Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 6s (2024)

Second Hankel Inequality for Certain Common Subclass of Classes of Starlike and Convex Functions

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Article History:

Received: *25-05-2024*

Revised: *28-06-2024*

Accepted: *20-07-2024*

Abstract:

In this paper we will define a new class S* C_{sin} (r) which is subordinate to function 1+sin z, we will find the Fekete - Szegö inequality for this class along with Fekete - Szegö inequality of the functions of this class defined S* C_{sin} (r,Θ) through Poisson distribution. Further we have solved the second Hankel determinant of this new class.

Keywords: Geometric function, Analytic univalent functions, Poisson distribution, subordination, Fekete - Szegö inequality, Upper bounds, Hankel determinant, coefficient inequalities.

1. Introduction

The class of all the analytic functions in a unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, whose Taylor's series expansion is of the form

$$h(z) := z + \sum_{n=2}^{\infty} a_n z^n = z + a_2 z^2 + a_3 z^3 + a_4 z^4 \cdots \ \forall z \in \mathbb{D} \tag{1}$$

and normalized by the conditions: $h(0) = 0, h'(0) = 1$, is denoted by $\mathcal{A}$.

Let $\mathcal{S}$ denotes a subclass of $\mathcal{A}$ of all univalent analytic functions In the unit disk $\mathbb{D}$. The class of Analytic -Univalent functions, with Taylor's series expansion of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{2}$$

in $\mathbb{D}$, such that $\Re e(P(z)) > 0$ is denoted by $\mathcal{P}$

In 1916 a German Mathematician Ludwig Bieberbach proposed a conjecture on the coefficients of analytic functions of from (1) in $\mathcal{S}$, i.e

$$|a_n| \leq n, \qquad n \in \mathbb{N}$$

This conjecture was known by the name of Bieberbach conjecture (1916) [2]. Until 1985 this conjecture was considered as very challenging problem in Geometric function theory of Complex Analysis.

after 69 years of this conjecture, a French-American Mathematician *Louis de Branges de Bourcia*(1985) solved this conjecture [3].Before de Branges's proof many scholars around the world tried to prove or disprove this conjecture, as a consequences of their these efforts they found multiple subfamilies of class $\mathcal{S}$.

The most common subfamilies of $\mathcal{S}$ are convex, star-like and close-to-convex functions whose set builder form is given by

$$C:=\left\{h\in\mathcal{S}:\Re e\left(\frac{(z(h'(z))'}{h'(z)}\right)>0,\forall z\in\mathbb{D}\right\}$$

$$S^*:=\left\{h\in\mathcal{S}:\Re e\left(\frac{zh'(z)}{h(z)}\right)>0,\forall z\in\mathbb{D}\right\}$$

$$R:=\{h\in\mathcal{S}:\Re e[h'(z)]>0,\forall z\in\mathbb{D}\}$$

Two functions h and g in $\mathcal{A}, h$ is said to be subordinated to g, or written as $h(z)\prec g(z)$, if we have a Schwarz functions $\omega(z)$ analytic over $\mathbb{D}$ with $\omega(0)=0$ and also $|\omega(z)|<1$, such that

$$h(z)=g\big(\omega(z)\big)\ \forall z\in\mathbb{D}$$

, but if function $g(z)$ is univalent in $\mathbb{D}$, then

$$h(z)\prec g(z) \text{ iff } h(0)=g(0) \text{ and } h(\mathbb{D})\subset g(\mathbb{D})$$

Ma and Minda [4] introduced two classes of analytic functions which are.

$$S^*(\phi):=\left\{h\in\mathcal{A}:\frac{zh'(z)}{h(z)}\prec\phi(z),\forall z\in\mathbb{D}\right\}$$

and

$$C(\phi):=\left\{h\in\mathcal{A}:1+\frac{zh''(z)}{h'(z)}\prec\phi(z),\forall z\in\mathbb{D}\right\}$$

The function $\phi(z)$ is an univalent analytic function with positive real part in the unit disk $\mathbb{D}$ such that $\phi(0)=1,\phi'(0)>0$ where ϕ maps the open unit disk onto a region starlike with respect to 1 and symmetric with respect to the real axis, several other classes can be formed by varying the function ϕ, some of the examples are as follows

- When $\phi=e^z$, this class is denoted by S_e^*, check out [5, 6] for more Details.
- When $\phi=1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$, we get a new class, for further details see [7]
- When $\phi=\frac{1+Cz}{1+Dz}(-1\leq D<<C\leq 1)$, we get the class $S^*(C,D)$. See [8] for more Details.
- When $\phi=\cosh(z)$, this new class is denoted by $S_{\cosh}^*$ see.[9]

- When $\phi = 1 + \sin(z)$, the class is denoted by $S^*_{\sin}$, For more details see [10, 11]

C. Pommerenke (1966-67), [12, 13] stated the p^{th} Hankel determinant for $p \geq 1$ and $n \geq 1$ where $p, n \in \mathbb{N}$ of functions h of form ??is defined as

$$\mathcal{H}_{(p,n)}(h) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+p-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+p} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+p-1} & a_{n+p} & \cdots & a_{n+2p-2} \end{vmatrix}$$

In Geometric function theory of Complex analysis finding upper bounds of Hankel determinant of various subfamilies of $\mathcal{A}$ is a widely famous and an interesting problem. Noonan(1976) and Noor(1983) [14, 15] studied the growth rate of $\mathcal{H}_{(p,n)}$ for fixed values of p and n, as $n \to \infty$ of different subfamilies of the univalent function of class $\mathcal{S}$.

where

$$\mathcal{H}_{(2,2)}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2$$

From past many years, a huge collection of research papers have been dedicated for finding the upper bounds for various orders of Hankel determinant, some recent work on second, third and fourth - order Hankel determinants see [[16] - [21]], Recently, Cho et al. [10] introduced the following function class $S^*_{\sin}$

$$S^*_{\sin} := \left\{ h \in \mathcal{A} : 1 + \frac{zh'(z)}{h(z)} \prec 1 + \sin\ z, \forall z \in \mathbb{D} \right\}$$

Various researchers established Fekete - Szegö inequality for various classes afterwards ([25] - [31])

. Lets define a new subclass $S^*C_{\sin}\,(r)$of $\mathcal{A}$. This subclass contains all those Analytic univalent functions in $\mathcal{A}$ which satisfies,

$$\left(\frac{zh'(z)}{h(z)}\right)^r \left(\frac{(zh'(z))'}{h'(z)}\right)^{1-r}, z \in \mathbb{D}$$

$$S^*C_{\sin}(r) := \left\{ h \in \mathcal{A} : \left(\frac{zh'(z)}{h(z)}\right)^r \left(\frac{(zh'(z))'}{h'(z)}\right)^{1-r} \prec 1 + \sin(z) \right\}$$

From above we have

$$S^*C_{\sin}(0) := C_{\sin} = C_{\sin} := \left\{ h \in \mathcal{A} : \left(\frac{(zh'(z))'}{h'(z)}\right) \prec 1 + \sin(z) \right\}$$

and

$$S^*C_{\sin}(1) := S^*_{\sin} := \left\{ h \in \mathcal{A} : \left(\frac{zh'(z)}{h(z)}\right) \prec 1 + \sin(z) \right\}$$

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ISSN: 1074-133X
Vol 31 No. 6s (2024)

2. Preliminary Lemmas.

Let $\mathcal{P}$ denotes the class of analytic functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, z \in \mathbb{D} \tag{3}$$

where$\Re(p(z)) > 0$ in $\mathbb{D}$.

Lemma 2.1. (see [13]) If $p(z) \in \mathcal{P}$, and c_n be the n^{th} cofficients of $P(z)$, then

$$|c_n| \leq 2 \ \textit{for all}\ n \in \mathbb{N} \tag{4}$$

$$|c_2 - \gamma c_1^2| \leq 2max\{1, |2\gamma - 1|\}\ \textit{where}\ \gamma \in \mathbb{C} \tag{5}$$

$$|c_{n+m} - \gamma c_n c_m| \leq 2, \gamma \in [0,1],\ \textit{Where}\ n, m \in \mathbb{N} \tag{6}$$

$$|c_2 - \delta c_1^2| \leq \begin{cases} -4\delta + 2, & \textit{if}\ \delta \leq 0, \\ 2, & \textit{if}\ 0 \leq \delta \leq 1, \\ 4\delta - 2, & \textit{if}\ 1 \leq \delta, \end{cases} \tag{7}$$

Lemma 2.2.(see[17]) If $P(z) \in \mathcal{P}$ then there exists $x, z \in \overline{\mathbb{D}}$ with $|x| \leq 1, |y| \leq 1$, such that

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{8}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - (4 - c_1^2)c_1 x^2 + 2(4 - c_1^2)(1 - |x|^2)y \tag{9}$$

3. Important Theorems

. **Theorem 3.1**. If the function $h(z) \in S^*C_{\sin}(r)$ and is of the form (1), then

$$|a_2| \leq \frac{1}{(2 - r)}$$

$$|a_3| \leq \frac{1}{2(3 - 2r)} \max\left\{1, \left|\frac{r^2 + 5r - 8}{2(-2 + r)^2}\right|\right\}$$

and

$$|a_3 - \delta a_2^2| \leq \frac{1}{2(3 - 2r)} \max\left\{1, \left|\frac{r^2 + 5r + 4\delta(3 - 2r) - 8}{2(r - 2)^2}\right|\right\}$$

Proof. From definition of subordination we have

$$S^*C_{\sin}(r) = \left\{h \in \mathcal{A}: \left(\frac{zh'(z)}{h(z)}\right)^r \left(\frac{(zh'(z))'}{h'(z)}\right)^{1-r} \prec 1 + \sin(z)\right\}$$

$$\left(\frac{zh'(z)}{h(z)}\right)^r \left(\frac{(zh'(z))'}{h'(z)}\right)^{1-r} = \frac{z^r [h'(z)]^{2r-1} \left[(zh'(z))'\right]^{1-r}}{[h(z)]^r} \tag{10}$$

After expanding $\frac{z^r [h'(z)]^{2r-1} \left[(zh'(z))'\right]^{1-r}}{[h(z)]^r}$, in a series form we have

$$=1+a_2(2-r)z+z^2\left(\frac{1}{2}a_2^2(r^2+5r-8)+2a_3(3-2r)\right)$$
$$+z^3\left(a_3a_2(4r^2+11r-18)+\frac{1}{6}a_2^3(-r^3-21r^2-20r+48)\right.$$
$$\left.+3a_4(4-3r)\right)+z^4(a_4a_2(9r^2+19r-32)+2a_3^2(4r^2+4r-9)$$
$$-2a_3a_2^2(r^3+16r^2+5r-24)+\frac{1}{24}a_2^4(r^4+46r^3+371r^2-58r-384)+\cdots \tag{11}$$

Using principal of subordination and expending $1+\sin(\omega(z))$ in series form

$$1+\sin(\omega(z))=1+\frac{c_1z}{2}+\left(\frac{c_2}{2}-\frac{c_1^2}{4}\right)z^2+\frac{1}{48}(5c_1^3-24c_2c_1+24c_3)z^3$$
$$+\frac{1}{32}(-c_1^4+10c_2c_1^2-16c_3c_1-8c_2^2+16c_4)z^4 \tag{12}$$

Comparing (11) and (12), we will get

$$a_2=\frac{-c_1}{2(r-2)} \tag{13}$$

$$a_3=\frac{3c_1^2r^2-4c_2r^2-3c_1^2r+16c_2r-16c_2}{16(r-2)^2(2r-3)} \tag{14}$$

$$a_4=\frac{1}{288(r-2)^3(6r^2-17r+12)}[-52c_1^3r^4+144c_1c_2r^4-96c_3r^4+165c_1^3r^3-780c_1c_2r^3+$$
$$720c_3r^3-205c_1^3r^2+1464c_1c_2r^2-2016c_3r^2+46c_1^3r-1104c_1c_2r+2496c_3r+48c_1^3+$$
$$288c_1c_2-1152c_3] \tag{15}$$

From equation (13) and (14)

$$|a_2|\le\frac{1}{2-r}$$
$$a_3=\frac{1}{4(3-2r)}\left[c_2-\frac{c_1^2(3r^2-3r)}{4(-2+r)^2}\right]$$
$$|a_3|\le\frac{1}{4(3-2r)}|c_2-\delta c_1^2|$$

Where $\delta=\frac{3r^2-3r}{4(-2+r)^2}$ on applying Lemma (3.1)

$$|a_3|=\frac{1}{2(3-2r)}\max\left\{1,\left|\frac{r^2+5r-8}{2(-2+r)^2}\right|\right\}$$

Now

$$a_3-\delta a_2^2=\frac{1}{4(3-2r)}\left[c_2-\frac{c_1^2\left(4\delta(3-2r)+3r^2-3r\right)}{4(r-2)^2}\right] \tag{16}$$

Again using lemma we will Have

$$|a_3-\delta a_2^2|\le\frac{1}{2(3-2r)}\max\left\{1,\left|\frac{r^2+5r+4\delta(3-2r)-8}{2(r-2)^2}\right|\right\}$$

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 6s (2024)

Theorem 3.2 : If function $h \in S^*C_{\sin}$, then

$$|a_3-\delta a_2^2| \le \begin{cases} \frac{-1}{4(3-2r)}\left[\frac{r^2+5r-8}{(-2+r)^2}+\frac{4\delta(3-2r)}{(-2+r)^2}\right] \text{ if } \delta \le \frac{-3r^2+3r}{4(3-2r)} \\ \frac{1}{2(3-2r)} \text{ if } \frac{-3r^2+3r}{4(3-2r)} \le \delta \le \frac{r^2-13r+16}{4(3-2r)} \\ \frac{1}{4(3-2r)}\left[\frac{r^2+5r-8}{(-2+r)^2}+\frac{4\delta(3-2r)}{(-2+r)^2}\right] \text{ if } \frac{(r^2-13r+16)}{4(3-2r)} \le \delta \end{cases}$$

Proof. From (16) we have

$$a_3-\delta a_2^2=\frac{1}{4(3-2r)}\left[c_2-\frac{c_1^2\left(4\delta(3-2r)+3r^2-3r\right)}{4(r-2)^2}\right] \quad (17)$$

this can be written as

$$a_3-\delta a_2^2=\frac{1}{4(3-2r)}[c_2-\delta c_1^2]$$

where$\delta := \frac{c_1^2\left(4\delta(3-2r)+3r^2-3r\right)}{4(r-2)^2}$

Using lemma (3.1) in (17) we have

$$|a_3-\delta a_2^2| \le \begin{cases} \frac{-1}{4(3-2r)}\left[\frac{r^2+5r-8}{(-2+r)^2}+\frac{4\delta(3-2r)}{(-2+r)^2}\right] \text{ if } \delta \le \frac{-3r^2+3r}{4(3-2r)} \\ \frac{1}{2(3-2r)} \text{ if } \frac{-3r^2+3r}{4(3-2r)} \le \delta \le \frac{r^2-13r+16}{4(3-2r)} \\ \frac{1}{4(3-2r)}\left[\frac{r^2+5r-8}{(-2+r)^2}+\frac{4\delta(3-2r)}{(-2+r)^2}\right] \text{ if } \frac{(r^2-13r+16)}{4(3-2r)} \le \delta \end{cases}$$

Function defined with Poisson Distribution

A discrete random variable ξ is said to be poisson distribution it it takes the values $0,1,2,3,4,5 \cdots$ with probabilitiese$e^{-\eta}, \eta\frac{e^{-\eta}}{1!}, \eta^2\frac{e^{-\eta}}{2!}, \eta^3\frac{e^{-\eta}}{3!}, \eta^4\frac{e^{-\eta}}{4!}+\eta^5\frac{e^{-\eta}}{5!}+\cdots$, respectively, where $\eta > 0$ is called the parameter. Thus

$$P(\xi=k)=\frac{\eta^k e^{-\eta}}{k!}, k=0,1,2,3,\cdots$$

A Power series with coefficients from probabilities of Poisson distribution, was introduced by Prowal, [17]

$$P(\eta,z):=z+\sum_{n=2}^{\infty}\frac{\eta^{n-1}}{(n-1)!}e^{-\eta}z^n, \qquad z\in\mathbb{C}$$

Where $\eta > 0$. By using ratio test we can easily establish that radius of convergence of above series is Infinity. Recent works by G. Murugusundaramoorthy, et al and S. Porwal et al [23, 24], let the linear Operator

$$P^{\eta}(z): \mathcal{A} \to \mathcal{A}$$

be given by

$$\begin{aligned} P^{\eta}h(z) &= P(\eta, z) \star h(z) \\ &= z + \sum_{n=2}^{\infty} \frac{\eta^{n-1}}{(n-1)!} e^{-\eta} a_n z^n \\ &= z + \sum_{n=2}^{\infty} \Theta_n(\eta) a_n z^n \end{aligned}$$

Where $\Theta_n = \Theta(\eta) = \frac{\eta^{n-1}}{(n-1)!} e^{-\eta}$ and $\star$ denote the convolution or the hadmard product of the two Series.

Here we will define a new class $S^{*}C_{\sin}(r, \Theta)$ whose functions having power series whose coefficients are probabilities of poisson Distribution Functions defined by Poisson distribution

$$S^{*}C_{\sin}(r, \Theta) := \left\{ h \in \mathcal{A}: \left[\frac{z[P^{\eta}h(z)]'}{P^{\eta}h(z)}\right]^{r} \left[\frac{[z(P^{\eta}h(z))']'}{(P^{\eta}h(z))'}\right]^{1-r} \prec 1 + \sin(z) \right\}$$

Theorem 3.3. Let $0 \le r \le 1, \delta \in \mathbb{C}$ if $h \in S^{*}C_{\sin}(r, \Theta)$, and $P^{\eta}h(z) = z + \Theta_2 a_2 z^2 + \Theta_3 a_3 z^3 + \Theta_4 a_4 z^4 + \Theta_5 a_5 z^5 + \cdots$ then we have

$$|a_3 - \delta a_2^2| \le \frac{1}{2(3-2r)\Theta_3} \max\left\{1, \left|\frac{3r^2 - 3r}{2(r-2)^2} - \frac{4\delta(2r-3)\Theta_3}{2(r-2)^2\Theta_2^2}\right|\right\}$$

Proof. We have $h \in S^{*}C_{\sin}(r, \Theta)$ which is defined as $P^{\eta}h(z) = z + \Theta_2 a_2 z^2 + \Theta_3 a_3 z^3 + \Theta_4 a_4 z^4 + \Theta_5 a_5 z^5 + \cdots$. From the definition we Have

$$\left[\frac{z[P^{\eta}h(z)]'}{P^{\eta}h(z)}\right]^{r} \left[\frac{[z(P^{\eta}h(z))']'}{(P^{\eta}h(z))'}\right]^{1-r} = 1 + \sin(w(z))$$

expansion of $\left[\frac{z[P^{\eta}h(z)]'}{P^{\eta}h(z)}\right]^{r} \left[\frac{[z(P^{\eta}h(z))']'}{(P^{\eta}h(z))'}\right]^{1-r}$ is

$$\begin{aligned} &1 + a_2\Theta_2(2-r)z + z^2\left(\frac{1}{2}a_2^2\Theta_2^2(r^2+5r-8) + 2a_3\Theta_3(3-2r)\right) \\ &+z^3\left(a_2a_3\Theta_3\Theta_2(4r^2+11r-18) + \frac{1}{6}a_2^3\Theta_2^3(-r^3-21r^2-20r+48) + 3a_4\Theta_4(4-3r)\right) \\ &+z^4(a_2a_4\Theta_4\Theta_2(9r^2+19r-32) + 2a_3^2\Theta_3^2(4r^2+4r-9) + \cdots \end{aligned} \tag{18}$$

comparing above with expansion of $1 + \sin(w(z))$

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ISSN: 1074-133X
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$$1+\frac{c_1 z}{2}+\left(\frac{c_2}{2}-\frac{c_1^2}{4}\right)z^2+\frac{1}{48}(5c_1^3-24c_2c_1+24c_3)z^3$$
$$+\frac{1}{32}(-c_1^4+10c_2c_1^2-16c_3c_1-8c_2^2+16c_4)z^4$$
$$+\frac{(c_1^5-480c_2c_1^3+1200c_3c_1^2+1200c_2^2c_1-1920c_4c_1-1920c_2c_3+1920c_5)z^5}{3840}+\cdots$$

We will get

$$a_2=\frac{-c_1}{2\Theta_2(r-2)}$$
$$a_3=\frac{3c_1^2r^2-4c_2r^2-3c_1^2r+16c_2r-16c_2}{16\Theta_3(r-2)^2(2r-3)}$$
$$a_4=\frac{1}{288(r-2)^3\Theta_4(6r^2-17r+12)}[-52c_1^3r^4+144c_1c_2r^4-96c_3r^4+165c_1^3r^3z$$
$$-780c_1c_2r^3+720c_3r^3-250c_1^3r^2+1464c_1c_2r^2-2016c_3r^2+46c_1^3r-1104c_1c_2r+2496c_3r+48c_1^3+288c_1c_2-1152c_3] \tag{19}$$

From above we will have

$$\left[\frac{z[P^\eta h(z)]'}{P^\eta h(z)}\right]^r\left[\frac{[z(P^\eta h(z))']'}{(P^\eta h(z))'}\right]^{1-r}=1+\sin(w(z))$$
$$a_3-\delta a_2^2=\frac{1}{4(3-2r)\Theta_3}\left[\mathrm{c}_2-\frac{c_1^2\left(4\delta(3-2r)\Theta_3+(3r^2-3r)\Theta_2^2r\right)}{4(r-2)^2\Theta_2^2}\right] \tag{20}$$

From lemma (3.1) we have

$$|a_3-\delta a_2^2|\le\frac{1}{2(3-2r)\Theta_3}\max\left\{1,\left|\frac{3r^2-3r}{2(r-2)^2}-\frac{4\delta(2r-3)\Theta_3}{2(r-2)^2\Theta_2^2}\right|\right\}$$

Theorem 3.4. Let $0\le r\le 1$and $P^\eta h(z)=z+\Theta_2a_2z^2+\Theta_3a_3z^3+\Theta_4a_4z^4+\Theta_5a_5z^5+\cdots$, with $\delta\in\mathbb{R}$ Then

$$|a_3-\delta a_2^2|\le\begin{cases}\dfrac{-1}{4(3-2r)\Theta_3}\left[\dfrac{r^2+5r-8}{(-2+r)^2}+\dfrac{4\delta(3-2r)\Theta_3}{(-2+r)^2\Theta_2^2}\right] & \text{if } \delta\le\dfrac{(-3r^2+3r)\Theta_2^2}{4(3-2r)\Theta_3}\\ \dfrac{1}{2(3-2r)\Theta_3} & \text{if } \dfrac{(-3r^2+3r)\Theta_3}{4(3-2r)\Theta_2^2}\le\delta\le\dfrac{(r^2-13r+16)\Theta_2^2}{4(3-2r)\Theta_3}\\ \dfrac{1}{4(3-2r)\Theta_3}\left[\dfrac{r^2+5r-8}{(-2+r)^2}+\dfrac{4\delta(3-2r)\Theta_3}{(-2+r)^2}\right] & \text{if } \dfrac{(r^2-13r+16)\Theta_2^2}{4(3-2r)\Theta_3}\le\delta\end{cases}$$

Proof. From (20) and along with using lemma (3.1) we will get the desired Result.

Second Hankel inequality for $h\in S^*C_{\sin}$

Theorem 3.5. If the function $h\in S^*C_{\sin}(r)$ then

$$|a_2a_4-a_3^2|\le\frac{1}{4(3-2r)^2}$$

Proof. From (15), we Have

$$a_4 = \frac{1}{288(r-2)^3(6r^2-17r+12)}[-52c_1^3r^4 + 144c_1c_2r^4 - 96c_3r^4 + 165c_1^3r^3 - 780c_1c_2r^3 + 720c_3r^3$$
$$-205c_1^3r^2 + 1464c_1c_2r^2 - 2016c_3r^2 + 46c_1^3r - 1104c_1c_2\,r + 2496c_3\,r + 48c_1^3 + 288c_1c_2 - 1152c_3 \quad (21)$$

Using (13), (14) and (15) we will have

$$a_2a_4 - a_3^2 = \frac{1}{2304}\left[-\frac{24c_2c_1^2(21r^3-77r^2+90r-36)}{(3-2r)^2(r-2)^2(3r-4)} + \frac{c_1^4(173r^5-1134r^4+2729r^3-2504r^2+168r+576)}{(3-2r)^2(r-2)^4(3r-4)} + \frac{192c_3c_1}{(r-2)(3r-4)} - \frac{144c_2^2}{(3-2r)^2}\right]$$

Without loss of generality we can say That $c := c_1$, where $|c_1| \le 2$ and substituting the values of c_2 and c_3 we have

$$(a_2a_4 - a_3^2) = \frac{1}{2304}\left[\frac{96(1-|x|^2)(4-c^2)cy}{(r-2)(3r-4)} - x^2(4-c^2)\left(\frac{48c^2}{(r-2)(3r-4)} + \frac{36(4-c^2)}{(3-2r)^2}\right) - \frac{12x(4-c^2)c^2(12-6r-13r^2 7r^3)}{(3-2r)^2(r-2)^2(3r-4)} + \frac{c^4(5r^5+78r^4-607r^3+1816r^2-2424r+1152)}{(3-2r)^2(r-2)^4(3r-4)}\right]$$

Replacing $|x|$ by b and using triangular inequality and the inequality $|y| \le 1$ in above we will get

$$|a_2a_4 - a_3^2| \le \frac{1}{2304}\left[\frac{96(1-b^2)(4-c^2)c}{(r-2)(3r-4)} + b^2(4-c^2)\left(\frac{48c^2}{(r-2)(3r-4)} + \frac{36(4-c^2)}{(3-2r)^2}\right) + \frac{12b(4-c^2)c^2(12-6r-3r^2+7r^3)}{(3-2r)^2(r-2)^2(3r-4)} + \frac{c^4(5r^5+78r^4-607r^3+1816r^2-2424r+1152)}{(3-2r)^2(r-2)^4(4-3r)}\right]$$

Let us assume that above is

$$I(c,b) = \frac{1}{2304}\left[\frac{96(1-b^2)(4-c^2)c}{(r-2)(3r-4)} + b^2(4-c^2)\left(\frac{48c^2}{(r-2)(3r-4)} + \frac{36(4-c^2)}{(3-2r)^2}\right) + \frac{12b(4-c^2)c^2(12-6r-13r^2+7r^3)}{(3-2r)^2(r-2)^2(3r-4)} + \frac{c^4(5r^5+78r^4-607r^3+1816r^2-2424r+1152)}{(3-2r)^2(r-2)^4(3r-4)}\right]$$

We have to maximize the function $I(c,b)$for $(c,b) \in [0,2] \times [0,1]$, now differentiation the above functions partially with respect to b we will have

$$\frac{\partial I}{\partial b} = \frac{1}{2304}\left[\frac{-192(4-c^2)c}{(r-2)(3r-4)} + 2b(4-c^2)\left(\frac{48c^2}{(r-2)(3r-4)} + \frac{36(4-c^2)}{(3-2r)^2}\right) + \frac{12(4-c^2)c^2(12-6r-13r^2+7r^3)}{(3-2r)^2(r-2)^2(3r-4)}\right]$$

For $0 \le b \le 1$ and for any $c \in [0,2]$, we can easily check that $\frac{\partial I}{\partial b} > 0$, from this we can conclude that $I(c,b)$ is an increasing function of c, and it will attain maximum value at $b = 1$.

By putting $b = 1$ in above we will have

$$I(c,1)=E(c)=\frac{1}{2304}\left[(4-c^2)\left(\frac{48c^2}{(r-2)(3r-4)}+\frac{36(4-c^2)}{(3-2r)^2}\right)\right.$$
$$\left.+\frac{12(4-c^2)c^2(12-6r-13r^2+7r^3)}{(3-2r)^2(r-2)^2(3r-4)}+\frac{c^4(5r^5+78r^4-607r^3+1816r^2-2424r+1152)}{(3-2r)^2(r-2)^4(3r-4)}\right]$$

We have

$$E'(c)$$
$$=\frac{c^3(-5r^5+150r^4-953r^3+2120r^2-1896r+576)-24c(r-2)^2(9r^3-29r^2+30r-12)}{576(3-2r)^2(r-2)^4(3r-4)}$$

From $E'(c)=0$ we will have $c=0$

$$E''(0)=-\frac{9r^3-29r^2+30r-12}{24(3-2r)^2(r-2)^2(3r-4)}$$

that we can easily conclude that $E''(0)$ is Negative, which implies that at $c=0$ the function will attain its maxima.

$$|a_2a_4-a_3^2|\le\frac{1}{4(3-2r)^2}$$

Corollary

When $r=0$, we have $h\in S^*C_{\sin}(0):=C_{\sin}$ and we get

$$|a_2a_4-a_3^2|\le\frac{1}{36}$$

When $r=1$ we have $h\in S^*C_{\sin}(1):=S^*_{\sin}$, and we get

$$|a_2a_4-a_3^2|\le\frac{1}{4}$$

4. Conclusions

In this paper we have introduced a new class $S^*C_{\sin}(r)$, where $(0\le r\le 1)$, and have worked on the Fekete-Szegö inequality along with upper bound of second Hankel determinant. We have stated the Fekete-Szegö inequality related to Poisson distribution of this new class. As $h\in S^*C_{\sin}(r)$ by fixing parameter $r=0$function $h\in C_{\sin}$, and when we fix $r=1$ then we have $h\in S^*_{\sin}$. Our this research paper is an extended work of research article [32], readers and interested scholars can extend our work further by working on higher order of Hankel determinant and coefficients inequalities on this newly introduced class.

5. Acknowledgment

The authors are very much thankful to the anonymous Editor and Referees for their valuable suggestions to improve the paper.

Funding: Not applicable

Conflict of interest: Not applicable

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