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Stability and Bifurcation in a Second-Order Difference System

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Abstract:
In the paper, we examine the system

$$x_{n+1} = \alpha_1 + a_1 e^{-x_{n-1}} + b_1 y_n e^{-y_{n-1}} + c_1 e^{-z_{n-1}},$$
$$y_{n+1} = \alpha_2 + a_2 e^{-y_{n-1}} + b_2 z_n e^{-z_{n-1}} + c_2 e^{-x_{n-1}}, \quad (1)$$
$$z_{n+1} = \alpha_3 + a_3 e^{-z_{n-1}} + b_3 x_n e^{-x_{n-1}} + c_3 e^{-y_{n-1}}, \quad n = 0,1,2,\ldots,$$

where $\alpha_1, \alpha_2, \alpha_3, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are positive real numbers and the initial conditions $x_{-1}, x_0, y_{-1}, y_0, , z_{-1}, z_0$ are arbitrary nonnegative numbers. We investigate the persistence, boundedness, convergence, invariance, and global asymptotic character of the positive solutions of (1). Bifurcation diagrams are then plotted to visualize the periodic character.

Keywords: persistence, boundedness, invariance, local property, global property, bifurcation.

AMS Subject Classification 2000: 39A22.

1. Introduction

In the study of dynamical systems, difference equations play a crucial role in modeling various phenomena across diverse scientific disciplines, including biology, economics, engineering, and physics.(See [16],[21],[22],[27], [31], [32]). Unlike differential equations, which describe continuous change, difference equations are discrete analogues that characterize systems evolving in distinct time steps.(See [6],[8],[13],[22],[23],[24]) Most of the popular models like SIRS and SEIRS are mainly of order one. Two species models are examined in [3], [11],[29] and [33]. Competition models of two species second order with exponents are analyzed in [10],[12] and [15]-[20]. In [5], the authors analyzed a food-chain model using a first order system with three variables. More first order system with three variables can be seen in [1],[2],[4], [5], [7] and [28] whereas [31] deals with second order systems with three variables.

The system (1) which we investigate is an extension of [30] where we focus on a system of three interdependent difference equations involving twelve parameters and three variables. We analyze the boundedness, persistence, invariance and convergence of the solutions of (1). We then plot few bifurcation diagrams to observe the periodic nature of the system. By exploring a three variable second order system, this work aims to contribute to the broader understanding of multi-parameter, multi-variable difference system. The increased number of parameters provide a high level of flexibility to model real world scenarios, which is also cructial for system control dynamics.

2. Main Results

Theorem 2.1 *The positive solution* (x_n, y_n, z_n) *of (1) persists.*

It is bounded whenever

$$B = b_1 b_2 b_3 e^{-\alpha_1-\alpha_2-\alpha_3} < 1. \quad (2)$$

Proof:

Clearly, the system persists because of the presence of α_i.

For $n = 4,5,...$, (1) becomes

$$x_{n+1} \leq \alpha_1 + a_1 e^{-\alpha_1} + c_1 e^{-\alpha_3} + b_1 e^{-\alpha_2}[\alpha_2 + a_2 e^{-\alpha_2} + c_2 e^{-\alpha_1} + b_2 a_2 e^{-\alpha_3} z_{n-1}]$$

substituting for z_{n-1}, we get

$$\leq A_1 + B x_{n-2}, \quad (3)$$

where $A_1 = \alpha_1 + a_1 e^{-\alpha_1} + c_1 e^{-\alpha_3} + b_1 \alpha_2 e^{-\alpha_2} + b_1 a_2 e^{-\alpha_2-\alpha_2} + b_1 c_2 e^{-\alpha_2-\alpha_1} + b_1 b_2 \alpha_3 e^{-\alpha_2-\alpha_3} + b_1 b_2 a_3 e^{-\alpha_2-\alpha_3-\alpha_3} + b_1 b_2 c_3 e^{-\alpha_2-\alpha_2-\alpha_3}$

and $B = b_1 b_2 b_3 e^{-\alpha_1-\alpha_2-\alpha_3}$.

Similarly,

$$y_{n+1} \leq A_2 + B y_{n-2}, \quad (4)$$

where $A_2 = \alpha_2 + a_2 e^{-\alpha_2} + c_2 e^{-\alpha_1} + b_2 \alpha_3 e^{-\alpha_3} + b_2 a_3 e^{-\alpha_3-\alpha_3} + b_2 c_3 e^{-\alpha_2-\alpha_3} + b_2 b_3 \alpha_1 e^{-\alpha_1-\alpha_3} + b_2 b_3 a_1 e^{-\alpha_1-\alpha_1-\alpha_3} + b_2 b_3 c_1 e^{-\alpha_3-\alpha_3-\alpha_1}$.

Also,

$$z_{n+1} \leq A_3 + B z_{n-2}, \quad (5)$$

where $A_3 = \alpha_3 + a_3 e^{-\alpha_3} + c_3 e^{-\alpha_2} + b_3 \alpha_1 e^{-\alpha_1} + b_3 a_1 e^{-\alpha_1-\alpha_1} + b_3 c_1 e^{-\alpha_3-\alpha_1} + b_3 b_1 \alpha_2 e^{-\alpha_1-\alpha_2} + b_1 b_3 a_2 e^{-\alpha_1-\alpha_2-\alpha_2} + b_3 b_1 c_2 e^{-\alpha_1-\alpha_1-\alpha_2}$.

Now, consider the difference equations

$$u_{n+1} = A_1 + B u_{n-2},$$

$$v_{n+1} = A_2 + B v_{n-2},$$

$$w_{n+1} = A_3 + B w_{n-2}, \quad n = 4,5,... \quad (6)$$

Solution (u_n, v_n, w_n) of (6) is of the form

$$u_n = r_1 B^{n/3} + r_2 B^{n/3}\cos(\tfrac{n\pi}{2}) + r_3 B^{n/3}\sin(\tfrac{n\pi}{2}) + \tfrac{A_1}{1-B}, \quad n = 5,6,..., \quad (7)$$

$$v_n = s_1 B^{n/3} + s_2 B^{n/3}\cos(\tfrac{n\pi}{2}) + s_3 B^{n/3}\sin(\tfrac{n\pi}{2}) + \tfrac{A_2}{1-B}, \quad n = 5,6 ..., \quad (8)$$

$$w_n = p_1 B^{n/3} + p_2 B^{n/3}\cos(\tfrac{n\pi}{2}) + p_3 B^{n/3}\sin(\tfrac{n\pi}{2}) + \tfrac{A_3}{1-B}, \quad n = 5,6 ..., \quad (9)$$

where $p_i, r_i, s_i, i = 1,2,3$ depend on w_4, u_4, v_4 respectively.

Hence, (u_n, v_n, w_n) is bounded.

Now we examine (u_n, v_n, w_n) such that the initial conditions of 6 and 1 are same. Clearly we can conclude that (x_n, y_n, z_n) is bounded.

Theorem 2.2 *Let (2) hold. Let A_1, A_2, A_3 be defined as in Theorem 2.1. Then* $[\alpha_1, \frac{A_1}{1-B}]$ $\times [\alpha_2, \frac{A_2}{1-B}] \times [\alpha_3, \frac{A_3}{1-B}]$ *is an invariant set for the system (1).*

Proof: Take $I_1 = [\alpha_1, \frac{A_1}{1-B}], I_2 = [\alpha_2, \frac{A_2}{1-B}]$ and $I_3 = [\alpha_3, \frac{A_3}{1-B}]$.

Let $x_{-1}, x_0 \in I_1$, $y_{-1}, y_0 \in I_2$ and $z_{-1}, z_0 \in I_3$.

Then $x_1 \le \alpha_1 + a_1 e^{-\alpha_1} + c_1 e^{-\alpha_3} + b_1 e^{-\alpha_2} y_0$

Since $y_0 \le \frac{A_2}{1-B}$, we get

$$x_1 \le [\frac{\alpha_1 + a_1 e^{-\alpha_1} + c_1 e^{-\alpha_3} - b_1 b_2 b_3 \alpha_1 e^{-\alpha_1-\alpha_2-\alpha_3} - b_1 b_2 b_3 a_1 e^{-\alpha_1-\alpha_1-\alpha_2-\alpha_3}}{1 - b_1 b_2 b_3 e^{-\alpha_1-\alpha_2-\alpha_3}}]$$

$$+[\frac{-b_1 b_2 b_3 c_1 e^{-\alpha_1-\alpha_2-\alpha_3-\alpha_3} + b_1 \alpha_2 e^{-\alpha_2} + b_1 a_2 e^{-\alpha_2-\alpha_2} + b_1 c_2 e^{-\alpha_1-\alpha_2}}{1 - b_1 b_2 b_3 e^{-\alpha_1-\alpha_2-\alpha_3}}]$$

$$+[\frac{b_1 b_2 \alpha_3 e^{-\alpha_2-\alpha_3} + b_1 b_2 a_3 e^{\alpha_3-\alpha_3} + b_1 b_2 c_3 e^{-\alpha_2-\alpha_2-\alpha_3} + b_1 b_2 b_3 \alpha_1 e^{-\alpha_2-\alpha_2-\alpha_3}}{1 - b_1 b_2 b_3 e^{-\alpha_1-\alpha_2-\alpha_3}}]$$

$$+[\frac{b_1 b_2 b_3 a_1 e^{-\alpha_1--\alpha_1-\alpha_2-\alpha_3} + b_1 b_2 b_3 c_1 e^{-\alpha_1-\alpha_2-\alpha_3-\alpha_3}}{1 - b_1 b_2 b_3 e^{-\alpha_1-\alpha_2-\alpha_3}}]$$

i.e., $x_1 \in I_1$. Similarly we get $y_1 \in I_2$ and $z_1 \in I_3$.

Hence the induction the proof follows.

Theorem 2.3 *Assume (2). Let A_1, A_2, A_3 be as in the Theorem 2.1. Let $I_4 = [\alpha_1, \frac{A_1+\epsilon}{1-B}]$, $I_5 = [\alpha_2, \frac{A_2+\epsilon}{1-B}]$ and $I_6 = [\alpha_3, \frac{A_3+\epsilon}{1-B}]$ where ϵ is arbitrary. Then $x_n \in I_4, y_n \in I_5$ and $z_n \in I_6$, for every $n \ge N$, $N \in \mathbb{N}$.*

Proof:

Given (x_n, y_n, z_n) be a positive solution of (1).

Theorem 2.1 implies,

$\limsup_{n\to\infty} x_n = P < \infty$, $\limsup_{n\to\infty} y_n = Q < \infty$ and $\limsup_{n\to\infty} z_n = R < \infty$.

Theorem 2.1 implies,

$x_{n+1} \le A_1 + b_1 b_2 b_3 x_{n-2} e^{-\alpha_1-\alpha_2-\alpha_3}$, $y_{n+1} \le A_2 + b_1 b_2 b_3 y_{n-2} e^{-\alpha_1-\alpha_2-\alpha_3}$

and

$$z_{n+1} \le A_3 + b_1 b_2 b_3 z_{n-2} e^{-\alpha_1-\alpha_2-\alpha_3}.$$

Hence, $P \le \frac{A_1}{1-b_1 b_2 b_3 e^{-\alpha_1-\alpha_2-\alpha_3}}$, $Q \le \frac{A_2}{1-b_1 b_2 b_3 e^{-\alpha_1-\alpha_2-\alpha_3}}$ and $R \le \frac{A_3}{1-b_1 b_2 b_3 e^{-\alpha_1-\alpha_2-\alpha_3}}$.

Hence, the result.

Here we state a lemma which is an an extension of Lemma 5 in [10] and a variation of Theorem 1.16 in [26].

Lemma 2.4 *Assume A, B, C, D, E, F represent reals. Let $f_1: [A, B] \times [C, D] \times [C, D] \times [E, F] \to [A, B]$, $f_2: [C, D] \times [E, F] \times [E, F] \times [A, B] \to [C, D]$ and $f_3: [E, F] \times [A, B] \times [A, B] \times [C, D] \to [E, F]$ be continuous. Examine*

$$\begin{aligned} x_{n+1} &= f_1(x_{n-1}, y_n, y_{n-1}, z_{n-1}), \\ y_{n+1} &= f_2(y_{n-1}, z_n, z_{n-1}, x_{n-1}), \\ z_{n+1} &= f_3(z_{n-1}, x_n, x_{n-1}, y_{n-1}), \quad n = 0,1,2, \ldots \end{aligned} \tag{10}$$

where $x_{-1}, x_0 \in [A, B], y_{-1}, y_0 \in [C, D]$ and $z_{-1}, z_0 \in [E, F]$. (or $x_{n_0}, x_{n_0+1} \in [A, B]$, $y_{n_0}, y_{n_0+1} \in [C, D], z_{n_0}, z_{n_0+1} \in [E, F], n_0 \in \mathbb{N}$). Assume the conditions given below holds.

1. If $f_1(x, y, z, u)$, $f_2(x, y, z, u)$ and $f_3(x, y, z, u)$ are nonincreasing in x, nondecreasing in y, nonincreasing in z and nonincreasing in u.

2. If $(m_1, M_1, m_2, M_2, m_3, M_3) \in [A, B]^2 \times [C, D]^2 \times [E, F]^2$ satisfies the systems $m_1 = f_1(M_1, m_2, M_2, M_3)$; $M_1 = f_1(m_1, M_2, m_2, m_3)$, $m_2 = f_2(M_2, m_3, M_3, M_1)$; $M_2 = f_2(m_2, M_3, m_3, m_1)$ and $m_3 = f_3(m_1, M_1, M_3, M_2)$; $M_3 = f_3(M_1, m_1, m_3, m_2)$ then $m_1 = M_1, m_2 = M_2$ and $m_3 = M_3$,

then $(\bar{x}, \bar{y}, \bar{z})$ is the unique equilibrium point of (10) where $\bar{x} \in [A, B]$,$\bar{y} \in [C, D]$ and $\bar{z} \in [E, F]$. And any other solution of (10) converges to $(\bar{x}, \bar{y}, \bar{z})$.

Theorem 2.5 *Let (2) hold. Suppose*

$$a_1 e^{-\alpha_1} < 1, a_2 e^{-\alpha_2} < 1, a_3 e^{-\alpha_3} < 1 \tag{11}$$

and

$$\frac{[D_2 D_3 + B_3 L_2][D_1 D_2 + B_2 L_1][D_3 D_1 + B_1 L_3]}{[B_2 B_3 - D_2 L_3][B_1 B_2 - D_1 L_2][B_3 B_1 - D_3 L_1]} < 1, \tag{12}$$

where $B_1 = 1 - a_1 e^{-\alpha_1}, B_2 = 1 - a_2 e^{-\alpha_2}, B_3 = 1 - a_3 e^{-\alpha_3}, D_1 = b_1 e^{-\alpha_2}(1 + \frac{A_2}{1-B}), D_2 = b_2 e^{-\alpha_3}(1 + \frac{A_3}{1-B}), D_3 = b_3 e^{-\alpha_1}(1 + \frac{A_1}{1-B}), L_1 = c_1 e^{-\alpha_3}, L_2 = c_2 e^{-\alpha_1}, L_3 = c_3 e^{-\alpha_2}$. Then $E(\bar{x}, \bar{y}, \bar{z})$ is the unique positive equilibrium of (1). And any solution of (1) converges to $E(\bar{x}, \bar{y}, \bar{z})$.

Proof:

Define $\quad f_1(x, y, z) = \alpha_1 + a_1 e^{-x} + b_1 y e^{-y} + c_1 e^{-z}$,

$f_2(x, y, z) = \alpha_2 + a_2 e^{-y} + b_2 z e^{-z} + c_2 e^{-x}$, $\quad f_3(x, y, z) = \alpha_3 + a_3 e^{-z} + b_3 x e^{-x} + c_3 e^{-yS}$.

Take $m_i \leq M_i, i = 1,2,3$ to denote positive reals where and

$$m_1 = \alpha_1 + a_1 e^{-M_1} + b_1 m_2 e^{-M_2} + c_1 e^{-M_3}, M_1 = \alpha_1 + a_1 e^{-m_1} + b_1 M_2 e^{-m_2} + c_1 e^{-m_3},$$

$$m_2 = \alpha_2 + a_2 e^{-M_2} + b_2 m_3 e^{-M_3} + c_1 e^{-M_1}, M_2 = \alpha_2 + a_2 e^{-m_2} + b_2 M_3 e^{-m_3} + c_1 e^{-m_1}$$

and

$$m_3 = \alpha_3 + a_3 e^{-M_3} + b_3 m_1 e^{-M_3} + c_1 e^{-M_2}, M_3 = \alpha_3 + a_3 e^{-m_3} + b_3 M_1 e^{-m_1} + c_1 e^{-m_2}. \quad (13)$$

Therefore, $M_1 - m_1 = a_1[e^{-m_1} - e^{-M_1}] + b_1[M_2 e^{-m_2} - m_2 e^{-M_2}] + c_1[e^{-m_3} - e^{-M_3}]$.

$$M_1 - m_1 = a_1[e^{-m_1} - e^{-M_1}] + b_1 e^{-m_2 - M_2}[M_2 e^{M_2} - m_2 e^{m_2}] + c_1[e^{-m_3} - e^{-M_3}]. \quad (14)$$

Here, there exists a ζ_1 , $M_2 \geq \zeta_1 \geq m_2$ satisfying

$$M_2 e^{M_2} - m_2 e^{m_2} = (1 + \zeta_1) e_1^{\zeta} (M_2 - m_2). \quad (15)$$

From (14) and (15) we get,

$$M_1 - m_1 = a_1[e^{-m_1} - e^{-M_1}] + b_1 e^{-m_2 - M_2 + \zeta_1}(1 + \zeta_1)[M_2 - m_2] + c_1[e^{-m_3} - e^{-M_3}]. \quad (16)$$

Now, $a_1[e^{-m_1} - e^{-M_1}] = a_1 e^{-m_1 - M_1}[e^{M_1} - e^{m_1}]$.

And, there exists a λ, $M_1 \geq \lambda \geq m_1$ satisfying

$$a_1[e^{-m_1} - e^{-M_1}] = a_1 e^{-m_1 - M_1 + \lambda}[M_1 - m_1]. \quad (17)$$

Since $m_1, M_1 \geq \alpha_1$,

$$a_1[e^{-m_1} - e^{-M_1}] \leq a_1 e^{-\alpha_1}[M_1 - m_1]. \quad (18)$$

Thus from (16) and (18) we get,

$$M_1 - m_1 \leq a_1 e^{-\alpha_1}[M_1 - m_1] + b_1 e^{-m_2 - M_2 + \zeta_1}(1 + \zeta_1)[M_2 - m_2] + c_1 e^{-\alpha_3}[M_3 - m_3]. \quad (19)$$

Since $m_2, M_2 \geq \alpha_2$, (19) becomes

$$M_1 - m_1 \leq a_1 e^{-\alpha_1}[M_1 - m_1] + b_1 e^{-\alpha_2}(1 + \zeta_1)[M_2 - m_2] + c_1 e^{-\alpha_3}[M_3 - m_3]. \quad (20)$$

i.e.,

$$[1 - a_1 e^{-\alpha_1}][M_1 - m_1] \leq b_1 e^{-\alpha_2}(1 + \zeta_1)[M_2 - m_2] + c_1 e^{-\alpha_3}[M_3 - m_3]. \quad (21)$$

Also, (13) can be written as

$$M_2 = \alpha_2 + a_2 e^{-m_2} + b_2[\alpha_3 + a_3 e^{-m_3} + b_3 M_1 e^{-m_1} + c_3 e^{-m_2}]e^{-m_3} + c_2 e^{-m_1}. \quad (22)$$

Substituting again for M_1 and simplifying we get

$$M_2 \leq \frac{A_2}{1 - b_1 b_2 b_3 e^{-\alpha_1 - \alpha_2 - \alpha_3}}. \quad (23)$$

Since $\zeta_1 \leq M_2$ we get,

$$\zeta_1 \leq \frac{A_2}{1 - b_1 b_2 b_3 e^{-\alpha_1 - \alpha_2 - \alpha_3}}. \quad (24)$$

Therefore, (21) becomes

$$[1 - a_1 e^{-\alpha_1}][M_1 - m_1] \leq b_1 e^{-\alpha_2}[1 + \frac{A_2}{1 - b_1 b_2 b_3 e^{-\alpha_1 - \alpha_2 - \alpha_3}}][M_2 - m_2] + c_1 e^{-\alpha_3}[M_3 - m_3]. \quad (25)$$

Similarly we get,

$$[1 - a_2 e^{-\alpha_2}][M_2 - m_2] \leq b_2 e^{-\alpha_3}[1 + \frac{A_3}{1 - b_1 b_2 b_3 e^{-\alpha_1 - \alpha_2 - \alpha_3}}][M_3 - m_3] + c_2 e^{-\alpha_1}[M_1 - m_1] \quad (26)$$

and

$$[1-a_3e^{-\alpha_3}][M_3-m_3] \le b_3e^{-\alpha_1}[1+\frac{A_1}{1-b_1b_2b_3e^{-\alpha_1-\alpha_2-\alpha_3}}][M_1-m_1]+c_3e^{-\alpha_2}[M_2-m_2] \quad (27)$$

From (25), (26) and (27) we get,

$$\frac{[B_1B_2-D_1L_2]}{B_2}[M_1-m_1] \le \frac{[D_1D_2+B_2L_1]}{B_2}[M_3-m_3].$$

Similarly,

$$\frac{[B_2B_3-D_2L_3]}{B_3}[M_2-m_2] \le \frac{[D_2D_3+B_3L_2]}{B_3}[M_1-m_1] \quad (28)$$

and

$$\frac{[B_3B_1-D_3L_1]}{B_1}[M_3-m_3] \le \frac{[D_3D_1+B_1L_3]}{B_1}[M_2-m_2] \quad (29)$$

Hence from (12) and (28), we get $M_1=m_1$.

Similarly we get $M_2=m_2$ and $M_3=m_3$.

Hence by Lemma 2.4, we get the required result.

Theorem 2.6 *Assume (2), (11) and (12) hold. If*

$$a_1e^{-\alpha_1}[1+a_2e^{-\alpha_2}]+a_2e^{-\alpha_2}[1+a_3e^{-\alpha_3}]+a_3e^{-\alpha_3}[1+a_1e^{-\alpha_1}]+a_1a_2a_3e^{-\alpha_1}e^{-\alpha_2}e^{-\alpha_3}$$
$$+\frac{B}{(1-B)^3}[(1-B)^3+(1-B)^2(A_1+A_2+A_3)$$
$$+(1-B)(A_1A_2+A_2A_3+A_1A_3)+A_1A_2A_3]<1, \quad (30)$$

where B, A_1, A_2, A_3 are as in Theorem 2.1, then $E(\bar{x},\bar{y},\bar{z})$ of (2.5) is globally asymptotically stable.

Proof:

We need to illustrate $E(\bar{x},\bar{y},\bar{z})$ is locally asymptotic. Construct the Jacobian $JF(\bar{x},\bar{y},\bar{z})$ about $E(\bar{x},\bar{y},\bar{z})$. Its characteristic equation is given by

$$\begin{aligned}\lambda^6+\lambda^4(a_1e^{-\bar{x}}+a_2e^{-\bar{y}}+a_3e^{-\bar{z}})+\lambda^3(b_2c_3e^{-\bar{y}-\bar{z}}+b_1c_2e^{-\bar{y}-\bar{x}}+b_3c_1e^{-\bar{z}-\bar{x}}+b_1b_2b_3e^{-\bar{x}-\bar{z}-\bar{y}})\\
+\lambda^2(a_2a_3e^{-\bar{z}-\bar{y}}-b_2c_3\bar{z}e^{-\bar{z}-\bar{y}}+a_1a_3e^{-\bar{z}-\bar{x}}-b_3c_1\bar{x}e^{-\bar{z}-\bar{x}}+a_1a_2e^{-\bar{x}}e^{-\bar{y}}\\
-b_1c_2\bar{y}e^{-\bar{x}}e^{-\bar{y}}+e^{-\bar{x}-\bar{y}-\bar{z}}[b_1b_2b_3\bar{x}+b_1b_2b_3\bar{y}+b_1b_2b_3\bar{z}])\\
+\lambda e^{-\bar{x}-\bar{y}-\bar{z}}(-b_1b_2b_3\bar{x}\bar{y}-b_1b_2b_3\bar{y}\bar{z}-b_1b_2b_3\bar{x}\bar{z}+a_3b_1c_2+a_1b_2c_3+a_2b_3c_1)\\
+e^{-\bar{x}-\bar{y}-\bar{z}}(a_1a_2a_3+b_1b_2b_3\bar{x}\bar{y}\bar{z}+c_1c_2c_3-a_2b_3c_1\bar{x}-a_3b_1c_2\bar{y}-a_1b_2c_3\bar{z})=0.\end{aligned}$$

Applying Remark 1.3.1 of [25],

$$\begin{aligned}|a_1e^{-\bar{x}}|+|a_2e^{-\bar{y}}|+|a_3e^{-\bar{z}}|+|b_1b_2b_3e^{-\bar{z}-\bar{y}-\bar{x}}|+|b_1b_2b_3\bar{x}e^{-\bar{z}-\bar{y}-\bar{x}}|+|b_1b_2b_3\bar{y}e^{-\bar{z}-\bar{y}-\bar{x}}|\\
+|b_1b_2b_3\bar{z}e^{-\bar{z}-\bar{y}-\bar{x}}|+|a_1a_2e^{-\bar{x}}e^{-\bar{y}}|+|a_1a_3e^{-\bar{x}}e^{-\bar{z}}|+|a_2a_3e^{-\bar{y}}e^{-\bar{z}}|\\
+|b_1b_2b_3\bar{x}\bar{y}e^{-\bar{z}-\bar{y}-\bar{x}}|+|b_1b_2b_3\bar{y}\bar{z}e^{-\bar{z}-\bar{y}-\bar{x}}|+|b_1b_2b_3\bar{x}\bar{z}e^{-\bar{z}-\bar{y}-\bar{x}}|\\
+|a_1a_2a_3e^{-\bar{z}-\bar{y}-\bar{x}}|+|b_1b_2b_3\bar{x}\bar{y}\bar{z}e^{-\bar{z}-\bar{y}-\bar{x}}|<1\end{aligned}$$

is satisfied whenever $a_1e^{-\alpha_1}[1+a_2e^{-\alpha_2}]+a_2e^{-\alpha_2}[1+a_3e^{-\alpha_3}]+a_3e^{-\alpha_3}[1+a_1e^{-\alpha_1}]+$

$$a_1a_2a_3e^{-\alpha_1}e^{-\alpha_2}e^{-\alpha_3}+B[\bar{x}\bar{y}\bar{z}+\bar{x}\bar{y}+\bar{x}\bar{z}+\bar{y}\bar{z}+\bar{z}+\bar{y}+\bar{x}+1]<1. \tag{31}$$

Clearly from Theorem 2.1,

$$\bar{x}\le\frac{A_1}{1-B}, \tag{32}$$

$$\bar{y}\le\frac{A_2}{1-B} \tag{33}$$

and

$$\bar{z}\le\frac{A_3}{1-B}. \tag{34}$$

Substitute (32), (33) and (34) in (31).

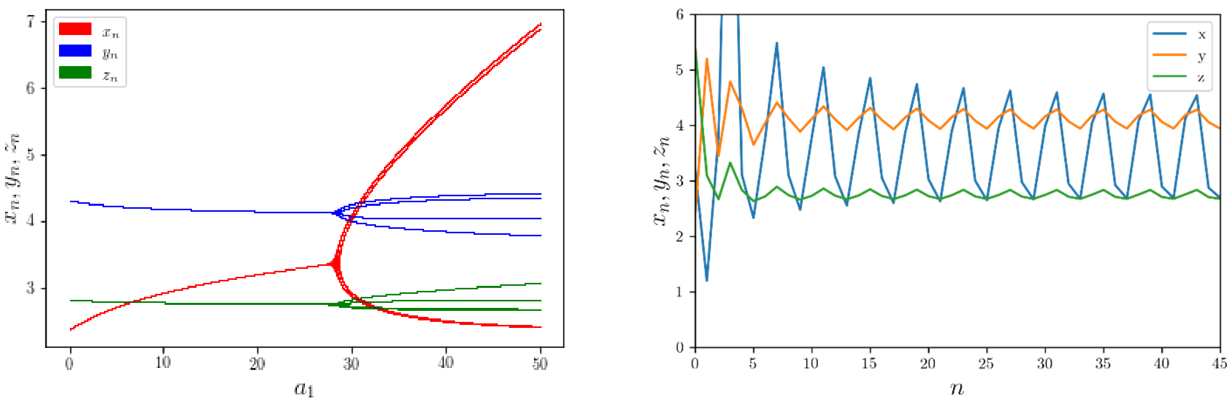
Use Remark 1.3.1 of [25] and Theorem 2.5 to get the result.

3. Numerical Analysis and Open Problem

In this section we observe the dynamics of the discrete model (1) numerically and propose an open problem. Figure (1a) shows the bifurcation diagram with a_1 as bifurcation parameter and figure (1b) shows the plots of x_n, y_n, z_n for a particular value,ie., $a_1 = 32.0$. Figure (1b) shows that the plot is eventually 4-periodic. Similarly figures(2a) and (3a) shows the bifurcation diagrams with a_2 and a_3 as bifurcation parameter, whereas figures(2b) and (3b) shows their corresponding plots for $a_2 = 10.0$ and $a_3 = 45.9$ respectively. Figures (2b) and (3b) shows that the plots are eventually 4-periodic.

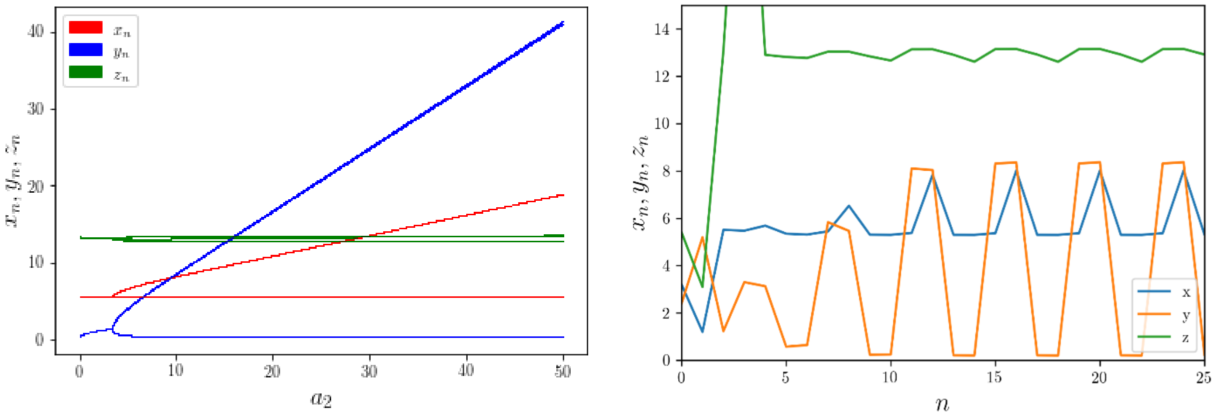
3.1 Open Problem

Derive the condition for (1) to be eventually 4-periodic.



a) [Bifurcation Diagrams of (1) with $a_1$as bifurcation parameter] b) [Plots of x_n, y_n, z_n with $a_1 = 32.0$]

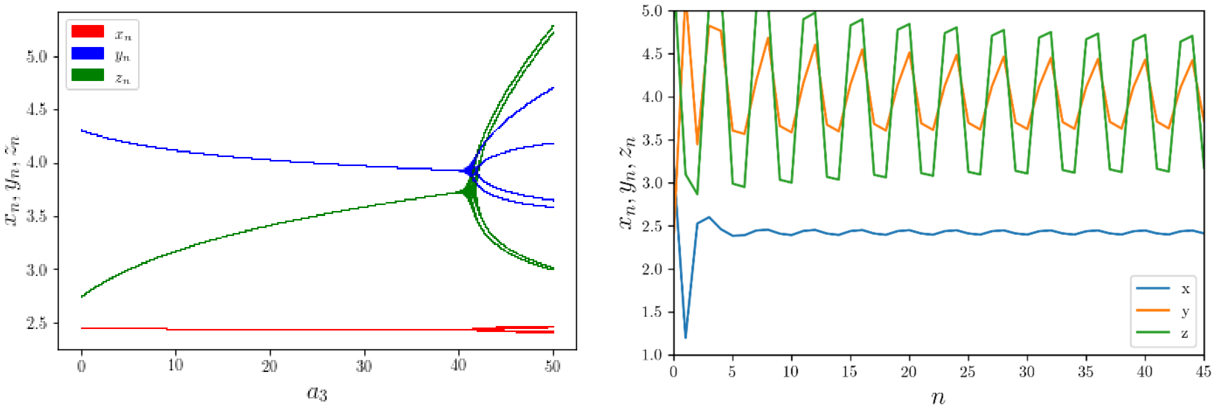
Figure 1: Here $\alpha_1 = 2.3, \alpha_2 = 3.2, \alpha_3 = 2.6, b_1 = .4, c_1 = .5, a_2 = 0.5, b_2 = 4.4, c_2 = 3.5, a_3 = 0.9, b_3 = 0.6, c_3 = 0.4$.



a) [Bifurcation Diagrams of (1) with a_2 as bifurcation parameter]

b) [Plots of x_n, y_n, z_n with $a_2 = 10.0$]

Figure 2: Here $\alpha_1 = 5.3, \alpha_2 = 0.2, \alpha_3 = 12.6, a_1 = 0.5, b_1 = .4, c_1 = .5, b_2 = 4.4, c_2 = 1.5, a_3 = 5.9, b_3 = 8.6, c_3 = 0.4$.



a) [Bifurcation Diagrams of (1) with a_3 as bifurcation parameter]

b) [Plots of x_n, y_n, z_n with $a_3 = 45.0$]

Figure 3: Here $\alpha_1 = 2.3, \alpha_2 = 3.2, \alpha_3 = 2.6, a_1 = 0.9, b_1 = .4, c_1 = .5, a_2 = 0.5, b_2 = 4.4, c_2 = 3.5, b_3 = 0.6, c_3 = 0.4$.

4. Conclusion

In this paper, we studied the dynamics of a second-order system defined by three variables, focusing on the existence of a unique positive equilibrium and its global stability. This study is particularly relevant in the context of population biology, where understanding the conditions for local asymptotic stability and global stability are crucial. We successfully established the conditions which assure the global asymptotic stability of the unique positive equilibrium.Moreover, we proposed an open problem that invite further investigation into the conditions necessary for the system to exhibit 4-periodic behavior. Addressing this problem will provide deeper insights into the periodic nature of the system and its long-term behavior.

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