteristic polynomial of A_Γ^-.

In this section, we will study some spectral properties of the digraph $\Gamma(n,k)$ using the out-adjacency matrix A_Γ^+ or in-adjacency matrix A_Γ^-. We define the eigenvalues of the digraph $\Gamma(n,k)$ as the eigenvalues of its out-adjacency matrix (or in-adjacency matrix) and the spectrum of the digraph $\Gamma(n,k)$ as the set of eigenvalues of $\Gamma(n,k)$ together with their algebraic multiplicities.

Example 4.1. Let us consider the digraph $\Gamma(9,11)$.

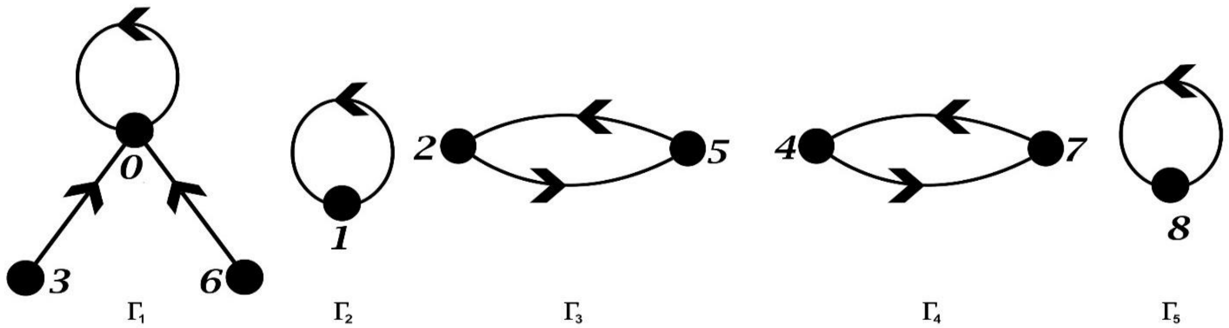

Figure 2: Digraph $\Gamma(9,11)$ with components $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$.

We have,

$$A_{\Gamma_1}^+ = \begin{array}{c} \\ 0 \\ 3 \\ 6 \end{array}\begin{array}{c} \begin{matrix} 0 & 3 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}, \quad A_{\Gamma_2}^+ = \begin{array}{c} \\ 1 \end{array}\begin{array}{c} 1 \\ [1] \end{array}, \quad A_{\Gamma_3}^+ = \begin{array}{c} \\ 2 \\ 5 \end{array}\begin{array}{c} \begin{matrix} 2 & 5 \end{matrix} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}, \quad A_{\Gamma_4}^+ = \begin{array}{c} \\ 4 \\ 7 \end{array}\begin{array}{c} \begin{matrix} 4 & 7 \end{matrix} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}, \text{ and } A_{\Gamma_5}^+ = \begin{array}{c} \\ 8 \end{array}\begin{array}{c} 8 \\ [1] \end{array}$$

Therefore, the characteristic polynomials of $A_{\Gamma_1}^+, A_{\Gamma_2}^+, A_{\Gamma_3}^+, A_{\Gamma_4}^+$ and $A_{\Gamma_5}^+$ are $\lambda^2(1-\lambda), (1-\lambda), (\lambda^2 - 1), (\lambda^2 - 1)$, and $(1-\lambda)$ respectively. And, eigenvalues of $A_{\Gamma_1}^+$, $A_{\Gamma_2}^+$, $A_{\Gamma_3}^+$, $A_{\Gamma_4}^+$, and $A_{\Gamma_5}^+$ are 0,0,1; 1; -1,1; -1,1 and 1 respectively.

$$\text{Also, } A_{\Gamma}^{+} = \begin{bmatrix} A_{\Gamma_1}^{+} & 0 & 0 & 0 & 0 \\ 0 & A_{\Gamma_2}^{+} & 0 & 0 & 0 \\ 0 & 0 & A_{\Gamma_3}^{+} & 0 & 0 \\ 0 & 0 & 0 & A_{\Gamma_4}^{+} & 0 \\ 0 & 0 & 0 & 0 & A_{\Gamma_5}^{+} \end{bmatrix} = \begin{array}{c} \\ 0 \\ 3 \\ 6 \\ 1 \\ 2 \\ 5 \\ 4 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 3 & 6 & 1 & 2 & 5 & 4 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Therefore, the characteristic polynomial of A_{Γ}^{+} is

$$-\lambda^9 + 3\lambda^8 - \lambda^7 - 5\lambda^6 + 5\lambda^5 + \lambda^4 - 3\lambda^3 + \lambda^2 = -\lambda^2(\lambda - 1)^5(\lambda + 1)^2$$

and, eigenvalues of A_{Γ}^{+} are $0, 0, -1, -1, 1, 1, 1, 1, 1$. So, the Spectrum of $\Gamma(9, 11)$ w.r.t. the adjacency matrix A_{Γ}^{+} is $Spec(\Gamma(9, 11)) = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 2 & 5 \end{pmatrix}$.

Moreover, the Characteristic polynomial of $A_{\Gamma}^{-} =$ The Characteristic polynomial of A_{Γ}^{+}. So, the characteristic polynomial of A_{Γ}^{-} is

$$-\lambda^9 + 3\lambda^8 - \lambda^7 - 5\lambda^6 + 5\lambda^5 + \lambda^4 - 3\lambda^3 + \lambda^2 = -\lambda^2(\lambda - 1)^5(\lambda + 1)^2$$

and, eigenvalues of A_{Γ}^{-} are $0, 0, -1, -1, 1, 1, 1, 1, 1$. So, the Spectrum of $\Gamma(9, 11)$ w.r.t. the adjacency matrix A_{Γ}^{-} is $Spec(\Gamma(9, 11)) = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 2 & 5 \end{pmatrix}$.

Result 4.1. The digraph $\Gamma(n, k)$ has n eigenvalues.

Proof. Let us consider the digraph $\Gamma(n, k)$. Clearly $|V(\Gamma)| = n$.

The characteristic polynomial of the digraph $\Gamma(n, k)$ is given as $P_{\Gamma}(\lambda) = |A_{\Gamma}^{+} - \lambda I_n|$, which is a polynomial of degree n in λ. By the Fundamental theorem of algebra, we know that every polynomial of degree n possesses precisely n roots, taking into account their multiplicities within the complex number field. Hence, $P_{\Gamma}(\lambda)$ has n roots. This shows that the digraph $\Gamma(n, k)$ has n- eigenvalues. □

Result 4.2. If $\Gamma_1, \Gamma_2, \Gamma_3, \cdots, \Gamma_s$ are the s-components of the digraph $\Gamma(n, k)$ then

$$P_{\Gamma}(\lambda) = P_{\Gamma_1}(\lambda) \cdot P_{\Gamma_2}(\lambda) \cdot P_{\Gamma_3}(\lambda) \cdots P_{\Gamma_s}(\lambda)$$

where $P_{\Gamma}(\lambda), P_{\Gamma_1}(\lambda), P_{\Gamma_2}(\lambda), P_{\Gamma_3}(\lambda), \cdots, P_{\Gamma_s}(\lambda)$ are the characteristic polynomials of the digraphs $\Gamma, \Gamma_1, \Gamma_2, \Gamma_3, \cdots, \Gamma_s$ respectively.

Proof. Let us consider the digraph $\Gamma(n, k)$, where $|V(\Gamma)| = n$.

The characteristic polynomial of the digraph $\Gamma(n, k)$ is given as $P_{\Gamma}(\lambda) = |A_{\Gamma}^{+} - \lambda I_n|$.

Let $A_{\Gamma_1}^{+}, A_{\Gamma_2}^{+}, A_{\Gamma_3}^{+}, \cdots, A_{\Gamma_s}^{+}$ be the out-adjacency matrices of the component digraphs $\Gamma_1, \Gamma_2, \Gamma_3, \cdots, \Gamma_s$ respectively. Also, let $|V(\Gamma_i(n, k))| = n_i, 1 \leq i \leq s$ such that $\sum_{i=1}^{s} n_i = n$. Then we have

$$A_\Gamma^+ = \begin{bmatrix} A_{\Gamma_1}^+ & 0 & 0 & \cdots & 0 \\ 0 & A_{\Gamma_2}^+ & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{\Gamma_s}^+ \end{bmatrix}$$

$$and, det(A_\Gamma^+ - \lambda I_n) = det(A_{\Gamma_1}^+ - \lambda I_{n_1}) \cdot det(A_{\Gamma_2}^+ - \lambda I_{n_2}) \cdot det(A_{\Gamma_3}^+ - \lambda I_{n_3}) \cdots det(A_{\Gamma_s}^+ - \lambda I_{n_s})$$

$$i.e. P_\Gamma(\lambda) = P_{\Gamma_1}(\lambda) \cdot P_{\Gamma_2}(\lambda) \cdot P_{\Gamma_3}(\lambda) \cdots \cdot P_{\Gamma_s}(\lambda).$$

□

Result 4.3. Let $\Gamma(n,k)$ be a digraph with s-components $\Gamma_1, \Gamma_2, \Gamma_3, \cdots, \Gamma_s$ then the spectrum of $\Gamma(n,k)$ is the union of the spectra of $\Gamma_1, \Gamma_2, \Gamma_3, \cdots, \Gamma_s$.

Proof. To prove this, we try to show that each eigenvalue of Γ is also an eigenvalue of at least one of the components Γ_i and conversely, each eigenvalue of Γ_i is an eigenvalue of Γ ; $1 \le i \le s$

Let $A_\Gamma^+, A_{\Gamma_1}^+, A_{\Gamma_2}^+, A_{\Gamma_3}^+, \cdots, A_{\Gamma_s}^+$ be the out-adjacency matrices of the digraphs $\Gamma, \Gamma_1, \Gamma_2, \Gamma_3, \cdots, \Gamma_s$ respectively. Then we have

$$A_\Gamma^+ = \begin{bmatrix} A_{\Gamma_1}^+ & 0 & 0 & \cdots & 0 \\ 0 & A_{\Gamma_2}^+ & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{\Gamma_s}^+ \end{bmatrix}$$

Let λ be an eigenvalue of the digraph Γ and let v be the corresponding eigenvector, then

$$A_\Gamma^+ \cdot v = \lambda \cdot v$$

$$\Rightarrow \begin{bmatrix} A_{\Gamma_1}^+ & 0 & 0 & \cdots & 0 \\ 0 & A_{\Gamma_2}^+ & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{\Gamma_s}^+ \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_s \end{bmatrix} = \lambda \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_s \end{bmatrix}, \text{ where } v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_s \end{bmatrix} \text{ is an eigenvector of } \Gamma.$$

$$\Rightarrow A_{\Gamma_1}^+ \cdot v_1 = \lambda \cdot v_1, \quad A_{\Gamma_2}^+ \cdot v_2 = \lambda \cdot v_2, \ \cdots, A_{\Gamma_s}^+ \cdot v_s = \lambda \cdot v_s$$

$$\Rightarrow A_{\Gamma_i}^+ \cdot v_i = \lambda \cdot v_i \, ; \quad i = 1, 2, \cdots, s.$$

This shows that λ is an eigenvalue of the component digraphs Γ_i with eigenvalue v_i. Since λ is an eigenvalue of at least one of the components Γ_i, it is included in the spectrum of Γ.

Conversely, let λ be an eigenvalue of a component Γ_i, then there exists a non-zero vector v_i such that $A_{\Gamma_i}^+ \cdot v_i = \lambda \cdot v_i$; $1 \le i \le s$

$$\Rightarrow \begin{bmatrix} A_{\Gamma_1}^+ & 0 & 0 & \cdots & 0 \\ 0 & A_{\Gamma_2}^+ & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{\Gamma_s}^+ \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ v_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ v_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow A_\Gamma^+ \cdot v^/ = \lambda \cdot v^/ \text{ ; where } v^/ = [0 \quad \cdots \quad 0 \quad v_i \quad 0 \quad \cdots \quad 0]^t.$$

This shows that λ is an eigenvalue of the digraph Γ. Thus we have shown that every eigenvalue of Γ is also an eigenvalue of at least one of the components Γ_i and conversely, every eigenvalue of Γ_i is an eigenvalue of Γ. This proves that the spectrum of Γ is the union of the spectra of Γ_i. □

Result 4.4. Let $\Gamma(n,k)$ be a digraph with s-components $\Gamma_1, \Gamma_2, \Gamma_3, \cdots, \Gamma_s$. Then 1 is an eigenvalue of each of the out-adjacency matrix $A^+_{\Gamma_i} (1 \leq i \leq s)$ with algebraic multiplicity one.

Proof. Let $A^+_{\Gamma_i}$ be the out-adjacency matrix of the component digraph Γ_i with n_i vertices, where $n_i \leq n$ and $1 \leq i \leq s$. Clearly $A^+_{\Gamma_i}$ is an $n_i \times n_i$ matrix. As the out-degree of each vertex in $\Gamma(n,k)$ is 1, so the out-degree of each vertex in Γ_i is also 1 and hence 1 appears exactly once in each row of $A^+_{\Gamma_i}$ with other entries as 0. We now consider the matrix $A^+_{\Gamma_i} - \lambda I_{n_i}$ and we apply the column operation $C_1 \rightarrow C_1 + C_2 + \cdots + C_{n_i}$ in the matrix $A^+_{\Gamma_i} - \lambda I_{n_i}$, then it can be easily seen that each element of C_1 is $(1-\lambda)$ and hence $(1-\lambda)$ will be a factor of $det(A^+_{\Gamma_i} - \lambda I_{n_i})$. This shows that 1 is an eigenvalue of $A^+_{\Gamma_i} (1 \leq i \leq s)$.

Next, to show that the algebraic multiplicity of 1 is one. If possible, let the algebraic multiplicity of 1 be greater than one. Then there exists at least two linearly independent vectors u and v with eigenvalue 1 such that $A^+_{\Gamma_i} \cdot u = 1 \cdot u$ and $A^+_{\Gamma_i} \cdot v = 1 \cdot v$ which is possible if u and v are scalar multiples of each other and in this case, u and v are linearly dependent, which is a contradiction. Hence, the algebraic multiplicity of 1 is one. □

Result 4.5. The algebraic multiplicity of 1 as an eigenvalue of A^+_{Γ} is the number of components of the digraph $\Gamma(n,k)$.

Proof. Let A^+_{Γ} be the out-adjacency matrix of the digraph $\Gamma(n,k)$ where $|V(\Gamma(n,k))| = n$. Suppose $\Gamma(n,k)$ has s-components $\Gamma_1, \Gamma_2, \cdots, \Gamma_s$ with their out-adjacency matrices $A^+_{\Gamma_1}, A^+_{\Gamma_2}, A^+_{\Gamma_3}, \cdots, A^+_{\Gamma_s}$ respectively. Also, let $|V(\Gamma_i(n,k))| = n_i, 1 \leq i \leq s$ such that $\sum_{i=1}^{s} n_i = n$. Then we have

$$A^+_{\Gamma} = \begin{bmatrix} A^+_{\Gamma_1} & 0 & 0 & \cdots & 0 \\ 0 & A^+_{\Gamma_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A^+_{\Gamma_s} \end{bmatrix}$$

By Result 4.4., each block matrices $A^+_{\Gamma_1}, A^+_{\Gamma_2}, \cdots, A^+_{\Gamma_s}$ has eigenvalue 1 with algebraic multiplicity 1.

Also, we have

$$det(A^+_{\Gamma} - \lambda I_n) = det(A^+_{\Gamma_1} - \lambda I_{n_1}) \cdot det(A^+_{\Gamma_2} - \lambda I_{n_2}) \cdot det(A^+_{\Gamma_3} - \lambda I_{n_3}) \cdots det(A^+_{\Gamma_s} - \lambda I_{n_s})$$

So, the algebraic multiplicity of 1 for the out-adjacency matrix A^+_{Γ} is the sum of the algebraic multiplicities of 1 for each $A^+_{\Gamma_1}, A^+_{\Gamma_2}, \cdots, A^+_{\Gamma_s}$. Then this sum is $\underbrace{1+1+1+\cdots+1}_{s-terms} = s$ (as $\Gamma(n,k)$ has s -components). This shows that the algebraic multiplicity of 1 as an eigenvalue of A^+_{Γ} is s, which is the number of components of the digraph $\Gamma(n,k)$. □

5. Conclusion

we introduced the adjacency matrix of the power digraph $\Gamma(n,k)$, defining the out-adjacency matrix (A_Γ^+) and the in-adjacency matrix (A_Γ^-). We demonstrated that these matrices are singular if certain conditions are met and discussed the spectral properties of $\Gamma(n,k)$. Additionally, we proved that the algebraic multiplicity of 1 as an eigenvalue of (A_Γ^+) corresponds to the number of components in the digraph.

Conflicts of Interest The authors declare no conflict of interest.

References

[1] A. Adbollahi, Determinants of Adjacency matrices of Graph, *Transactions on Combinatorics*, **1** no. 4 (2012) 9-16.

[2] R. B. Bapat, Graphs and Matrices, *Springer* 2010.

[3] E. Blanton Jr., S. Hurd, and J. McCranie, On a digraph defined by square modulo n, *Fibonacci Quarterly*, **34** (1992) 322–334.

[4] S. Bryant, Groups, graphs and Fermat's last theorem, *Amer. Math. Monthly.*, **74** (1967) 152–156.

[5] T. Ju, M. Wu, On iteration digraph and zero-divisor graph of the ring $\mathbb{Z}_n$, *Czechoslovak Mathematical Journal*, **64** (2008) 611–628.

[6] W. Carlip and M. Mincheva, Symmetry of iteration graphs, *Czechoslovak Mathematical Journal*, **58** (2008) 131-145.

[7] P. M. Cvetkovic, M. Doob, H. Sachs A, Spectra of Graphs: Theory and Application, *Academic Press*, 1980.

[8] P. Goswami, S. K. Thakur, and G. C. Ray, The structure of the power digraph connected with the congruence $a^{11} \equiv$ b (mod n), *Proyecciones Journal of Mathematics*, **42** no. 2 (2023) 457-477.

[9] W. Y. Jiang, T. G. Hua, The square mapping graphs of the ring $\mathbb{Z}_n$ [i], *Journal of Math (PRC)*, **36** no. 4 (2016) 676-682.

[10] C. Lucheta, E. Miller, and C. Reiter, Digraphs from Powers modulo p, *Fibonacci Quart.*, **34** (1996) 226-239.

[11] M. Haris Mateen, and M. Khalid Mahmood, Power Digraphs Associated with the Congruence $x^k \equiv$ y (mod n), *Punjab Univ. j. math.*, **51** (2019) 93-102.

[12] E. A. Osba, S. A. Addasi and N.A. Jaradeh, Zero divisor graph for the ring of Gaussian integers modulo n, Taylor & Francis, *Communication in Algebra*, **36** (2008) 3865-3877.

[13] M. Rahmati, Some digraphs attached with congruence $x^k \equiv$ y (mod n), *Journal of Mathematical Extension*, **11** no. 1 (2017) 47-56.

[14] T. D. Rogers, The graph of the square mapping on the prime fields, *Discrete Math.*, **148** (1996) 317-324.

[15] J. Skowronek-Kaziow, Some digraphs arising from number theory and remarks on the zero-divisor graph of the ring $\mathbb{Z}_n$, *Information Processing Letters*, **108** (2008) 165-169.

[16] J. Skowronek-Kaziow, Z. Gora, Properties of digraphs connected with some congruence relations, *Czechoslovak Mathematical Journal*, **59** no. 134 (2009) 39-49.

[17] L. Somer, and M. Krizek, On a connection of number theory with graph theory, *Czechoslovak Mathematical Journal*, **54** (2004) 465-485.

[18] L. Somer, and M. Krizek, Structure of digraphs associated with quadratic congruences with composite moduli, *Discrete Mathematics*, **306** (2006) 2174-2185.

[19] L. Szalay, A discrete iteration in number theory, *BDTF Tud. Közl.*, **8** (1992) 71-91 (in Hungarian).

[20] S. K. Thakur, P. Goswami, and G. C. Ray, Enumeration of cyclic vertices and components over the congruence $a^{11} \equiv$b (mod n), *Notes on Number Theory and Discrete Mathematics*, **29** no. 3 (2023) 525-537.

[21] S. K. Thakur, P. Goswami, and G. C. Ray, Some results on the degree of vertices of the digraphs $\Gamma(n,2)$ and its complement digraph $\overline{\Gamma(n,2)}$, (Communicated).

[22] C. Vasudeva, Graph Theory with Applications, *New Age International (p) Limited Publishers*, ISBN 81-224-1737-X.