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A Class of Analytic Functions with respect to Symmetric Points Involving Multiplicative Derivative

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Abstract:
Here we explore the behaviour and deviations of the geometric properties of a class of univalent functions when the classical derivative is replaced with a multiplicative derivative. The primary question that we will be addressing here is that given a more versatile calculus of Newton and Euler, why we need a study involving such a restrictive calculus so called as multiplicative calculus. Precisely, we introduce and study a new subclass of analytic function with respect to symmetric points using multiplicative derivative. We obtain the estimates for the initial coefficients and Fekete-Szegő inequalities of the same. We have included some examples to establish the inclusion and closure properties of our defined class. Further, we obtain the logarithmic and inverse coefficients for the defined function class.

Keywords: multiplicative calculus, starlike function, convex function, close-to-convex function, subordination.

1. Introduction

For $\mathcal{U} = \{\omega \in \mathbb{C};\ |\omega| < 1\}$, we let $\mathcal{A}$ to denote the class of functions analytic with normalization $\varphi(0) = 0 = \varphi'(0) - 1$. We denote the classes of starlike and convex function by $\mathcal{S}^*(\gamma)$ and $\mathcal{C}(\gamma)$ respectively. It is well-known that $\mathcal{S}^*(\gamma)$ and C(γ) satisfies the condition

$$Re\left(\frac{\omega\varphi'(\omega)}{\varphi(\omega)}\right) > \gamma \quad and\ \ Re\left(1 + \frac{\omega\varphi''(\omega)}{\varphi'(\omega)}\right) > \gamma, (\omega \in \mathcal{U}; 0 \le \gamma < 1),$$

respectively. Let $\mathcal{P}$ signify the category of functions that are analytic in $\mathcal{U}$ with $p(0) = 1$ and $Re\{p(\omega)\} > 0$ for all $\omega \in \mathcal{U}$. Let $\mathcal{S}$ denote the class of functions $\varphi \in \mathcal{A}$ which are univalent in $\mathcal{U}$. The class $\mathcal{S}$ is not preserved under even the most basic operations like addition or subtraction. However, the class is preserved under $k-$root transformation. It is well known that if $\varphi \in \mathcal{A}$ is in $\mathcal{S}$, then \$$[\varphi(\omega^k)]^{\frac{1}{k}}$, ($k$ is a positive integer) is also in $\mathcal{S}$. Refer to [9, pg. 18] for the formal definition of k-symmetric function. For every integer k, let $\varphi_k(\omega)$ be defined by the following equality

$$\varphi_k(\omega) = \frac{1}{k}\sum_{\nu=0}^{k-1}\frac{\varphi(\varepsilon^{\nu}\ \omega)}{\varepsilon^{\nu}}, \qquad (\varphi \in \mathcal{A}). \tag{1.1}$$

From (1.1), we see that $\varphi_k(\omega)$ satisfies the linearity conditions. Sakaguchi [22] defined the class $\mathcal{S}_s^*(\gamma)$, the class of function starlike with respect to symmetric points as follows

$$Re\left(\frac{2\omega\varphi'(\omega)}{\varphi(\omega)-\varphi(-\omega)}\right) > \gamma, \qquad (\omega \in \mathcal{U}; 0 \le \gamma < 1)\,.$$

The functions belonging to the class $\mathcal{S}_s^*(\gamma)$ are univalent (see [22]). Extending the Sakaguchi class of starlike function, the class of starlike functions with respect to k −symmetric points denoted by $\mathcal{S}_s^k(\gamma)$ was introduced and is known to satisfy the analytic characterization

$$Re\left(\frac{2\omega\varphi'(\omega)}{\varphi_k(\omega)}\right) > \gamma, \qquad (k = 1, 2, 3, \dots)\,, \qquad (1.2)$$

where $\varphi_k(\omega) = \frac{1}{k}\sum_{\nu=0}^{k-1}\frac{\varphi(\varepsilon^\nu\,\omega)}{\varepsilon^\nu}$, $(\varphi \in \mathcal{A})$. For developments and study of various subclasses of analytic functions with respect to symmetric points, refer to [12, 13, 23, 24, 25, 26, 27, 28].

Bashirov, Kurpinar and őzyapĭ in [5] (also see [6, 7, 21]) studied the properties of a calculus titled *Multiplicative calculus* which has been a useful mathematical tool in economics and finance. For a positive real valued function $\varphi: \mathcal{R} \to \mathcal{R}$, the multiplicative derivative φ^* is defined as follows

$$\varphi^*(x) = \lim_{h\to 0}\left(\frac{\varphi(x+h)}{\varphi(x)}\right)^{\frac{1}{h}} = e^{\frac{\varphi'(x)}{\varphi(x)}} = e^{[\ln\varphi(x)]'}$$

where $\varphi'(x)$ is the ordinary derivative. The $*$-derivative of φ at ω belonging to a small neighbourhood of a domain in a complex plane where φ is non-vanishing differentiable, is given by

$$\varphi^*(\omega) = e^{\frac{\varphi'(\omega)}{\varphi(\omega)}} \; and \; \varphi^{*(n)}(\omega) = e^{\left[\frac{\varphi'(\omega)}{\varphi(\omega)}\right]^{(n)}}, \;\; n = 1, 2, \dots.$$

Influenced by the definition of multiplicative derivative, recently Karthikeyan and Murugusundaramoorthy in [10] introduced and studied a class of analytic functions $\mathcal{R}(\chi)$ satisfying the subordination condition

$$\frac{\omega\, e^{\frac{\omega^2\varphi'(\omega)}{\varphi(\omega)}}}{\varphi(\omega)} \prec \chi(\omega) \qquad (1.3)$$

where $\chi \in \mathcal{P}$ and $\chi(\mathcal{U})$ is symmetric with respect to the real axis which has a series expansion of the form

$$\chi(\omega) = 1 + L_1\omega + L_2\omega^2 + L_3\omega^3 + \cdots, \qquad (L_1 > 0; \;\; \omega \in \mathcal{U}). \qquad (1.4)$$

The class is non-empty and possess good geometrical implications but it does not reduce to well-known subclasses of $\mathcal{S}$. For the detailed analysis and closure properties of the class $\mathcal{R}(\chi)$, refer to [10, 11].

Throughout this paper, we let

$$\Gamma_{n,k} = \frac{1}{k}\sum_{\nu=0}^{k-1}\left[\exp\frac{2\pi\, i}{k}\right]^{(n-1)\nu}$$

and

$$\varphi_k(\omega) = \omega + \sum_{n=2}^{\infty}\Gamma_{n,k}a_n\omega^n.$$

Motivated by $\mathcal{R}(\chi)$, here we will introduce and study a new subclass of starlike functions with respect to symmetric points. The definition of the new function class are as follows.

Definition 1.1. Let k be chosen such that $\Gamma_{n,k} = (n-1)$. A function $\varphi \in \mathcal{A}$ is said to be in $\mathcal{M}_k(\chi)$, if it satisfies the following condition

$$\frac{\omega\, \mathrm{F}^*(\omega)}{e\, \varphi_k(\omega)} \prec \chi(\omega), \quad (\omega \in \mathcal{U};\ \mathrm{e} = \backslash\exp(1)) \qquad (1.5)$$

where $F^*(\omega) = e^{\frac{\omega\varphi'(\omega)}{\varphi(\omega)}}$, $\varphi_k(\omega) = \frac{1}{k}\sum_{\nu=0}^{k-1}\frac{\varphi(\varepsilon^\nu\,\omega)}{\varepsilon^\nu}$, $\chi \in \mathcal{P}$ and $\chi(\mathcal{U})$ is defined as in (1.4).

The class $\mathcal{M}_k(\chi)$ has been defined by replacing the classical derivative with a multiplicative derivative in (1.2). Note that $k = 1$ is not admissible in $\mathcal{M}_k(\chi)$, so the class of function $\varphi \in \mathcal{A}$ satisfying

$$Re\left(\frac{\omega\, F^*(\omega)}{e\, \varphi(\omega)}\right) > 0, \qquad \left(F^*(\omega) = e^{\frac{\omega\varphi'(\omega)}{\varphi(\omega)}}\right)$$

fails to exist. The reason for imposing $k \neq 1$ in $\mathcal{M}_k(\chi)$ is that we would be unable to work within the existing framework, since the requirement of the condition of L_1 to be non-zero would be violated.

Alternatively, we will now define a class which would be defined for $k = 1$.

Definition 1.2. Let k be chosen such that $\Gamma_{n,k} \neq 0$. A function $\varphi \in \mathcal{A}$ is said to be in $\mathcal{L}_k(\chi)$, if it satisfies the following condition

$$\frac{\omega\, e^{\frac{\omega^2\varphi'(\omega)}{\varphi(\omega)}}}{\varphi_k(\omega)} \prec \chi(\omega), \qquad (\omega \in \mathcal{U}), \qquad (1.6)$$

where $\varphi_k(\omega)$ is defined as in (1.1), $\chi \in \mathcal{P}$ and $\chi(\mathcal{U})$\$ is defined as in (1.4).

From the study of [10], we find that classes involving the multiplicative derivative does not have any well-known classes as its special cases. But these classes had very good geometric behaviour when compared to various other subclasses of analytic functions.

Letting $k = 2$ and $\chi(\omega) = \frac{1+\omega}{1-\omega}$ in (1.5), we get the following familiar analytic characterizations

$$Re\left(\frac{2\omega\, e^{\frac{\omega\varphi'(w)}{\varphi(\omega)}-1}}{\varphi(\omega)-\varphi(-\omega)}\right) > 0. \qquad (1.7)$$

Notice that the expression in (1.7) is similar to the analytic characterization of $\mathcal{S}_s^*(0)$. Also letting $k = 1$ in definition 1.2, the class $\mathcal{L}_k(\chi)$ reduces to the class $\mathcal{R}(\chi)$ studied by Karthikeyan and Murugusundaramoorthy in [10].

2. Coefficients Inequalities Of Functions In $\mathcal{M}_k(\chi)$ And $\mathcal{L}_k(\chi)$

Now we will find the solution to the Fekete-Szegő problem for $\varphi \in \mathcal{M}_k(\chi)$.

Lemma 2.1 [15] If $d(\omega) = 1 + \sum_{k=1}^{\infty} d_k\, \omega^k \in \mathcal{P}$, and ρ is complex number, then

$$|d_2 - \rho\, d_1^2| \leq 2\max\{1; |2\rho - 1|\},$$

and the result is sharp.

Theorem 2.1. If $\varphi(\omega) \in \mathcal{M}_k(\chi)$, then we have

$$|a_2| \leq \frac{L_1}{|1-\Gamma_{2,k}|} \tag{2.1}$$

$$|a_3| \leq \frac{L_1}{|2-\Gamma_{3,k}|} \max\left\{1;\ \left|\frac{L_2}{L_1} - L_1 \frac{\left(\Gamma_{2,k}^2 - \Gamma_{2,k} - \frac{1}{2}\right)}{\left(1-\Gamma_{2,k}\right)^2}\right|\right\} \tag{2.2}$$

and for all $\rho \in \mathbb{C}$

$$|a_3 - \rho\, a_2^2| \leq \frac{L_1}{|2-\Gamma_{3,k}|} \max\left\{1;\ \left|\frac{L_2}{L_1} - L_1 \frac{\left(\Gamma_{2,k}^2 - \Gamma_{2,k} - \frac{1}{2}\right)}{\left(1-\Gamma_{2,k}\right)^2} - \frac{L_1\rho\left(2-\Gamma_{3,k}\right)}{\left(1-\Gamma_{2,k}\right)^2}\right|\right\}. \tag{2.3}$$

The inequality is sharp for each $\rho \in \mathbb{C}$.

Proof. As $\varphi \in \mathcal{M}_k(\chi)$, by (1.5) we have

$$\frac{\omega\, F^*(\omega)}{e\, \varphi_k(\omega)} = \chi[w(\omega)]. \tag{2.4}$$

Thus, let $\vartheta \in \mathcal{P}$ be of the form $\vartheta(\omega) = 1 + \sum_{k=1}^{\infty} \vartheta_k \omega^k$ and defined by

$$\vartheta(\omega) = \frac{1+w(\omega)}{1-w(\omega)}, \qquad \omega \in \mathcal{U}\,.$$

On computation, the right hand side of (2.4)

$$\chi[w(\omega)] = 1 + \frac{\vartheta_1 L_1}{2}\omega + \frac{L_1}{2}\left[\vartheta_2 - \frac{\vartheta_1^2}{2}\left(1 - \frac{L_2}{L_1}\right)\right]\omega^2 + \cdots. \tag{2.5}$$

The left hand side of (2.4) will be of the form

$$\frac{\omega\, F^*(\omega)}{e\, \varphi_k(\omega)} = 1 + a_2[1-\Gamma_{2,k}]\omega + \left[\left(\Gamma_{2,k}^2 - \Gamma_{2,k} - \frac{1}{2}\right)a_2^2 + \left(2-\Gamma_{3,k}\right)a_3\right]\omega^2 + \cdots. \tag{2.6}$$

From (2.5) and (2.6), we obtain

$$a_2 = \frac{1}{\left(1-\Gamma_{2,k}\right)}\left[\frac{\vartheta_1 L_1}{2}\right] \tag{2.7}$$

and

$$a_3 = \frac{L_1}{2\left(2-\Gamma_{3,k}\right)}\left[\vartheta_2 - \frac{\vartheta_1^2}{2}\left(1 - \frac{L_2}{L_1} - L_1\frac{\left(\Gamma_{2,k}^2 - \Gamma_{2,k} - \frac{1}{2}\right)}{\left(1-\Gamma_{2,k}\right)^2}\right)\right] \tag{2.8}$$

Equations (2.1) can be obtained by applying the well-known result of $|\vartheta_1| \leq 2$ in (2.7). Applying Lemma 2.1 in (2.8), we get (2.2).

Now to prove the Fekete-Szegő inequality for the class $\mathcal{M}_k(\chi)$, we consider

$$|a_3 - \rho\, a_2^2| = \left|\frac{L_1}{2\left(2-\Gamma_{3,k}\right)}\left[\vartheta_2 \; - \frac{\vartheta_1^2}{2}\left(1 - \frac{L_2}{L_1} - L_1 \frac{\left(\Gamma_{2,k}^2 - \Gamma_{2,k} - \frac{1}{2}\right)}{\left(1-\Gamma_{2,k}\right)^2}\right)\right] - \frac{\rho\, \vartheta_1^2\, L_1^2}{4\left(1-\Gamma_{2,k}\right)^2}\right|$$

$$= \left|\frac{L_1}{2\left(2-\Gamma_{3,k}\right)}\left[\vartheta_2 \; - \frac{\vartheta_1^2}{2}\left(1 - \frac{L_2}{L_1} - L_1 \frac{\left(\Gamma_{2,k}^2 - \Gamma_{2,k} - \frac{1}{2}\right)}{\left(1-\Gamma_{2,k}\right)^2} + \frac{L_1 \rho\left(2-\Gamma_{3,k}\right)}{\left(1-\Gamma_{2,k}\right)^2}\right)\right]\right|.$$

Using the triangle inequality and Lemma 2.1 in the above equality, we can obtain (2.3).

Let $k = 2$ in Theorem 2.1, we have the following.

Corollary 2.1. Let $\varphi \in \mathcal{M}_2(\chi)$. Then,

$$|a_2| \leq\; L_1, \qquad\qquad |a_3\,| \leq\; L_1 \max\left\{1;\; \left|\frac{L_2}{L_1} + \frac{L_1}{2}\right|\right\}$$

and for a complex number ρ,

$$|a_3 - \rho\, a_2^2| \leq\; L_1 \max\left\{1; \left|\frac{L_2}{L_1} + \frac{L_1}{2}(1-2\rho)\right|\right\}$$

The inequality is sharp for each $\rho \in \mathbb{C}$.

Proof. By the definition of $\varphi_k(\omega)$, we have

$$\varphi_k(\omega) = \frac{1}{k}\sum_{\nu=0}^{k-1} \frac{\varphi(\varepsilon^\nu\, \omega)}{\varepsilon^\nu} = \omega + \sum_{n=2}^{\infty} \Gamma_{n,k}\, a_n\, \omega^n,$$

where $\Gamma_{n,k} = \frac{1}{k}\sum_{\nu=0}^{k-1}\left[\exp\left(\frac{2\pi i}{k}\right)\right]^{(n-1)\nu}$. It can be easily seen that

$$\Gamma_{2,2} = \frac{1}{2}\sum_{\nu=0}^{1}[\exp(\pi i)]^{\nu} = 0, \qquad \Gamma_{3,2} = \frac{1}{2}\sum_{\nu=0}^{1}[\exp(\pi i)]^{2\nu} = 1.$$

Substituting the above expression in (2.1), (2.2) and (2.3), we obtain the assertion of the corollary.

Fixing $\chi(\omega)$ to be well-known conic regions, we can obtain several applications of our result. But here we will restrict to pointing out the case when $\chi(\omega)$ is known to be extremal.

Letting $\chi(\omega) = \frac{1+\omega}{1-\omega}$ in Corollary 2.1, we get

Corollary 2.2. Let $\varphi \in \mathcal{A}$ satisfy the condition

$$Re\left(\frac{2\omega\, e^{\frac{\omega\varphi'(\omega)}{\varphi(\omega)}-1}}{\varphi(\omega) - \varphi(-\omega)}\right) > 0.$$

Then,

$$|a_2| \leq\; 2, \quad |a_3| \leq\; 4$$

and for a complex number ρ,

$$|a_3 - \rho\, a_2^2| \le 2\max\{1;\ 2|1-\rho|\}$$

Theorem 2.2. If $\varphi(\omega) \in \mathcal{L}_k(\chi)$, then we have

$$|a_2| \le \frac{1}{|\Gamma_{2,k}|}|L_1 + 1|. \tag{2.9}$$

$$|a_3| \le \frac{L_1}{|\Gamma_{3,k}|}\left\{\max\left\{1, \left|\frac{L_2}{L_1} - L_1\right|\right\} + \left|\frac{1}{\Gamma_{2,k}} + 1\right| + \frac{3}{2|L_1|}\right\} \tag{2.10}$$

and for all $\rho \in \mathbb{C}$

$$|a_3 - \rho\, a_2^2\,| \le \frac{L_1}{|\Gamma_{3,k}|}$$

$$\left[\max\left\{1, \left|\frac{L_2}{L_1} - L_1 + \frac{\rho L_1 \Gamma_{3,k}}{\Gamma_{2,k}^2}\right|\right\} + \left|\frac{1}{\Gamma_{2,k}} + 1 - \frac{2\rho\Gamma_{3,k}}{\Gamma_{2,k}^2}\right| + \frac{1}{2|L_1|}\left|3 - -\frac{2\rho\Gamma_{3,k}}{\Gamma_{2,k}^2}\right|\right] \tag{2.11}$$

The inequality is sharp for each $\rho \in \mathbb{C}$.

Proof. Expanding the left hand side of (1.6) and simplifying the expansion we get

$$\frac{\omega\, e^{\frac{\omega^2\varphi'(\omega)}{\varphi(\omega)}}}{\varphi_k(\omega)} = 1 + (1 - a_2\Gamma_{2,k})\omega + \left(\frac{1}{2} + a_2 - a_2\Gamma_{2,k} + a_2^2\Gamma_{2,k}^2 - a_3\Gamma_{3,k}\right)\omega^2 + \cdots. \tag{2.12}$$

Given $\varphi \in \mathcal{L}_k\ (\chi)$, so the right hand side of the expansion of (1.6) is the same as (2.5).

From (2.12) and (2.5), we obtain

$$a_2 = -\frac{1}{\Gamma_{2,k}}\left[\frac{\vartheta_1 L_1}{2} - 1\right] \tag{2.13}$$

and

$$a_3 = -\frac{1}{\Gamma_{3,k}}\left\{\frac{L_1}{2}\left[\vartheta_2 - \frac{\vartheta_1^2}{2}\left(1 - \frac{L_2}{L_1} + L_1\right)\right] + \frac{\vartheta_1 L_1}{2}\left[\frac{1}{\Gamma_{2,k}} + 1\right] - \frac{3}{2}\right\}. \tag{2.14}$$

Equations (2.9) can be obtained by applying the well-known result of $|\vartheta_1| \le 2$ in (2.13) Applying Lemma 2.1 in (2.14), we get (2.10).

Now to prove the Fekete-Szegő inequality for the class $\mathcal{L}_k(\chi)$, we consider

$$|a_3 - \rho\, a_2^2| = \left|\frac{1}{\Gamma_{3,k}}\left\{\frac{L_1}{2}\left[\vartheta_2 - \frac{\vartheta_1^2}{2}\left(1 - \frac{L_2}{L_1} + L_1\right)\right] + \frac{\vartheta_1 L_1}{2}\left[\frac{1}{\Gamma_{2,k}} + 1\right] - \frac{3}{2}\right\} + \frac{\rho}{\Gamma_{2,k}^2}\left[\frac{\vartheta_1^2\, L_1^2}{4} - \vartheta_1 L_1 + 1\right]\right|.$$

Using the triangle inequality and Lemma 2.1 in the above equality, we can obtain (2.11).

Corollary 2.3. [10] If $\varphi(\omega) \in \mathcal{L}_1(\chi)$, then we have

$$|a_2| \leq 1 + L_1.$$

$$|a_3| \leq L_1 \left[\max\left\{1, \left|\frac{L_2}{L_1} - L_1\right|\right\} + \frac{3}{2|L_1|} + 2\right]$$

and for all $\rho \in \mathbb{C}$

$$|a_3 - \rho\, a_2^2| \leq L_1 \left[\max\left\{1, \left|\frac{L_2}{L_1} - L_1(1-\rho)\right|\right\} + \frac{1}{2|L_1|}|3 - 2\rho| + 2|1-\rho|\right].$$

The inequality is sharp for each $\rho \in \mathbb{C}$.

Letting $\chi(\omega) = \frac{1+\omega}{1-\omega}$ in Corollary 2.3, we get

Corollary 2.4 Let $\varphi \in \mathcal{A}$ satisfy the condition

$$Re\left(\frac{\omega e^{\frac{\omega^2\varphi'(\omega)}{\varphi(\omega)}}}{\varphi(\omega)}\right) > 0.$$

Then,

$$|a_2| \leq 3, \quad |a_3| \leq \frac{15}{2}$$

and for a complex number ρ,

$$|a_3 - \rho\, a_2^2| \leq 2\left[\max\{1, |2\rho - 1|\} + 2|1-\rho| + \frac{1}{4}|3 - 2\rho|\right].$$

The inequality is sharp for each $\rho \in \mathbb{C}$.

3. Coefficient Estimates For The Inverse Functions.

In this section, we will find the coefficient estimates for the inverse functions of φ belonging to the classes $\mathcal{M}_k(\chi)$ and $\mathcal{L}_k(\chi)$. Refer to [14 18] for its relevance and application in the field of univalent function theory. The following result would help us to obtain the coefficient estimates for φ^{-1} (provided it exists), form the coefficient estimates of φ.

Lemma 3.1. [9, p. 56] If the function $\varphi \in \mathcal{A}$ and $\varphi^{-1} = g(w)$ given by

$$g(w) = w + \sum_{k=2}^{\infty} b_k\, w^k \qquad (3.1)$$

are inverse functions, then for $k \geq 2$

$$b_k = \frac{(-1)^{k+1}}{k!}\begin{bmatrix} ka_2 & 1 & 0 & \cdots & 0 \\ 2ka_3 & (k+1)a_2 & 2 & \cdots & 0 \\ 3ka_4 & (2k+1)a_3 & (k+1)a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & (k-2) \\ (k-1)ka_k & [k(k-2)+1]a_{k-1} & [k(k-3)+2]a_{k-2} & \cdots & (2k-2)a_2 \end{bmatrix} \qquad (3.2)$$

The elements of the determinant in (3.2) are given by

$$\Theta_{ij} = \begin{cases} [(i-j+1)k+j-1]a_{i-j+2}, & \text{if } i+1 \geq j \\ 0, & \text{if } i+1 < j \end{cases}.$$

The functions in $\mathcal{M}_k(\chi)$ need not be univalent, but since $\varphi'(0) = 1 \neq 0$ for all $\varphi \in \mathcal{M}_k(\chi)$ and $\varphi(0) = 0$, there exist an inverse function in some small disk with center at $w = 0$.

Theorem 3.1. Let $\varphi \in \mathcal{M}_k(\chi)$ and let φ^{-1} be the inverse of φ defined by

$$\varphi^{-1}(w) = w + \sum_{k=2}^{\infty} b_k\, w^k, \qquad (|\,w| < r;\ r \geq \frac{1}{4}),$$

then

$$|\,b_2| \leq \frac{L_1}{|\,1-\Gamma_{2,k}|}$$

$$|b_3| \leq \frac{L_1}{|\,2-\Gamma_{3,k}|}\max\left\{1;\ \left|\frac{L_2}{L_1} - \frac{L_1\left(\Gamma_{2,k}^2 - \Gamma_{2,k} - \frac{1}{2}\right)}{\left(1-\Gamma_{2,k}\right)^2} - \frac{2L_1\left(2-\Gamma_{3,k}\right)}{\left(1-\Gamma_{2,k}\right)^2}\right|\right\}$$

and for a complex number τ,

$$|b_3 - \tau\, a_2^2\,| \leq \frac{L_1}{|\,2-\Gamma_{3,k}|}\max\left\{1;\ \left|\frac{L_2}{L_1} - \frac{L_1\left(\Gamma_{2,k}^2 - \Gamma_{2,k} - \frac{1}{2}\right)}{\left(1-\Gamma_{2,k}\right)^2} - \frac{(\tau-2)L_1\left(2-\Gamma_{3,k}\right)}{\left(1-\Gamma_{2,k}\right)^2}\right|\right\}$$

Proof. From $\varphi(\omega) = \omega + \sum_{n=2}^{\infty} a_n \omega^n$ and (3.1), we have

$$b_2 = -a_2 \quad and \quad b_3 = 2a_2^2 - a_3.$$

The estimate for $|\,b_2| = |\,a_2|$ follows immediately from (2.7). Letting $\rho = 2$ in (2.3), we get the estimate $|b_3|$.

To find the Fekete-Szegő inequality for φ^{-1}, consider

$$|b_3 - \tau\, b_2^2| = |2a_2^2 - a_3 - \tau\, a_2^2| = |a_3 - (\tau-2)a_2^2|.$$

Changing $\rho = (\tau - 2)$ in the (2.3), we get the desired result.

Analogous to the results obtained in Theorem 3.2, we can easily get the following result.

Theorem 3.2 Let $\varphi \in \mathcal{L}_k(\chi)$$ and let φ^{-1} be the inverse of φ defined by

$$\varphi^{-1}(w) = w + \sum_{k=2}^{\infty} b_k\, w^k, \qquad (|\,w| < r;\ r \geq \frac{1}{4}),$$

then$$

$$|\, b_2| \le \frac{1}{\left|\,\Gamma_{2,k}\right|}[L_1 + 1],$$

$$|b_3| \le \frac{L_1}{\left|\,\Gamma_{3,k}\right|}\left[\max\left\{1;\ \left|\frac{L_2}{L_1} - L_1 + \frac{2L_1\Gamma_{3,k}}{\Gamma_{2,k}^2}\right|\right\} + \left|1 + \frac{1}{\Gamma_{2,k}} - \frac{4\Gamma_{3,k}}{\Gamma_{2,k}^2}\right| + \frac{1}{2|\, L_1|}\left|3 - \frac{4\Gamma_{3,k}}{\Gamma_{2,k}^2}\right|\right]$$

and for a complex number τ,

$$|b_3 - \tau\, a_2^2| \le \frac{L_1}{\left|\,\Gamma_{3,k}\right|}\left[\max\left\{1;\ \left|\frac{L_2}{L_1} - L_1 + \frac{(\tau-2)L_1\Gamma_{3,k}}{\Gamma_{2,k}^2}\right|\right\} + \left|1 + \frac{1}{\Gamma_{2,k}} - \frac{2(\tau-2)\Gamma_{3,k}}{\Gamma_{2,k}^2}\right| + \frac{1}{2|\, L_1|}\left|3 - \frac{2(\tau-2)\Gamma_{3,k}}{\Gamma_{2,k}^2}\right|\right].$$

4. Logarithmic Coefficients

Milin in [16] studied the properties of the logarithmic coefficients to obtain the bounds of the Taylor coefficients of univalent functions. The Milin conjuncture about the inequalities of the logarithmic coefficients garnered the attention several researchers in those period of time, because proving Milin conjuncture would imply proving Robertson conjecture and the Bieberbach conjecture. Refer to Ponnusamy et al. [18, 19 20] and [1, 2, 3, 4, 17] for the detailed study on properties and significance of the logarithmic coefficients.

The logarithmic coefficients d_n of a function $\varphi \in \mathcal{A}$ such that $\frac{\varphi(\omega)}{\omega} \neq 0$ for all $\omega \in \mathcal{U}$ is defined by

$$\log \varphi(\omega) \ = 2\sum_{n=1}^{\infty} d_n \omega^n.$$

Note that for all functions $\varphi(\omega) \in \mathcal{M}_k(\chi)$ and $\mathcal{L}_k(\chi)$, the relation (4.1) is well-defined.

Theorem 4.1. If $\varphi(\omega) \in \mathcal{M}_k(\chi)$, with the logarithmic coefficients given by (4.1), then,

$$|d_1| \le \frac{L_1}{2\left|\, 1 - \Gamma_{2,k}\right|}$$

$$|d_2| \le \frac{L_1}{2\left|\, 2 - \Gamma_{3,k}\right|}\max\left\{1;\ \left|\frac{L_2}{L_1} - \frac{L_1\left(\Gamma_{2,k}^2 - \Gamma_{2,k} - \frac{1}{2}\right)}{\left(1 - \Gamma_{2,k}\right)^2} - \frac{L_1\left(2 - \Gamma_{3,k}\right)}{2\left(1 - \Gamma_{2,k}\right)^2}\right|\right\}.$$

For $\mu \in \mathbb{C}$, we have

$$|d_2 - \mu d_1^2| \le \frac{L_1}{2\left|\, 2 - \Gamma_{3,k}\right|}\max\left\{1;\ \left|\frac{L_2}{L_1} - \frac{L_1\left(\Gamma_{2,k}^2 - \Gamma_{2,k} - \frac{1}{2}\right)}{\left(1 - \Gamma_{2,k}\right)^2} - \frac{L_1(1+\mu)\left(2 - \Gamma_{3,k}\right)}{2\left(1 - \Gamma_{2,k}\right)^2}\right|\right\}. \quad (4.2)$$

Proof. From $\varphi(\omega) = \omega + \sum_{n=2}^{\infty} a_n \omega^n$ and equating the first two coefficients of the relation (4.1), we get

$$d_1 = \frac{a_2}{2}, \quad d_2 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{2}\right).$$

Using (2.1) and (2.3) in the above expression, we can find the estimates for d_1 and d_2. To find the estimate (4.2), consider

$$|d_2 - \mu\, d_1^2| = \frac{1}{2}\left[a_3 - \frac{1+\mu}{2} a_2^2\right]$$

Changing $\rho = \frac{(1+\backslash mu)}{2}$ in (2.3) and simplifying, we get the desired result.

For completeness, we just state the following.

Theorem 4.2. If $\varphi(\omega) \in L_k(\chi)$, with the logarithmic coefficients given by (4.1), then,

$$|d_1| \leq \frac{1}{2|\,\Gamma_{2,k}|}[L_1 + 1]$$

$$|d_2| \leq \frac{L_1}{2|\,\Gamma_{3,k}|}\left[\max\left\{1;\ \left|\frac{L_2}{L_1} - L_1 + 2\frac{L_1\Gamma_{3,k}}{\Gamma_{2,k}^2}\right|\right\} + \left|1 + \frac{1}{\Gamma_{2,\mathrm{k}}} - \frac{4\Gamma_{3,\mathrm{k}}}{\Gamma_{2,\mathrm{k}}^2}\right| + \frac{1}{2|\,\mathrm{L}_1|}\left|3 - \frac{4\Gamma_{3,\mathrm{k}}}{\Gamma_{2,\mathrm{k}}^2}\right|\right].$$

For $\mu \in \mathbb{C}$, we have

$$\left|d_2 - \mu d_1^2\right| \leq \frac{L_1}{2|\,\Gamma_{3,k}|}\left[\max\left\{1;\ \left|\frac{L_2}{L_1} - L_1 + \frac{(1+\mu)L_1\Gamma_{3,k}}{\Gamma_{2,k}^2}\right|\right\} + \left|1 + \frac{1}{\Gamma_{2,\mathrm{k}}} - \frac{(1+\mu)\Gamma_{3,\mathrm{k}}}{\Gamma_{2,\mathrm{k}}^2}\right| \right.$$
$$\left. + \frac{1}{2|\,\mathrm{L}_1|}\left|3 - \frac{(1+\mu)\Gamma_{3,\mathrm{k}}}{\Gamma_{2,\mathrm{k}}^2}\right|\right].$$

Conclusions: The function class $\mathcal{M}_k(\chi)$ was defined by replacing the classical derivative with a multiplicative derivative in the well-known class of starlike functions with respect to symmetric points. The definition $\mathcal{M}_k(\chi)$ is not defined for all integers k. In fact the class exist only for the integers values of k, for which $\Gamma_{n,k} \neq (n-1)$. Hence we defined a class $\mathcal{L}_k(\chi)$, influenced by the multiplicative derivative which will be defined for $\Gamma_{n,k} = (n-1)$.

Letting χ to be a specific conic region and varying parameters involved in the Definitions 1.1 and 1.2, the function classes $\mathcal{M}_k(\chi)$ and $\mathcal{L}_k(\chi)$ will reduce to classes having good geometry. Our main results have wide applications. The classical starlike functions with respect to k-symmetric points are known to be univalent. But the classes $\mathcal{M}_k(\chi)$ and $\mathcal{L}_k(\chi)$ are neither a subclass nor a generalized class of univalent functions. So, the scope of further research of this paper are to explore the relationship and closure properties with the known classes like spirallike, starlike, and convex.

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