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Outcomes for The Analysis of The Existence of Fuzzy Fractional Differential Equations

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Abstract:

In this article, a Cauchy problem for a fuzzy q-fractional differential equation of order α has been considered.The results for generalized Hukuhara q-differentiablity of a fuzzy function are established. This work has led to the study of existence and uniqueness of the fuzzy function with Caputo Hukuhara q-differentiability and generalized Banach fixed point theorem. A crucial qualitative property of the differential equation, which is continuous dependence on initial conditions and the functions involved is analysed. An illustrative example is given which ensures the result.

Keywords: Fuzzy Fractional Differential Equations, Fuzzy q-Fractional Differential Equation, Existence of Solution, Fixed point theorem, Continuous dependence.

1 Introduction

The generalized form of differential equations is termed as fractional differential equations (FDE), which emerged as an application of fractional calculus. FDE engages as an important tool to perform many physical phenomena which includes viscoelasticity, control theory of dynamical systems, optics and signal processing etc. FDE attracted many scientists and mathematicians. The existence theory of fractional differential equations of nonlinear type finds its consistent study by many researchers.

In recent years, Cauchy problem for nonlinear FDE has become the most interesting field to study the existence, uniqueness, long time behavior etc. Fixed point theory has created extensive interest in the researchers to utilize it for the existence of solution of the fractional differential equation. Fractional derivative is derived mostly by two operators Riemann-Liouville and Caputo operators. But the Caputo operator has advantages for initial value problems.

The core attention towards the analysis of existence and uniqueness of the solution of the FDEs is prolific to study. In particular, the study of the existence of solution to Fuzzy Fractional Differential Equation is quite interesting. In addition, fixed point theory is an inevitable tool to study the existence and uniqueness of some mathematical models, which has been studied since many decades. Research is progressing extensively to study the Cauchy problems, with the utilization of fixed point theory approach.

Further, the study of uncertainty becomes the most interesting study in these days. Also the dynamics itself is uncertain due to its dependence on time. Basically, the proposed ideas are a generalisation of the theory and solution of fuzzy differential equations . However, the authors considered fuzzy fractional differential equations under the Riemann-Liouville H-derivative[4]. Again, it requires a quantity of fractional H-derivative of an unknown solution at the fuzzy initial point. Also, the q-calculus appears to be the connection between mathematics and physics. It has several applications in the areas like quantum theory, hypo geometric functions and electronics[1, 3, 8, 9]. Z.Noeiaghdam et.al and Obaidat et.al [6, 7] study the existence of q-fractional differential equations with uncertainty. The existence of solution of a fractional q-integro differential equation with q-nonlocal condition has been examined by Ibrahim et.al [5]. Z.Noeiaghdam et.al analysed the fuzzy q-derivative and fuzzy q-fractional derivative in Caputo sense and used generalized Hukuhara difference. Fuzzy q-fractional differential equation solving creates strenuous attention to the researchers [8]. Motivated by those, this work studies the existence of integro differential equations

$$^{C}D^{\alpha}(\phi(\tau)) = \lambda\phi(\tau) + f(\tau, \phi(\tau), \mathcal{G}\phi(\tau), \mathcal{S}\phi(\tau)), \tag{1.1}$$

$$\phi(\tau_0) = \phi_0 \tag{1.2}$$

where

$$\mathcal{G}\phi(\tau) = \int_{\tau_0}^{\tau} \mathcal{K}(\tau, s)\phi(s)ds, \quad \mathcal{K} \in C[\mathcal{D}, \mathbb{R}^+],$$

$$where \quad \mathcal{D} = \{(\tau, s) \in \mathbb{R}^2 : 0 \le s \le \tau \le \mathbb{T}\},$$

$$\mathcal{S}\phi(\tau) = \int_{\tau_0}^{\tau} \mathcal{H}(\tau, s)\phi(s)ds, \quad \mathcal{H} \in C[\mathcal{D}_0, \mathbb{R}^+],$$

$$where \quad \mathcal{D}_0 = \{(\tau, s) \in \mathbb{R}^2 : 0 \le \tau, s \le \mathbb{T}\}$$

where $0 < \alpha \le 1, \tau \in [\tau_0, \mathbb{T}]$ λ is a constant and $f: \mathbb{T}_q \times \mathbb{R}_{\mathbb{F}} \times \mathbb{R}_{\mathbb{F}} \times \mathbb{R}_{\mathbb{F}} \to \mathbb{R}_{\mathbb{F}}$ are continuous and $f(\tau, \phi(\tau), G\phi(\tau), S\phi(\tau))$ satisfies the following condition

$$(H1)\|f(\tau, \phi, \mathcal{G}\phi, \mathcal{S}\phi) - f(\tau, \psi, \mathcal{G}\psi, \mathcal{S}\psi)\| \le \mathcal{L}[\|\phi - \psi\| + \|\mathcal{G}u - \mathcal{G}v\| + \|\mathcal{S}u - \mathcal{S}v\|]$$

with $0 < \mathcal{L} < 1$.

2 Preliminaries

2.1 Basic Fuzzy Concepts

Definition 2.1 *[2] Let $\mathbb{R}_{\mathbb{F}}$ be the set of all fuzzy valued functions. Let $\phi: \mathbb{R}_{\mathbb{F}} \to [0,1]$ be satisfying the following conditions.*

1. ϕ is upper semi-continuous on $\mathbb{R}$
2. ϕ is fuzzy convex
3. ϕ is normal
4. Closure of $\{t \in \mathbb{R}_{\mathbb{F}} | \phi(t) > 0\}$ is compact.

Let $\mathbb{R}_{\mathbb{F}}$ be the space of above said fuzzy numbers.

Also, for $0 < r \le 1$, denote $\phi(r) = \{t \in \mathbb{R}^n | \phi(t) > r\}$, which is known as the r-level set, which is closed for all $r \in [0,1]$. In $\mathbb{R}_{\mathbb{F}}$, for arbitrary $\phi, \psi \in \mathbb{R}_{\mathbb{F}}$ and a scalar k, define the following binary operations, namely, addition and scalar multiplication, respectively.

Addition:

$$(\phi \oplus \psi)(r) = \phi(r) + \psi(r)$$

Scalar multiplication:

$$(k \odot \phi)(r) = \left(k\underline{\phi}(r), k\overline{\phi}(r)\right),\ k \geq 0$$

$$(k \odot \phi)(r) = \left(k\overline{\phi}(r), k\underline{\phi}(r)\right),\ k \leq 0$$

2.2 Hukuhara Difference

We utilise Hukuhara difference from [?], for $\phi, \psi \in \mathbb{R}_{\mathbb{F}}$, as follows.

i.e., $\phi = \psi + \omega$, if $w \in \mathbb{R}_{\mathbb{F}}$.

Then ω is called as the Hukuhara difference of ϕ and ψ.

Also, the generalized Hukuhara difference, denoted as ${}_gH$ in short, is also defined as,

$$\phi \ominus_{gH} \psi = \omega \Leftrightarrow \begin{cases} (i)\phi = \psi + \omega \\ (ii)\phi = \psi + (-1)\omega \end{cases}$$

It is obvious that (i) and (ii) are true, if and only if ω is a crisp number.

2.3 q Calculus

The usual derivative of a function $'f(\tau)'$ is defined as $\underbrace{lim}_{\tau \to \tau_0} \frac{f(\tau)-f(\tau_0)}{\tau-\tau_0}$. Now we can study q-derivative of f. For that, we use the following definitions from [?]. Let $\mathbb{T}_q$ be the time scale, for $0 < q < 1$.

$$\mathbb{T}_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$$

$$\mathbb{T}_q{}^n = \{q^{\alpha+n} : n \in \mathbb{Z}\} \cup \{0\}$$

Consider an arbitrary function $f: \mathbb{T}_q \to \mathbb{R}$.

Its q-differential is

$$d_q f(\tau) = f(\tau) - f(q\tau)$$

Then the q-derivative of f is given by,

$$D_q f(\tau) = \frac{d_q f(\tau)}{d_q \tau} = \frac{f(\tau)-f(q\tau)}{(1-q)\tau}, \quad \tau \in \mathbb{T}_q - \{0\}$$

The q-gamma function denoted by $\Gamma_q(.)$ can be defined as,

$$\Gamma_q(\alpha) = \frac{(1-q)_q^{\alpha-1}}{(1-q)^{\alpha-1}}, \quad \alpha \in \mathbb{R} - \{0\} \cup \mathbb{Z},\ 0 < q < 1,$$

which is satisfied in the following relation

$$\Gamma_q(\alpha+1) = \frac{(1-q)^{\alpha}}{(1-q)} \Gamma_q(\alpha)$$

$$\Gamma_q(\alpha) = 1,\ \alpha > 0$$

Definition 2.2 *[6] Let for* $\alpha > 0,\ f(\tau)$ *is a* q*-integrable function, the fractional* q*-integral of order* α *is defined by,*

$$ {}_{q}I_{a}^{\alpha} f(\tau) = \frac{1}{\Gamma_{q}(\alpha)} \int_{a}^{\tau} (\tau - qs)_{q}^{\alpha-1} f(s) d_{q}s $$

Definition 2.3 *[6] For an arbitrary fuzzy valued function* $f: \mathbb{T}_{q} \to \mathbb{R}_{\mathbb{F}}$*, the* q*-differential by* ${}_{g}H$ *difference is* ${}_{g}Hd_{q}f(\tau) = f(\tau) \ominus_{gH} f(q\tau)$*. If* ${}_{g}Hd_{q}f(\tau) = f(\tau) \ominus_{gH} f(q\tau)$ *exists, the fuzzy generalized Hukuhara* q*-derivative of* f *is defined by,*

$$ {}^{F}D_{q}f(\tau) = \frac{{}_{g}Hd_{q}f(\tau)}{d_{q}\tau} = \frac{f(\tau) \ominus_{gH} f(q\tau)}{(1-q)\tau}, \quad \tau \in \mathbb{T}_{q} - \{0\} $$

Definition 2.4 *[6] Let* $f: \mathbb{T}_{q} \to \mathbb{R}_{\mathbb{F}}$ *be a* q*-integrable function. The fuzzy* q*-fractional integral of order* α *for* f *is defined by,*

$$ {}^{F}I_{a}^{\alpha} f(\tau) = \frac{1}{\Gamma_{q}(\alpha)} \odot \int_{a}^{\tau} (\tau - qs)_{q}^{\alpha-1} \odot f(s) d_{q}s $$

Definition 2.5 *[6] Let* $\forall m,\ {}_{q}^{F}D^{m}f$ *be continuous and* q*-integrable functions in* $\mathbb{T}_{q}$*. The fuzzy Caputo q-fractional derivative of order* α *of fuzzy valued function is defined,*

$$ {}_{q}^{FC}D^{\alpha} f(t) = {}_{a}^{F}I_{a}^{m-\alpha}({}^{F}D_{q}^{m}f)(\tau) $$

$$ = \frac{1}{\Gamma_{q}(m-\alpha)} \odot \int_{a}^{\tau} (\tau - qs)_{q}^{m-\alpha-1} \odot_{q}^{F} D^{m}f(s) d_{q}s $$

where $m-1 < \alpha \leq m,\ m \in \mathbb{N},\ t > a$.

Definition 2.6 *[6] For* $0 < \alpha \leq 1$*, the fuzzy Caputo* q*-fractional* ${}_{g}H$ *derivative will be,*

$$ {}_{q}^{FC}D^{\alpha} f(\tau) = {}_{a}^{F}I_{a}^{1-\alpha}({}^{F}D_{q}f)(\tau) = \frac{1}{\Gamma_{q}(1-\alpha)} \odot \int_{a}^{\tau} (\tau - qs)_{q}^{-\alpha} \odot_{q}^{F} Df(s) d_{q} $$

for $\tau > a$.

Definition 2.7 *[7] For a fuzzy valued function* $f(\tau)$*, where* $\tau \in T_{q}$*,*

the Caputo q-integral of order α is

$$ {}_{q}I_{a}^{\alpha} f(\tau) = \frac{1}{\Gamma_{q}(\alpha)} \int_{a}^{\tau} (\tau - qs)_{q}^{\alpha-1} f(s) d_{q}s $$

Definition 2.8 *[6] For a fuzzy valued function* $f(\tau)$*, where* $\tau \in T_{q}$*,*

$$ {}_{q}I_{a}^{\alpha}({}_{q}^{C}D^{\alpha})(f(\tau)) = f(\tau) - \sum_{k=0}^{m-1} \frac{(\tau - a)_{q}^{k}}{\Gamma_{q}(k+1)} D_{q}^{k} f(a) $$

where $m-1 < \alpha \leq m,\ m \in \mathbb{N}$ and for the special case $0 < \alpha \leq 1$,

$$ {}_{q}I_{a}^{\alpha}({}_{q}^{C}D)(f(\tau)) = f(\tau) - f(a) $$

Definition 2.9 *[6] Let $f:\mathbb{T}_q \to \mathbb{R}_{\mathbb{F}}$ be Caputo differentiable at $\phi_0 \in \mathbb{T}_q$. We say that f is Caputo $[(i)-_gH]$ differentiable at ϕ_0,*

$$\text{if } (i)\,{}^{C}_{gH}D^{\alpha}f(\phi_0;r) = \left[{}^{C}_{gH}D^{\alpha}\underline{f}(\phi_0;r), {}^{C}_{gH}D^{\alpha}\overline{f}(\phi_0;r)\right]$$

and f is Caputo $[(ii)-_gH]$ differentiable at ϕ_0, if

$$(ii)\,{}^{C}_{gH}D^{\alpha}f(\phi_0;r) = \left[{}^{C}_{gH}D^{\alpha}\overline{f}(\phi_0;r), {}^{C}_{gH}D^{\alpha}\underline{f}(\phi_0;r)\right], \text{ for } 0 \le r \le 1$$

Definition 2.10 *The function $E_\alpha(\alpha > 0)$ defined by*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$$

is called Mittag-Leffler function of order α.

Lemma 2.1 *Suppose that $f:\mathbb{T}_q \to \mathbb{R}_{\mathbb{F}}$ be a fuzzy valued function and f is $_gH$ q-differentiable and q-integrable in $\mathbb{T}_q$, then for $0 < \alpha \le 1$, then*

$${}^{F}_{q}I^{\alpha}_{a}({}^{FC}_{q}D^{\alpha}f(\tau)) = f(\tau) \ominus_{gH} f(a)$$

Lemma 2.2 *Let $v:[\tau_0,T] \to [0,+\infty)$ be a real function and $w(.)$ is a non negative, locally integrable function on $[\tau_0,T]$. Assume that there is a constant α such that $0 < \alpha \le 1$.*

$$v(\tau) \le w(\tau) + a\int_0^{\tau}(\tau-s)^{-\alpha}v(s)ds.$$

Then, there exists a constant $K = K(\alpha)$ such that

$$v(\tau) \le w(\tau) + Ka\int_0^{\tau}(\tau-s)^{-\alpha}w(s)ds$$

for every $\tau \in [\tau_0,T]$.

We consider here the generalized Banach fixed point theorem, which is used to prove the existence results.

Theorem 2.3 *Let U be a nonempty closed subset of a Banach space B, and let $\alpha_n \ge 0$, $n \in \mathbb{N} \cup \{0\}$, be a sequence such that $\sum_{n=0}^{\infty}\alpha_n$ converges. Moreover, let the mapping $F:U \to U$ satisfy the inequality*

$$\|F^n u - F^n v\| \le \alpha_n \|u - v\|$$

$\forall\ n \in \mathbb{N} \cup \{0\}$, and for every $u,v \in U$. Then F has a uniquely defined fixed point u^*.Furthermore, the sequence $\{F^n u_0\}_{n=1}^{\infty}$ converges to the fixed point u^* for every $u_0 \in U$.

3 Results on Fuzzy q-fractional derivative by $_gH$ difference

Let X be a nonempty closed subset of a $\mathbb{R}$. Consider an arbitrary fuzzy valued function $h:X \to \mathbb{R}_F$, where $\mathbb{R}$ is the set of all real numbers, and $[h(\tau)]_r = [\underline{h}(\tau;r), \overline{h}(\tau;r)]$ is called as r-cut or parametric form of the fuzzy valued function f.

The results on Fuzzy $\mathfrak{q}$ -fractional derivative by $_{g}H$ difference of the fuzzy function $f(\tau, u(\tau), \mathcal{G}u(\tau), \mathcal{S}u(\tau))$ has been studied by Z.Noeiaghdam et.al in [?]. For an arbitrary fuzzy valued function $f: \mathbb{T}_{\mathfrak{q}} \to \mathbb{R}_{\mathbb{F}}$, q-differential by $_{g}H$ difference is

$$d_{q,gH} = f(\tau) \ominus_{gH} f(\mathfrak{q}\tau)$$

if $f(\tau) \ominus_{gH} f(\mathfrak{q}\tau)$ exists. Then the fuzzy generalized Hukuhara $\mathfrak{q}$ derivative of f is defined by

$$D_{F,q} f(\tau) = \frac{d_{q,gH}}{d_q \tau} = \frac{f(\tau) \ominus_{gH} f(\mathfrak{q}\tau)}{(1-\mathfrak{q})\tau}, \quad \tau \in \mathbb{T}_{\mathfrak{q}} - 0$$

Lemma 3.1 *The function f is $_{g}H$ differentiable if and only if, $\underline{f}(\tau, \phi(\tau), \mathcal{G}\phi(\tau), \mathcal{S}\phi(\tau); r)$ and $\overline{f}(\tau, \phi(\tau), \mathcal{G}\phi(\tau), \mathcal{S}\phi(\tau))$ are differentiable with respect to τ for all $r \in [0,1]$ and*

$$\begin{aligned} &f'_{gH}(\tau, \phi(\tau), \mathcal{G}\phi(\tau), \mathcal{S}\phi(\tau); r) = \\ &\Big[\min\Big(\underline{f}(\tau, \phi(\tau), \mathcal{G}\phi(\tau), \mathcal{S}\phi(\tau); r), \overline{f}(\tau, \phi(\tau), \mathcal{G}\phi(\tau), \mathcal{S}\phi(\tau); r)\Big), \\ &\max\Big(\underline{f}(\tau, \phi(\tau), \mathcal{G}\phi(\tau), \mathcal{S}\phi(\tau); r), \overline{f}(\tau, \phi(\tau), \mathcal{G}\phi(\tau), \mathcal{S}\phi(\tau); r)\Big)\Big] \end{aligned}$$

Theorem 3.2 *Let $f: \mathbb{T}_{\mathfrak{q}} \times \mathbb{R}_{\mathbb{F}} \times \mathbb{R}_{\mathbb{F}} \times \mathbb{R}_{\mathbb{F}} \to \mathbb{R}_{\mathbb{F}}$ be a fuzzy valued function on $\mathbb{J}$.*

(a) If f is $\left[(i)-_{gH}\right]$ differentiable at $\phi_0 \in \mathbb{T}_{\mathfrak{q}}$, then f is Caputo $\left[(i)-_{gH}\right]$ differentiable at ϕ_0

(b) If f is $\left[(ii)-_{gH}\right]$ differentiable at $\phi_0 \in \mathbb{T}_{\mathfrak{q}}$, then f is Caputo $\left[(ii)-_{gH}\right]$ differentiable at ϕ_0

Proof.

$$\begin{aligned} C\ D^{\alpha}(f(\tau_0, \phi(\tau_0), \mathcal{G}\phi(\tau_0), \mathcal{S}\phi(\tau_0)); r) &= I^{1-\alpha}(f')(\tau_0, \phi(\tau_0), \mathcal{G}\phi(\tau_0), \mathcal{S}\phi(\tau_0)); r) \\ &= \Big[\frac{1}{\Gamma(1-\alpha)} \int \frac{\underline{f'}(\tau_0, \phi(\tau_0), \mathcal{G}\phi(\tau_0), \mathcal{S}\phi(\tau_0)); r)}{(\tau_0 - s)^{\alpha}}\, ds, \\ &\quad \frac{1}{\Gamma(1-\alpha)} \int \frac{\overline{f'}(\tau_0, \phi(\tau_0), \mathcal{G}\phi(\tau_0), \mathcal{S}\phi(\tau_0)); r)}{(\tau_0 - s)^{\alpha}}\, ds\Big] \\ &= \Big[\ {}^{C}_{gH}D^{\alpha}\underline{f}(\tau_0, \phi(\tau_0), \mathcal{G}\phi(\tau_0), \mathcal{S}\phi(\tau_0)), \\ &\quad {}^{C}_{gH}D^{\alpha}\overline{f}(\tau_0, \phi(\tau_0), \mathcal{G}\ \phi(\tau_0), \mathcal{S}\phi(\tau_0))\Big], \quad 0 \le r \le 1 \end{aligned}$$

Therefore, f is $\left[(i)-_{gH}\right]$ Caputo differentiable. Similarly we can prove that f is $\left[(ii)-_{gH}\right]$ Caputo differentiable.

4 Existence result for fuzzy $\mathfrak{q}$-fractional differential equations

Theorem 4.1 *If the function $f: \mathbb{T}_{\mathfrak{q}} \times \mathbb{R}_{\mathbb{F}} \times \mathbb{R}_{\mathbb{F}} \times \mathbb{R}_{\mathbb{F}} \to \mathbb{R}_{\mathbb{F}}$ is a continuous fuzzy valued function, then the given equation* (1.1) *with the initial condition* (1.2) *is equivalent to the following integral equation*

$\phi(\tau) \ominus_{gH} \phi_0 = \frac{\lambda}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1}\phi(s)d_q s + \frac{1}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1} f(s, \phi(s), \mathcal{G}\phi(s), \mathcal{S}\phi(s))d_q s$ (4.1)

Proof. Let $\phi(\tau)$ be a solution of (1.1).

We claim that $\phi(\tau)$ is also a solution of (4.1).

To assure that, let us consider,

$r(\tau) = \lambda\phi(\tau) + f(\tau, \phi(\tau), \mathcal{G}\phi(\tau), \mathcal{S}\phi(\tau))$.

Then we have, $r(\tau) = {}^C D^{\alpha}\phi(\tau)$.

Assume that here $\phi(\tau)$ is a monotone function for $\tau \in \mathbb{T}_q$. Using Lemma (2.1), we have

$$I^{\alpha}({}^C D^{\alpha})(\phi(\tau)) = \phi(\tau) \ominus_{gH} \phi(\tau_0)$$

$$= \phi(\tau) \ominus_{gH} \phi_0$$

$$\text{Hence } I^{\alpha} r(\tau) = \phi(\tau) \ominus_{gH} \phi_0$$

i.e.,

$$\phi(\tau) \ominus_{gH} \phi_0 = \frac{\lambda}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1}\phi(s)d_q s$$

$$+ \frac{1}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1} f(s, \phi(s), \mathcal{G}\phi(s), \mathcal{S}\phi(s))d_q s$$

The necessary condition is satisfied.

Next, let us consider that $\phi(\tau)$ is a monotone function for $\tau \in \mathbb{T}_q$, such that (4.1) is satisfied. Operating fuzzy Caputo $_{gH}$ derivative of order α, by employing Definition (2.5), on both the sides of equation (4.1), we get

$${}^C D^{\alpha}(\phi(\tau)) = \lambda\phi(\tau) + f(\tau, \phi(\tau), \mathcal{G}\phi(\tau), \mathcal{S}\phi(\tau))$$

for $\tau \in \mathbb{T}_q$. Now, this part is sufficient to prove the theorem.

Theorem 4.2 *Let* $\mathbb{T}_q \geq 0$ *and let* $\epsilon \geq 0$ *be a constant such that* $0 \in [\phi_0 - \epsilon, \phi_0 + \epsilon]$. *Assume that* $f: \mathbb{T}_q \times [\phi_0 - \epsilon, \phi_0 + \epsilon] \times [\phi_0 - \epsilon, \phi_0 + \epsilon] \times [\phi_0 - \epsilon, \phi_0 + \epsilon] \to [\phi_0 - \epsilon, \phi_0 + \epsilon]$ *satisfies the condition* $(H1)$.

$$\kappa = min\left\{\mathbb{T}_q, \left[\frac{\Gamma_q(\alpha+1)\epsilon}{(\epsilon + \|\phi_0\|)(\lambda + L(1 + K^* + H^*) + M}\right]^{\frac{1}{\alpha}}\right\},$$

where $M = Sup_{\tau \in [\tau_0, T]}|f(\tau, 0,0,0)|$. Then the cauchy problem for (1.1) has a unique solution $u: [0, \kappa] \to \mathbb{R}_{\mathbb{F}}$.

Proof. Define the set $\mathbb{B} = \{\phi \in C([\tau_0, \kappa], \mathbb{R}_{\mathbb{F}}): \phi(\tau_0) = \phi_0, \|\phi - \phi_0\| \leq \epsilon\}$.

Since $\phi_0 \in \mathbb{B}$, $\mathbb{B}$ is nonempty. Also, $\mathbb{B}$ is a closed, bounded and convex subset of Banach space $C([0, \kappa], \mathbb{R}_{\mathbb{F}})$.

On $\mathbb{B}$, define an operator $\mathcal{K}$ by

$$\mathcal{K}\phi(\tau) \ominus_{gH} \phi_0 = \frac{\lambda}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - \mathrm{q}s)^{\alpha-1}\phi(s) d_{\mathrm{q}}s$$
$$+ \frac{1}{\Gamma_{\mathrm{q}}(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - \mathrm{q}s)^{\alpha-1} f(s, u(s), \mathcal{G}u(s), \mathcal{S}u(s)) d_{\mathrm{q}}s$$

Now, it needs to prove that $\mathcal{K}$ maps $\mathbb{B}$ to itself.

Consider any $\phi \in \mathbb{B}$ and $\tau \in [0, \kappa]$.

$$\left\|\mathcal{K}\phi(\tau) \ominus_{gH} \phi_0\right\| \leq \frac{\lambda}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - \mathrm{q}s)^{\alpha-1} \|\phi(s)\| d_{\mathrm{q}}s$$
$$+ \frac{1}{\Gamma_{\mathrm{q}}(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - \mathrm{q}s)^{\alpha-1} \|f(s, \phi(s), \mathcal{G}\phi(s), \mathcal{S}\phi(s))\| d_{\mathrm{q}}s$$
$$\leq \frac{\lambda}{\Gamma_{\mathrm{q}}(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - \mathrm{q}s)^{\alpha-1} (\|\phi(s) - \phi_0\| + \|\phi_0\|) d_{\mathrm{q}}s$$
$$+ \frac{1}{\Gamma_{\mathrm{q}}(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - \mathrm{q}s)^{\alpha-1} \left(\left\|f\left(s, \phi(s), \mathcal{G}u(s), \mathcal{S}u(s)\right) - f(s, 0,0,0)\right\|\right) d_{\mathrm{q}}s$$
$$+ \frac{1}{\Gamma_{\mathrm{q}}(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - \mathrm{q}s)^{\alpha-1} \|f(s, 0,0,0)\| d_{\mathrm{q}}s$$

By the assumption $(H1)$ and the definition of $\mathbb{B}$ for any $\tau \in [0, \kappa]$,

$$\|\phi(\tau)\| \leq \|\phi(\tau) - \phi_0\| + \|\phi_0\| \leq \epsilon + \|\phi_0\| \tag{4.2}$$

Consider,

$$\frac{1}{\Gamma_{\mathrm{q}}(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - \mathrm{q}s)^{\alpha-1} (\|f(s, \phi(s), \mathcal{G}\phi(s), \mathcal{S}\phi(s)) - f(s, 0,0,0)\|) d_{\mathrm{q}}s$$
$$\leq \frac{1}{\Gamma_{\mathrm{q}}(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - \mathrm{q}s)^{\alpha-1} \mathcal{L}[\|\phi\| + \|\mathcal{G}\phi\| + \|\mathcal{S}\phi\|] d_{\mathrm{q}}s$$
$$\frac{\mathcal{L}}{\Gamma_{\mathrm{q}}(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - \mathrm{q}s)^{\alpha-1} \|\phi\| (1 + K^* + H^*) d_{\mathrm{q}}s$$

where $max\|\mathcal{G}\phi\| = K^*$ and $max\|\mathcal{S}\phi\| = H^*$ Therefore,

$$\left\|\mathcal{K}\phi(\tau) \ominus_{gH} \phi_0\right\| \leq \frac{\lambda}{\Gamma_{\mathrm{q}}(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - \mathrm{q}s)^{\alpha-1} (\epsilon + \|\phi_0\|) d_{\mathrm{q}}s$$
$$+ \frac{\mathcal{L}}{\Gamma_{\mathrm{q}}(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - \mathrm{q}s)^{\alpha-1} (\epsilon + \|\phi_0\|)$$
$$+ \frac{M}{\Gamma_{\mathrm{q}}(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - \mathrm{q}s)^{\alpha-1} d_{\mathrm{q}}s$$

$$= \frac{(\lambda+\mathcal{L}(1+K^*+H^*))(\epsilon+\phi_0)+M}{\Gamma_q(\alpha)}\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha+1)}(\tau-\tau_0)^\alpha$$

$$\leq \frac{(\lambda+\mathcal{L}(1+K^*+H^*))(\epsilon+\phi_0)+M}{\Gamma_q(\alpha)}\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha+1)}(\kappa)^\alpha$$

$$= \epsilon$$

Hence $\mathcal{K}$ maps $\mathbb{B}$ to itself.

Next we consider that, for $0 \leq \tau_1 \leq \tau_2 \leq \kappa$,

$$\|\mathcal{K}\phi(\tau_1) - \mathcal{K}\phi(\tau_2)\| \leq \frac{\lambda}{\Gamma_q(\alpha)} \odot \left\|\int_{\tau_0}^{\tau_1}(\tau_1 - qs)^{\alpha-1}\phi(s)\, d_q s - \int_{\tau_0}^{\tau_2}(\tau_2 - qs)^{\alpha-1}\phi(s)\, d_q s\right\|$$

$$+\frac{1}{\Gamma_q(\alpha)} \odot \left\|\int_{\tau_0}^{\tau_1}(\tau_1 - qs)^{\alpha-1} f(s,\phi(s),\mathcal{G}u(s),\mathcal{S}u(s))\, d_q s\right.$$

$$\left.-\int_{\tau_0}^{\tau_2}(\tau_2 - qs)^{\alpha-1} f(s,\phi(s),\mathcal{G}\phi(s),\mathcal{S}\phi(s))\, d_q s\right\|$$

Consider

$$\frac{\lambda}{\Gamma_q(\alpha)} \odot \left\|\int_{\tau_0}^{\tau_1}(\tau_1 - qs)^{\alpha-1}\phi(s) d_q s - \int_{\tau_0}^{\tau_2}(\tau_2 - qs)^{\alpha-1}\phi(s) d_q s\right\|$$

$$\leq \frac{\lambda}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau_1}((\tau_1 - qs)^{\alpha-1} - (\tau_2 - qs)^{\alpha-1})\|\phi(s)\| d_q$$

$$+\frac{\lambda}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau_2}((\tau_2 - qs)^{\alpha-1})\|\phi(s)\| d_q s$$

$$\leq \frac{\lambda}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau_1}((\tau_1 - qs)^{\alpha-1} - (\tau_2 - qs)^{\alpha-1})(\|\phi(s)-\phi_0\| + \|\phi_0\|) d_q s$$

$$+\frac{\lambda}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau_2}((\tau_2 - qs)^{\alpha-1})(\|\phi(s)-\phi_0\| + \|\phi_0\|) d_q s$$

$$\leq \frac{\lambda}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau_1}((\tau_1 - qs)^{\alpha-1} - (\tau_2 - qs)^{\alpha-1})(\epsilon + \|\phi_0\|) d_q s$$

$$+\frac{\lambda}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau_2}((\tau_2 - qs)^{\alpha-1})(\epsilon + \|\phi_0\|) d_q s$$

$$\leq \frac{\lambda(\epsilon+\|\phi_0\|)}{\Gamma_q(\alpha)}$$

Consider

$$\frac{1}{\Gamma_q(\alpha)} \odot \left\|\begin{matrix}\int_{\tau_0}^{\tau_1}(\tau_1 - qs)^{\alpha-1} f(s,\phi(s),\mathcal{G}\phi(s),\mathcal{S}\phi(s)) d_q s \\ -\int_{\tau_0}^{\tau_2}(\tau_2 - qs)^{\alpha-1} f(s,\phi(s),\mathcal{G}\phi(s),\mathcal{S}\phi(s)) d_q s\end{matrix}\right\|$$

$$\leq \frac{1}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau_1}((\tau_1 - qs)^{\alpha-1} - (\tau_2 - qs)^{\alpha-1})\big\|f\big(s,\phi(s),\mathcal{G}\phi(s),\mathcal{S}\phi(s)\big) - f(s,0,0,0)\big\| d_q s$$

$$+\frac{1}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau_1}(\tau_2 - qs)^{\alpha-1}\|f(s,0,0,0)\| d_q s$$

$$+\frac{1}{\Gamma_{\mathrm{q}}(\alpha)}\odot\int_{\tau_1}^{\tau_2}((\tau_1-\mathrm{q}s)^{\alpha-1}-(\tau_2-\mathrm{q}s)^{\alpha-1})\left\|f\big(s,\phi(s),\mathcal{G}\phi(s),\mathcal{S}\phi(s)\big)-f(s,0,0,0)\right\|d_{\mathrm{q}}s$$

$$\frac{1}{\Gamma_{\mathrm{q}}(\alpha)}\odot\int_{\tau_1}^{\tau_2}(\tau_2-\mathrm{q}s)^{\alpha-1}\|f(s,0,0,0)\|d_{\mathrm{q}}s$$

$$\leq\frac{\mathcal{L}(1+K^*+H^*)}{\Gamma_{\mathrm{q}}(\alpha)}\odot\int_{\tau_0}^{\tau_1}((\tau_1-\mathrm{q}s)^{\alpha-1}-(\tau_2-\mathrm{q}s)^{\alpha-1})(\epsilon+\|\phi_0\|)d_{\mathrm{q}}s$$

$$+\frac{M}{\Gamma_q(\alpha)}\odot\int_{\tau_0}^{\tau_1}((\tau_1-\mathrm{q}s)^{\alpha-1}-(\tau_2-\mathrm{q}s)^{\alpha-1})d_{\mathrm{q}}s$$

$$+\frac{\mathcal{L}(1+K^*+H^*)}{\Gamma_{\mathrm{q}}(\alpha)}\odot\int_{\tau_1}^{\tau_2}((\tau_1-\mathrm{q}s)^{\alpha-1}-(\tau_2-\mathrm{q}s)^{\alpha-1})(\epsilon+\|\phi_0\|)d_{\mathrm{q}}s$$

$$+\frac{M}{\Gamma_{\mathrm{q}}(\alpha)}\odot\int_{\tau_1}^{\tau_2}((\tau_1-\mathrm{q}s)^{\alpha-1}-(\tau_2-\mathrm{q}s)^{\alpha-1})d_{\mathrm{q}}s$$

Then

$$\|\mathcal{K}\phi(\tau_1)-\mathcal{K}\phi(\tau_2)\|\leq\frac{(\lambda+\mathcal{L}(1+K^*+H^*))(\epsilon+\phi_0)+M}{\Gamma_q(\alpha)}\odot\int_{\tau_0}^{\tau_1}\begin{pmatrix}(\tau_1-\mathrm{q}s)^{\alpha-1}\\-(\tau_2-\mathrm{q}s)^{\alpha-1}\end{pmatrix}d_{\mathrm{q}}s$$

$$+\frac{(\lambda+\mathcal{L}(1+K^*+H^*))(\epsilon+\phi_0)+M}{\Gamma_{\mathrm{q}}(\alpha)}\odot\int_{\tau_1}^{\tau_2}(\tau_2-\mathrm{q}s)^{\alpha-1}d_{\mathrm{q}}s$$

$$=\frac{(\lambda+\mathcal{L}(1+K^*+H^*))(\epsilon+\phi_0)+M}{\Gamma_q(\alpha)}\odot(2(\tau_2-\tau_1)^{\alpha}+\tau_1^{\alpha}-\tau_2^{\alpha})$$

Hence $\mathcal{K}\phi$ is continuous.

Hence for any $\phi\in\mathbb{B}$, we have $\mathcal{K}\phi\in C([0,\kappa],\mathbb{R}_{\mathbb{F}})$ i.e. $\mathcal{K}\phi(\tau_0)=\phi_0$ and $\|\mathcal{K}\phi-\phi_0\|\leq\epsilon$. This concludes that $\mathcal{K}\phi\in\mathbb{B}$ whenever $u\in\mathbb{B}$. i.e. $\mathcal{K}\phi$ maps $\mathbb{B}$ into itself. The next step is to prove that, for every $n\in\mathbb{N}\cup\{0\}$, and every $\phi,\psi\in\mathbb{B}$, we have

$$\|\mathcal{K}^n\phi-\mathcal{K}^n\psi\|\leq\frac{[\lambda+\mathcal{L}(1+K^*+H^*)]^n}{\Gamma_q(n\alpha+1)}\|\phi-\psi\|,\tau\in[0,\kappa] \tag{4.3}$$

This can be seen by induction. For $n=0$, the inequality (4.4) is trivially true. We assume that (4.4) is true for $n=m-1$ and prove it for $n=m$. By using definition of operator $\mathcal{K}$, we have

$$\|\mathcal{K}^m\phi(\tau)-\mathcal{K}^m\psi(\tau)\|\leq\frac{\lambda}{\Gamma_{\mathrm{q}}(\alpha)}\odot\int_{\tau_0}^{\tau}(\tau-s)^{\alpha-1}\|\mathcal{K}^{m-1}\phi(s)-\mathcal{K}^{m-1}\psi(s)\|d_{\mathrm{q}}s$$

$$+\frac{1}{\Gamma_{\mathrm{q}}(\alpha)}\odot\int_{\tau_0}^{\tau}(\tau-s)^{\alpha-1}\left\|\begin{matrix}f\big(s,\mathcal{K}^{m-1}\phi(s),\mathcal{K}^{m-1}\mathcal{G}\phi(s),\mathcal{K}^{m-1}\mathcal{S}\phi(s)\big)\\-f(s,\mathcal{K}^{m-1}\psi(s),\mathcal{K}^{m-1}\mathcal{G}\psi(s),\mathcal{K}^{m-1}\mathcal{S}\psi(s))\end{matrix}\right\|d_qs$$

For $n=m-1$, we get

$$\|\mathcal{K}^m\phi(\tau)-\mathcal{K}^m\psi(\tau)\|\leq\frac{\lambda}{\Gamma_q(\alpha)}\frac{(\lambda+\mathcal{L}(1+K^*+H^*))^{m-1}}{\Gamma_q((m-1)\alpha+1)}\|\phi-\psi\|\odot\int_{\tau_0}^{\tau}(\tau-s)^{\alpha-1}s^{m\alpha-\alpha}d_{\mathrm{q}}s$$

$$+\frac{\mathcal{L}(1+K^*+H^*)}{\Gamma_{\mathrm{q}}(\alpha)}\odot\int_{\tau_0}^{\tau}(\tau-s)^{\alpha-1}\|\mathcal{K}^{m-1}\phi(s)-\mathcal{K}^{m-1}\psi(s)\|d_{\mathrm{q}}s$$

$$\leq\frac{(\lambda+\mathcal{L})}{\Gamma_{\mathrm{q}}(\alpha)}\frac{(\lambda+\mathcal{L}(1+K^*+H^*))^{m-1}}{\Gamma_{\mathrm{q}}((m-1)\alpha+1)}\|\phi-\psi\|\odot\int_{\tau_0}^{\tau}(\tau-s)^{\alpha-1}s^{m\alpha-\alpha}d_{\mathrm{q}}s$$

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$$= \frac{[(\lambda+\mathcal{L}(1+K^*+H^*))\tau^{\alpha}]^m}{\Gamma_q(m\alpha+1)}\|\phi-\psi\|$$

which is our desired inequality (4.4). Hence we have

$$\|\mathcal{K}^n\phi-\mathcal{K}^n\psi\| \leq \frac{[(\lambda+\mathcal{L}(1+K^*+H^*))\kappa^{\alpha}]^n}{\Gamma_q(m\alpha)+1}\|\phi-\psi\|$$

By definition(2.10), we have

$$\sum_{n=0}^{\infty}\frac{[(\lambda+\mathcal{L}(1+K^*+H^*))\kappa^{\alpha}]^n}{\Gamma_q(n\alpha+1)} = E(\lambda+\mathcal{L}(1+K^*+H^*))\kappa^{\alpha}$$

We have proved that the operator $\mathcal{K}$ satisfies all the conditions of theorem (2.3) and hence $\mathcal{L}$ has a unique fixed point $\phi: [0,\kappa] \to \mathbb{R}_{\mathbb{F}}$ which is the solution of (1.1).

5 Continuous Dependence and Uniqueness of Solutions

Theorem 5.1 *Suppose that the function* $f: \mathbb{T}_q \times \mathbb{R}_{\mathbb{F}} \times \mathbb{R}_{\mathbb{F}} \times \mathbb{R}_{\mathbb{F}} \to \mathbb{R}_{\mathbb{F}}$. *Let* $\phi_1(\tau)$ *and* $\phi_2(t)$ *be the solutions of equation,*

$$^{C}D^{\alpha}\phi(\tau) = \lambda\phi(\tau) + f(\tau,\phi(\tau),\mathcal{G}\phi(\tau),\mathcal{S}\phi(\tau)), \tau \in \mathbb{T}_q \tag{5.1}$$

corresponding to $\phi_1(0) = \phi_0$ and $\phi_2(0) = \phi_0^*$ respectively.

Then

$$\|\phi_1-\phi_2\| \leq \left\{1+\frac{K(\lambda+\mathcal{L}(1+K^*+H^*))}{\Gamma_q(\alpha)}T^{\alpha}\right\}\|\phi_0-\phi_0^*\|, \tau \in \mathbb{T}_q$$

Let $\phi_1(\tau)$ and $\phi_2(\tau)$ be the solutions of equation (5.1) corresponding to $\phi_1(0) = \phi_0$ and $\phi_2(0) = \phi_0^*$ respectively.

$$^{C}D^{\alpha}\phi_1(\tau) = \lambda\phi_1(\tau) + f(\tau,\phi_1(\tau),\mathcal{G}\phi_1(\tau),\mathcal{S}\phi_1(\tau)),$$
$$\phi_1(0) = \phi_0,$$
$$^{C}D^{\alpha}\phi_2(\tau) = \lambda\phi_2(\tau) + f(\tau,\phi_2(\tau),\mathcal{G}\phi_2(\tau),\mathcal{S}\phi_2(\tau)),$$
$$\tau_2(0) = \phi_0^*$$

This implies that,

$$\phi_1(\tau) = \phi_0 + \frac{\lambda}{\Gamma_q(\alpha)} \odot \int_0^{\tau}(\tau-qs)^{\alpha-1}\phi_1(s)d_qs +$$
$$\frac{1}{\Gamma_q(\alpha)} \odot \int_0^{\tau}(\tau-qs)^{\alpha-1}f(s,\phi_1(s),\mathcal{G}\phi_1(s),\mathcal{S}\phi_1(s))d_qs$$

and

$$\phi_2(\tau) = \phi_0^* + \frac{\lambda}{\Gamma_q(\alpha)} \odot \int_0^{\tau}(\tau-qs)^{\alpha-1}\phi_2(s)d_qs +$$
$$\frac{1}{\Gamma_q(\alpha)} \odot \int_0^{\tau}(\tau-s)^{\alpha-1}f(s,\phi_2(s),\mathcal{G}\phi_2(s),\mathcal{S}\phi_2(s))d_qs$$

Using the hypothesis $(H1)$, for any $\tau \in [\tau_0, T]$, we obtain

$$\|\phi_1(\tau) - \phi_2(\tau)\| \le \|\phi_0 - \phi_0^*\| + \frac{\lambda}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1} \|\phi_1(s) - \phi_2(s)\| d_q s$$

$$+ \frac{1}{\Gamma(\alpha)} \odot \int_0^{\tau} (\tau - qs)^{\alpha-1} \big\| f\big(s, \phi_1(s), \mathcal{G}\phi_1(s), \mathcal{S}\phi_1(s)\big) - f(s, \phi_2(s), \mathcal{G}\phi_2(s), \mathcal{S}u_2(s)) \big\| d_q s$$

$$\le \|\phi_0 - \phi_0^*\| + K(\lambda + \mathcal{L}(1 + K^* + H^*)) \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1} \|\phi_0 - \phi_0^*\| d_q s$$

$$= \|\phi_0 - \phi_0^*\| \left\{ 1 + \frac{K\big(\lambda + \mathcal{L}(1 + K^* + H)^*)\big)}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - s)^{\alpha-1} d_q s \right\}$$

$$\le \|\phi_0 - \phi_0^*\| \left\{ 1 + \frac{K\big(\lambda + \mathcal{L}(1 + K^* + H)^*)\big)}{\Gamma_q(\alpha)} T^{\alpha} \right\},$$

$\tau \in \mathbb{T}_q$. This proves the uniqueness of the solution of (1.1).

6 Continuous Dependence on Functions involved and Parameters

Consider (1.1) and

$$^{C}D^{\alpha}\chi(\tau) = \lambda\chi(\tau) + f^*(\tau, \chi(\tau), \mathcal{G}\chi(\tau), \mathcal{S}\chi(\tau)) \tag{6.1}$$

$$\chi(0) = \chi_0,$$

where $f: \mathbb{T}_q \times \mathbb{R}_{\mathbb{F}} \times \mathbb{R}_{\mathbb{F}} \times \mathbb{R}_{\mathbb{F}} \to \mathbb{R}_{\mathbb{F}}$

Theorem 6.1 *Suppose that* f *in* (1.1) *satisfies the hypothesis* $(H1)$. *Let* $\chi(t)$ *be a solution of* (6.1) *and suppose that*

$$\|f(\tau, \chi(\tau), \mathcal{G}\chi(\tau), \mathcal{S}\chi(\tau)) - f^*(\tau, \chi(\tau), \mathcal{G}\chi(t), \mathcal{S}\chi(t))\| \le \epsilon, \quad \tau \in \mathbb{T}_q$$

and

$$\|\phi_0 - \psi_0\| < \delta$$

where $\epsilon, \delta > 0$ are arbitrary small constants. Then the solution of $\phi(\tau)$ of (1.1) depends continuously on the functions involved therein.

Proof. Let $\phi(\tau)$ and $\psi(\tau)$ be solutions of (1.1) and (6.1) respectively.

$$^{C}D^{\alpha}\phi(\tau) = \lambda\phi(\tau) + f(\tau, \phi(\tau), \mathcal{G}\phi(\tau), \mathcal{S}\phi(t)), \phi(0) = \phi_0$$

$$^{C}D^{\alpha}\psi(\tau) = \lambda\psi(\tau) + f^*(\tau, \psi(\tau), \mathcal{G}\psi(\tau), \mathcal{S}\psi(\tau)), \psi(0) = \psi_0$$

This implies that,

$$\phi(\tau) = \phi_0 + \frac{\lambda}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1} \phi(s) ds$$

$$+ \frac{1}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1} f(s, \phi(s), \mathcal{G}\phi(s), \mathcal{S}\phi(s)) d_q s$$

and

$$\psi(t) = \psi_0 + \frac{\lambda}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1}\psi(s) d_q s$$
$$+ \frac{1}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1} f^*(s, \psi(s), \mathcal{G}\psi(s), \mathcal{S}\psi(s)) d_q s$$

$$\|\phi(\tau) - \psi(\tau)\| \leq \|\phi_0 - \psi_0\| + \frac{\lambda}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1} \|\phi(s) - \psi(s)\| d_q s$$
$$+ \frac{1}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1} \|f(s, \phi(s), \mathcal{G}\phi(s), \mathcal{S}\phi(s)) - f^*(s, \phi(s), \mathcal{G}\phi(s), \mathcal{S}\phi(s))\| d_q s$$
$$\leq \|\phi_0 - \psi_0\| + \frac{\lambda}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1} \|\phi(s) - \psi(s)\| d_q s$$
$$+ \frac{1}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1} \left\| f\big(s, \phi(s), \mathcal{G}\phi(s), \mathcal{S}\phi(s)\big) - f(s, \psi(s), \mathcal{G}\psi(s), \mathcal{S}\psi(s)) \right\| d_q s$$
$$+ \frac{1}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1} \left\| f\big(s, \psi(s), \mathcal{G}\psi(s), \mathcal{S}\psi(s)\big) - f^*(s, \psi(s), \mathcal{G}\psi(s), \mathcal{S}\psi(s)) \right\| d_q s$$
$$\leq \|\phi_0 - \psi_0\| + \frac{\lambda}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1} \|\phi(s) - \psi(s)\| d_q s$$
$$+ \frac{1}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1} \mathcal{L}[\|\phi - \psi\| + \|\mathcal{G}\phi - \mathcal{G}\psi\| + \|\mathcal{S}\phi - \mathcal{S}\psi\|] d_q s$$
$$+ \frac{\epsilon}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1} d_q s$$
$$\leq \|\phi_0 - \psi_0\| + \frac{\lambda}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1} \|\phi - \psi\| d_q s$$
$$+ \frac{1}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1} \mathcal{L}(1 + K^* + H^*) \|\phi - \psi\| d_q s$$
$$+ \frac{\epsilon}{\Gamma_q(\alpha)} \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1} d_q s$$
$$\leq \delta + \frac{\epsilon}{\Gamma_q(\alpha+1)} \tau^{\alpha} + \left(\frac{\lambda + \mathcal{L}(1 + K^* + H^*)}{\Gamma_q(\alpha)}\right) \odot \int_{\tau_0}^{\tau} (\tau - qs)^{\alpha-1} \|\phi - \psi\| d_q s$$

By Lemma (2.2),

$$\|\phi(t) - \psi(t)\| \leq \delta + \frac{\epsilon}{\Gamma_q(\alpha+1)} \tau^{\alpha} + \frac{K\big(\lambda + \mathcal{L}(1 + K^* + H^*)\big)\delta}{\Gamma_q(\alpha+1)} \tau^{\alpha} + K \frac{\lambda + L(1 + K^* + H^*)}{\Gamma_q(2\alpha+1)} \tau^{2\alpha},$$

$\tau \in \mathbb{T}_q$. From the above inequality it comes to know that the solution $\phi(\tau)$ depends continuously on the functions involved in the given FDE (1.1). If $\epsilon = 0$, the continuous dependence of solutions of the inequality is on the initial conditions. Here it is noted that as $\epsilon, \delta > 0$ were arbitrary, by taking $\epsilon, \delta > 0$, we have $\phi \to \psi$, where $\phi: \mathbb{T}_q \to \mathbb{R}_{\mathbb{F}}$ and $\psi: \mathbb{T}_q \to \mathbb{R}_{\mathbb{F}}$ are the solutions of (1.1) and (6.1) respectively.

7 Illustrative example

The incorporation of an illustrative example can serve as a powerful tool to amplify the overall impact of the presented idea in the article. To accomplish the same, we provide an example to illustrate the execution of our main result. We consider

$$^{C}D^{\alpha}[\phi(\tau)] = \frac{\tau+1}{11}\phi(\tau) + \frac{\tau^2}{2} + \int_0^1 \frac{1}{(s+\tau+2)^2}\phi(s)ds + \int_0^1 \frac{\tau}{s+1}\phi(s)ds,\ \tau \in [\tau_0, T] \quad (7.1)$$

$$\phi(0) = 0$$

Define $f(\tau, \phi, \mathcal{G}\phi, \mathcal{S}\phi) = \frac{\tau+1}{11}u(\tau) + \frac{\tau^2}{2} + \int_0^1 \frac{1}{(s+\tau+2)^2}\phi(s)ds + \int_0^1 \frac{\tau}{s+1}\phi(s)ds$

Clearly the function f and g continuous, since

$$\|f(\tau, \phi(\tau), \mathcal{G}\phi(\tau), \mathcal{S}\phi(\tau)) - f(\tau, \psi(\tau), \mathcal{G}\psi(\tau), \mathcal{S}\psi(\tau))\| \leq \frac{169}{264}\|\phi - \psi\|$$

Here by (1.2), we get $\mathcal{L} = \frac{169}{264} \leq 1$.

Therefore by theorem(3.2) this problem has a unique solution.

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