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Matthews Partial Metric Space Using $\mathcal{F}$-Contraction

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Abstract:

Following the notion of partial metric space (briefly PMS) established by Matthaws [1], in this present research article, we proved common fixed point theorem for two pair of self maps using the weakly compatible mappings through $\mathcal{F}$ -contraction. In addition, we give an illustrative example.

Keywords: Partial Metric Space **(**PMS), $\mathcal{F}$-contraction, weakly compatible (WC).

MSC (2000): 54H25;47H10.

1. Introduction

As a generalization of metric space, the concept of partial metric space emphasized by Matthews [1] in the year 1994.Recently, many fixed point theory researchers established fixed point theorems using different contractions with different weaker conditions. The PME play an vital role in study of data flows network and also theory of computation in the computer science. Some authors prove fixed point theorems in PMS like, [2],[3],[4],[6], and [8].

In the metric space the notion of F-contraction proposed by Wardowski [5], which is generalization of well known Banach contraction principle. Recently, Nazam M et.al, proved common fixed point theorems using one and two self mappings concerning F-contraction in PMS [7].

On the other hand, Sessa initiated the notation of weakly commuting maps which generalized the concept of commuting mappings consequently Jungck G, Rhoades B E [9] generalized this idea first to compatible mappings and later to weakly compatible (shortly WC) mappings.

The aim of the research article is to establish existence of unique common fixed point theorem for four self mappings through F -contraction using the idea of WC mappings in PMS.

Now we recall useful fundamental Definitions, Lemmas of PMS.

Definition 1.1[1]: A Partial metric on non-empty set $\mathfrak{X}$ is a function $\mathcal{P}:\mathfrak{X} \times \mathfrak{X} \to \mathbb{R}^+$ such that for all $\lambda, \mu, \xi \ in \ \mathfrak{X}$:

$(\mathcal{PMS}_1)$: $\mathcal{P}(\lambda, \lambda) = \mathcal{P}(\lambda, \mu) = \mathcal{P}(\mu, \mu) \Leftrightarrow \lambda = \mu$

$(\mathcal{PMS}_2)$: $\mathcal{P}(\lambda, \lambda) \leq \mathcal{P}(\lambda, \mu)$

$(\mathcal{PMS}_3)$: $\mathcal{P}(\lambda,\mu) = \mathcal{P}(\mu,\lambda)$

$(\mathcal{PMS}_4)$: $\mathcal{P}\ (\lambda,\mu) \le \mathcal{P}(\lambda,\xi) + \mathcal{P}(\xi,\mu) - \mathcal{P}(\xi,\xi)$.

The pair $(\mathfrak{X},\mathcal{P})$ is called partial metric space (briefly PMS) and $\mathcal{P}$ is a partial metric on $\mathfrak{X}$.

A mapping $\mathcal{P}^s::\mathfrak{X}\times\mathfrak{X} \to \mathbb{R}^+$ is defined by $\mathcal{P}^s(\lambda,\mu) = 2\mathcal{P}(\lambda,\mu) - \mathcal{P}(\lambda,\lambda) - \mathcal{P}(\mu,\mu)$

is usual metric, where $\mathcal{P}$ is partial metric on $\mathfrak{X}$.

Example 1.2[2,3]: Suppose that $\mathfrak{X} = \mathbb{R}^+ \cup \{\ 0\ \}$and we defined $\mathcal{P}(\lambda,\mu) = \max\{\lambda,\mu\}\ \ \forall \lambda,\mu \in \mathfrak{X}$.Then $(\mathfrak{X},\mathcal{P})$ is PMS but not (usual) metric space.

Definition 1.3[1]: Let $(\mathfrak{X},\mathcal{P})$ be a PMS. A sequence $\{\lambda_n\} \subseteq \mathcal{X}$ converges to $\lambda \in \mathfrak{X}$ if and only if $\mathcal{P}(\lambda,\lambda) = \lim_{\eta\to\infty} \mathcal{P}(\lambda,\lambda_\eta)$. Also cauchy sequence $\Leftrightarrow \lim_{\eta,\zeta\to\infty} \mathcal{P}(\lambda_\eta,\lambda_\zeta)$ exists finitely.

Lemma 1.4 [1]: Let $(\mathfrak{X},\mathcal{P})$ be a PMS and then,

(i) $\{\lambda_\eta\}$ is Cauchy sequence in $(\mathfrak{X},\mathcal{P})$ if and only if it is a cauchy in metric space $(\mathfrak{X},\mathcal{P}^s)$.

(ii) $(\mathfrak{X},\mathcal{P})$ is complete PMS if and only if $(\mathfrak{X},\mathcal{P}^s)$ is complete. Moreover, $\lim_{\eta\to\infty} \mathcal{P}^s(\lambda,\ \lambda_\eta) = 0 \Leftrightarrow \mathcal{P}(\lambda,\ \lambda) = \lim_{\eta\to\infty} \mathcal{P}\left(\lambda,\ \lambda_\eta\right) = \lim_{\eta,\zeta\to\infty} \mathcal{P}\left(\lambda_\eta,\lambda_\zeta\right)$.

Lemma1.5 [4]: Assume that $\lambda_n \longrightarrow \xi$ as $\eta \to \infty$ in $(\mathfrak{X},\mathcal{P})$ with $\mathcal{P}(\xi,\xi) = 0$ then $\lim_{\eta\to\infty} \mathcal{P}(\lambda_n,\ \mu) = \mathcal{P}(\xi,\mu)\ \ \forall\ \mu \in \mathfrak{X}$.

Lemma 1.6[4]: Let $(\mathfrak{X},\mathcal{P})$ be a PMS.

(i) If $\mathcal{P}\ (\ \lambda,\mu) = 0$ then $\lambda = \mu$. (ii) if $\lambda \ne \mu$ then $\mathcal{P}(\ \lambda,\mu) > 0$.

Definition 1.7[5]: A mapping $\mathcal{F}: \mathbb{R}^+ \to \mathbb{R}$ is said to be $\mathcal{F}$ -contraction if it satisfying following conditions

$(\mathcal{F}_1)$: if $\lambda,\mu \in \mathbb{R}^+$ such that $\lambda < \mu \Rightarrow \mathcal{F}(\lambda) < \mathcal{F}(\mu)$

$(\mathcal{F}_2)$: for each $\{\alpha_\eta\}_{\eta\in\mathbb{N}} \in \mathbb{R}^+$, $\lim_{\eta\to\infty} \alpha_\eta = 0$ if and only if $\lim_{\eta\to\infty} \mathcal{F}(\alpha_\eta) = -\infty$.

$(\mathcal{F}_3)$ $\exists$ real number $\theta \in (0,1)$ such that $\lim_{\alpha\to 0+} \alpha^\theta \mathcal{F}(\alpha) = 0$.

Notation 1.8[5]: We symbolize the collection of all functions which satisfy the above specified constraints $\mathcal{F}_1$ to $\mathcal{F}_3$ by $\Delta_\mathcal{F}$.

Definition 1.9[5]: A self mapping $\mathfrak{A}: \mathfrak{X} \to \mathfrak{X}$ is said to $\mathcal{F}$ - contraction if there exists a $\tau > 0$ such that for all $\lambda,\mu \in \mathfrak{X}$, $d(\mathfrak{A}\lambda,\mathfrak{A}\mu) > 0$ and we have $\tau + \mathcal{F}(d(\mathfrak{A}\lambda,\mathfrak{A}\mu)) \le \mathcal{F}(d(\lambda,\mu))$.

Example 1.10: Let $\mathcal{F}: \mathbb{R}^+ \to \mathbb{R}$ be given by $\mathcal{F}(\alpha) = \log_e \alpha$ and satisfies $\mathcal{F}_1$ to $\mathcal{F}_3$.Each mapping $\mathfrak{A}: \mathfrak{X} \to \mathfrak{X}$ is an $\mathcal{F}$ -contraction such that for all $\lambda,\mu \in \mathfrak{X}, \mathfrak{A}\lambda \ne \mathfrak{A}\mu, d(\mathfrak{A}\lambda,\mathfrak{A}\mu) \le e^{-\tau} d(\lambda,\mu)$.

It is clear that $\lambda,\mu \in \mathfrak{X}$ such that $\mathfrak{A}\lambda = \mathfrak{A}\mu$ then $d(\mathfrak{A}\lambda,\mathfrak{A}\mu) \le e^{-\tau} d(\lambda,\mu)$ also satisfying i.e. $\mathfrak{A}$ is Banach contraction.

Example 1.11: Defined the complete PMS $(\mathfrak{X}, \mathcal{P})$ by $\mathcal{P}(\lambda, \mu) = max\{\lambda, \mu\}$ and also complete metric space $(\mathfrak{X}, d)$ by $d(\lambda, \mu) = |\lambda - \mu|$ for all $\lambda, \mu \in \mathfrak{X}$.

Define mappings $\mathcal{F}: \mathbb{R}^+ \longrightarrow \mathbb{R}$ and $\mathcal{F}(\alpha) = \log_e \alpha$ and $\mathfrak{A}$ by $\mathfrak{A}(\lambda) = \begin{cases} \frac{\lambda}{2} & \text{if } \lambda \in [0,1) \\ \frac{3}{4} & \text{if } \lambda = 1 \end{cases}$ then

$\mathfrak{A}$ is not a $\mathcal{F}$ -contraction in metric space certainly for λ =1 and μ =1/2,

$d(\mathfrak{A}\lambda, \mathfrak{A}\mu) >0$ and we have $\tau + \mathcal{F}\big(d(\mathfrak{A}\lambda, \mathfrak{A}\mu)\big) \leq \mathcal{F}\big(d(\lambda, \mu)\big)$

$$\Rightarrow \tau + \mathcal{F}\left(d\left(\mathfrak{A}(1), \mathfrak{A}\left(\frac{1}{2}\right)\right)\right) \leq \mathcal{F}\left(d\left(1, \frac{1}{2}\right)\right)$$

$$\Rightarrow \tau + \left|\frac{3}{4} - \frac{1}{4}\right| \leq \left|1 - \frac{1}{2}\right|$$

$$\Rightarrow \tau + \frac{1}{2} \leq \frac{1}{2} \text{ which is a contradiction for all } \tau > 0.$$

Now if we studying in PMS $(\mathfrak{X}, \mathcal{P})$ we get,

$$\tau + \mathcal{F}(\mathcal{P}(\mathfrak{A}\lambda, \mathfrak{A}\mu)) \leq \mathcal{F}(\mathcal{P}(\lambda, \mu))$$

$$\Rightarrow \tau + \mathcal{F}\left(\mathcal{P}\left(\mathfrak{A}(1), \mathfrak{A}\left(\frac{1}{2}\right)\right)\right) \leq \mathcal{F}\left(\mathcal{P}\left(1, \frac{1}{2}\right)\right)$$

$$\Rightarrow \tau + \mathcal{F}\left(\max\left\{\frac{3}{4}, \frac{1}{4}\right\}\right) \leq \mathcal{F}\left(\max\left\{1, \frac{1}{2}\right\}\right)$$

$$\Rightarrow \tau + \mathcal{F}\left(\frac{3}{4}\right) \leq \mathcal{F}(1) \text{ which is true.}$$

In similar manner our assertion is true for every other points in $\mathfrak{X}$.

Definition 1.12: Let a pair if self mappings $\mathfrak{f}$ and $\mathfrak{g}$ are defined on a set $\mathfrak{X}$ is weakly compatible (WC) if a point $\lambda \in \mathfrak{X}$ is such that $\mathfrak{f}\lambda = \mathfrak{g}\lambda$ implies $\mathfrak{f}\mathfrak{g}\lambda = \mathfrak{g}\mathfrak{f}\lambda$.

The aim of this paper is to develop a fixed point theorem for $\mathcal{F} -$ contraction in PMS $(\mathfrak{X}, \mathcal{P})$ using the notation of weakly compatibility.

In the next section we present our main result.

3.Main Results

Theorem 3.1: Let $(\mathfrak{X}, \mathcal{P})$ be a complete PMS. Suppose that $\mathfrak{f}, \mathfrak{g}, \mathfrak{S}$ and $\mathfrak{T}: \mathfrak{X} \rightarrow \mathfrak{X}$ are four self mappings satisfying

(i) $\mathfrak{f}(\mathfrak{X}) \subseteq \mathfrak{T}(\mathfrak{X})$ and $\mathfrak{g}(\mathfrak{X}) \subseteq \mathfrak{S}(\mathfrak{X})$

(ii) two pairs $(\mathfrak{f}, \mathfrak{S})$ and $(\mathfrak{g}, \mathfrak{T})$ are WC mappings

(iii) $\mathfrak{f}(\mathfrak{X})$ or $\mathfrak{T}(\mathfrak{X})$ or $\mathfrak{g}(\mathfrak{X})$ or $\mathfrak{S}(\mathfrak{X})$ is closed subset of $(\mathfrak{X}, \mathcal{P})$

(iv) assume that there exists $\mathcal{F} \in \Delta_{\mathcal{F}}$ and $\tau > 0$ for $\lambda, \mu \in \mathfrak{X}$ such that

$\mathcal{P}(\mathfrak{f}\lambda, \mathfrak{g}\mu) > 0 \implies \tau + \mathcal{F}(\mathcal{P}(\mathfrak{f}\lambda, \mathfrak{g}\mu)) \leq \mathcal{F}\big(\mathcal{M}(\lambda, \mu)\big)$…………(3.1.1)

where $\mathcal{M}(\lambda, \mu) = max\{\mathcal{P}(\mathfrak{S}\lambda, \mathfrak{T}\mu), \mathcal{P}(\mathfrak{f}\lambda, \mathfrak{S}\lambda), \mathcal{P}(\mathfrak{g}\mu, \mathfrak{T}\mu), \frac{1}{2}[\mathcal{P}(\mathfrak{f}\lambda, \mathfrak{T}\mu) + \mathcal{P}(\mathfrak{g}\mu, \mathfrak{S}\lambda)].$

Then $\mathfrak{S}$, $\mathfrak{T}, \mathfrak{f}$ and $\mathfrak{g}$ have a unique common fixed point in $\mathfrak{X}$.

Proof: Let $\lambda_0 \in \mathfrak{X}$ be any point. By the (i) of (3.1), we construct sequences $\{\lambda_\eta\}$ *and* $\{\mu_\eta\}$ *in* $\mathfrak{X}$ satisfying

$\mathfrak{T}\lambda_{2\eta+1} = \mathfrak{f}\lambda_{2\eta} = \mu_{2\eta+1}$ and $\mathfrak{S}\lambda_{2\eta+2} = \mathfrak{g}\lambda_{2\eta+1} = \mu_{2\eta+2}$ ………………… .(3.1.2)

for η =0,1, 2 ,…….

Step-I: To prove that $\mathcal{P}(\mu_\eta, \mu_{\eta+1}) \longrightarrow 0$ as $\eta \longrightarrow 0$.

$\tau + \mathcal{F}(\mathcal{P}(\mu_{2\eta+1}, \mu_{2\eta+2})) \leq \mathcal{F}(\mathcal{M}(\lambda_{2\eta}, \lambda_{2\eta+1}))$…………………………….. .(3.1.3)

It follows from $(\mathcal{PMS}_2)$and $(\mathcal{PMS}_4)$ that

$$\mathcal{M}(\lambda_{2\eta}, \lambda_{2\eta+1}) = \mathcal{M}(\lambda, \mu)$$

$$= \max\{\mathcal{P}(\mathfrak{S}\lambda_{2\eta}, \mathfrak{T}\lambda_{2\eta+1}), \mathcal{P}(\mathfrak{f}\lambda_{2\eta}, \mathfrak{S}\lambda_{2\eta}), \mathcal{P}(\mathfrak{g}\lambda_{2\eta+1}, \mathfrak{T}\lambda_{2\eta+1}), \frac{1}{2}[\mathcal{P}(\mathfrak{f}\lambda_{2\eta}, \mathfrak{T}\lambda_{2\eta+1}) + \mathcal{P}(\mathfrak{g}\lambda_{2\eta+1}, \mathfrak{S}\lambda_{2\eta})]$$

$$= \max\{\mathcal{P}(\mu_{2\eta n}, \mu_{2n\eta+1}), \mathcal{P}(\mu_{2n\eta+1}, \mu_{2\eta n}), \mathcal{P}(\mu_{2n\eta+2}, \mu_{2n\eta+1}), \frac{1}{2}[\mathcal{P}(\mu_{2\eta n}, \mu_{2\eta+2}) + \mathcal{P}(\mu_{2\eta+1}, \mu_{2\eta+1})]\}$$

$$\leq max\left\{\mathcal{P}(\mu_{2\eta}, \mu_{2\eta+1}), \mathcal{P}(\mu_{2\eta+2}, \mu_{2\eta+1}), \frac{1}{2}[\mathcal{P}(\mu_{2\eta+1}, \mu_{2\eta+1}) + \mathcal{P}(\mu_{2\eta}, \mu_{2\eta+1}) + \mathcal{P}(\mu_{2\eta+1}, \mu_{2\eta+2}) - \mathcal{P}(\mu_{2\eta+1}, \mu_{2\eta+1})]\right\}$$

$$\leq max\{\mathcal{P}(\mu_{2\eta}, \mu_{2\eta+1}), \mathcal{P}(\mu_{2\eta+2}, \mu_{2\eta+1})\}$$

If $max\{\mathcal{P}(\mu_{2\eta}, \mu_{2\eta+1}), \mathcal{P}(\mu_{2\eta+2}, \mu_{2\eta+1})\} = \mathcal{P}(\mu_{2\eta+1}, \mu_{2\eta+2})$ then

$\tau + \mathcal{F}(\mathcal{P}(\mu_{2\eta+1}, \mu_{2\eta+2})) \leq \mathcal{F}(\mathcal{P}(\mu_{2\eta+1}, \mu_{2\eta+2}))$ this implies

$\mathcal{F}(\mathcal{P}(\mu_{2\eta+1}, \mu_{2\eta+2})) \leq \mathcal{F}(\mathcal{P}(\mu_{2\eta+1}, \mu_{2\eta+2})) - \tau$ which is congtradiction to ($\mathcal{F}$ -1).

Thus $max\{\mathcal{P}(\mu_{2\eta}, \mu_{2\eta+1}), \mathcal{P}(\mu_{2\eta+2}, \mu_{2\eta+1})\} = \mathcal{P}(\mu_{2\eta}, \mu_{2\eta+1})$ for all $\eta \in N$.

From (10), $\tau + \mathcal{F}(\mathcal{P}(\mu_{2\eta+1}, \mu_{2\eta+2})) \leq \mathcal{F}(\mathcal{P}(\mu_{2\eta}, \mu_{2\eta+1}))$… (3.1.4)

Constituting this way, it follows that

$\mathcal{F}(\mathcal{P}(\mu_{2\eta}, \mu_{2\eta+1})) \leq F\left(\mathcal{P}(\mu_{2\eta-1}, \mu_{2\eta})\right) - \tau$ …… (3.1.5).

Using (3.1.4) and (3.1.5)

$$\mathcal{F}(\mathcal{P}(\mu_{2\eta+1}, \mu_{2\eta+2})) \leq \mathcal{F}(\mathcal{P}(\mu_{2\eta}, \mu_{2\eta+1})) - \tau \leq \mathcal{F}(\mathcal{P}(\mu_{2\eta-1}, \mu_{2\eta})) - 2\tau$$
$$\leq \mathcal{F}(\mathcal{P}(\mu_{2\eta-2}, \mu_{2\eta-1})) - 3\tau \leq \mathcal{F}(\mathcal{P}(\mu_0, \mu_1)) - (2\eta+1)\tau \text{…….(3.1.6)}$$

And $\mathcal{F}(\mathcal{P}(\mu_{2\eta}, \mu_{2\eta+1})) \leq \mathcal{F}(\mathcal{P}(\mu_{2\eta-1}, \mu_{2\eta})) - \tau$

$$\leq \mathcal{F}\left(\mathcal{P}((\mu_{2\eta-2}, \mu_{2\eta-1})\right) - 2\tau$$

$$\leq \mathcal{F}(\mathcal{P}((\mu_0, \mu_1)) - (2\eta)\tau \text{.....(3.1.7)}$$

Repeating,

$\mathcal{F}(\mathcal{P}(\mu_\eta, \mu_{\eta+1})) \leq \mathcal{F}(\mathcal{P}((\mu_0, \mu_1)) - \eta\tau$ then it follows $\lim_{n\eta\to\infty} \mathcal{F}(\mathcal{P}(\mu_\eta, \mu_{\eta+1})) = -\infty$ by $\mathcal{F} \in \Delta_{\mathcal{F}}$ and ($\mathcal{F}$ -2) we have $\lim_{\eta\to\infty} \mathcal{P}(\mu_\eta, \mu_{\eta+1}) = 0$.......(3.1.8)

Step-II

Now we prove that $\{\mu_\eta\}$ is $\mathcal{P}$ -Cauchy sequence. By $\mathcal{F} \in \Delta_{\mathcal{F}}$ and ($\mathcal{F}$ -3) thereexists $\mathcal{k} \in (0,1)$ such that

$$\lim_{\eta\to\infty} \mathcal{P}(\mu_\eta, \mu_{\eta+1})^{\mathcal{k}}\ \mathcal{F}(\mathcal{P}(\mu_\eta, \mu_{\eta+1})) = 0 \text{.... (3.1.9)}$$

By (3.1.6) and (3.1.7)

$$\lim_{\eta\to\infty} \mathcal{P}(\mu_\eta, \mu_{\eta+1})^{\mathcal{k}}\ \mathcal{F}(\mathcal{P}(\mu_\eta, \mu_{\eta+1})) = 0$$

$$\lim_{\eta\to\infty} \mathcal{P}(\mu_{2\eta+1}, \mu_{2\eta+2})^{\mathcal{k}}\ \mathcal{F}\left(\mathcal{P}(\mu_{2\eta+1}, \mu_{2\eta+2})\right) - \mathcal{F}(\mathcal{P}(\mu_0, \mu_1)$$

$$\leq \mathcal{P}(\mu_{2\eta+1}, \mu_{2\eta+2})^{\mathcal{k}} - (2\eta+1)\tau \leq 0 \text{-----(3.1.10)}$$

And

$$\lim_{\eta\to\infty} \mathcal{P}(\mu_{2\eta}, \mu_{2\eta+1})^{\mathcal{k}}\ \mathcal{F}\left(\mathcal{P}(\mu_{2\eta}, \mu_{2\eta+1})\right) - \mathcal{F}(\mathcal{P}(\mu_0, \mu_1)$$

$$\leq \mathcal{P}(\mu_{2\eta}, \mu_{2\eta+1})^{\mathcal{k}} - (2\eta)\tau \leq 0 \text{----- (3.1.11)}$$

Using the above inequality and (3.1.9)

$$\lim_{\eta\to\infty} \eta\ \mathcal{P}(\mu_{2\eta}, \mu_{2\eta+1})^{\mathcal{k}} = 0.$$

Therefore, there exists $\eta \in \mathbb{N}$ such that $\eta.\mathcal{P}(\mu_{2\eta}, \mu_{2\eta+1})^{\mathcal{k}} < 1$ for all $\eta \geq \eta_1$.

(Or) $\mathcal{P}(\mu_n, \mu_{n+1}) < \frac{1}{\eta^{1/\mathcal{k}}}$...........(3.1.12)

Let $\zeta, \eta \in \mathbb{N}$ with $\zeta > \eta > \eta_1$ using triangular inequality we have

$$\mathcal{P}(\mu_\eta, \mu_\zeta) = \mathcal{P}(\mu_{n\eta}, \mu_{\eta+1}) + \mathcal{P}(\mu_{\eta+1}, \mu_{\eta+2}) \ldots. + \mathcal{P}(\mu_{\zeta-1}, \mu_\zeta)$$

$$-[\mathcal{P}(\mu_{\eta+1}, \mu_{\eta+1}) + \mathcal{P}(\mu_{\eta+2}, \mu_{\eta+2}) + \cdots . \mathcal{P}(\mu_{\zeta-1}, \mu_{\zeta-1})]$$

$$\leq \sum_{i=1}^{\zeta-1} \mathcal{P}(\mu_i, \mu_{i+1}) \leq \sum_{i=1}^{\infty} \mathcal{P}(\mu_i, \mu_{i+1}) \leq \sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{\mathcal{k}}}}.$$

As $\mathcal{k} \in (0,1)$ the infinite series $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{\mathcal{k}}}}$ converges, consequently we get $\lim_{\zeta,\eta\to\infty} \mathcal{P}(\mu_\eta, \mu_\zeta) = 0$. This proves that $\{\mu_\eta\}$ is a Cauchy sequence in $(\mathfrak{X}, \mathcal{P})$ and consequently we get $\{\mu_\eta\}$ is Cauchy in $(\mathfrak{X}, d_{\mathcal{P}})$.

Since $(\mathfrak{X}, \mathcal{P})$ is complete PMS, this ensures that $(\mathfrak{X}, d_{\mathcal{P}})$ is complete metric space, then $\exists\ u_{\wp} \in \mathfrak{X}$ such that $\lim_{\eta\to\infty} d_{\mathcal{P}}(\mu_\eta, u_{\wp}) = 0$.

Moreover $\mathcal{P}(u_{\wp}, u_{\wp}) = \lim_{n\to\infty} \mathcal{P}(\mu_\eta, u_{\wp}) = \lim_{\zeta,,\eta\to\infty} \mathcal{P}(\mu_\eta, \mu_\zeta) = 0$.......(3.1.13)

Since $\mu_\eta \to u_{\wp}$ then $\mathfrak{T}\lambda_{2\eta+1}$, $\mathfrak{f}\lambda_{2\eta}$ $\mathfrak{S}\lambda_{2\eta+2}$ and $\mathfrak{g}\lambda_{2\eta+1}$ converges to $u_{\wp}$.

Step-III

Now we claim that the mappings $\mathfrak{S}$, $\mathfrak{T}, \mathfrak{f}$ and $\mathfrak{g}$ have a common fixed point.

Suppose that the Range $\mathfrak{T}(\mathfrak{X})$ is closed, then $\exists\ \nu_{\wp} \in \mathfrak{X}$ such that $\mathfrak{T}\nu_{\wp} = u_{\wp}$……. (3.1.14)

Now assuming that $\mathfrak{g}\nu_{\wp} \neq u_{\wp}$ and put $\lambda = \lambda_{2\eta}$, $\mu = \nu_{\wp}$ in (3.1.1) then

$(\mathfrak{f}\lambda_{2\eta}, \mathfrak{g}\nu_{\wp}) > 0 \implies \tau + \mathcal{F}(\mathcal{P}(\mathfrak{f}\lambda_{2\eta}, \mathfrak{g}\nu_{\wp})) \leq \mathcal{F}\left(\mathcal{M}(\lambda_{2\eta}, \nu_{\wp})\right)$…..(3.1.15)

where

$\mathcal{M}(\lambda_{2\eta}, \nu_{\wp}) = max\{\mathcal{P}(\mathfrak{S}\lambda_{2\eta}, \mathfrak{T}\nu_{\wp}), \mathcal{P}(\mathfrak{f}\lambda_{2\eta}, \mathfrak{S}\lambda_{2\eta}), \mathcal{P}(\mathfrak{g}\nu_{\wp}, \mathfrak{T}\nu_{\wp}), \frac{1}{2}[\mathcal{P}(\mathfrak{f}\lambda_{2\eta}, \mathfrak{T}\nu_{\wp}) + \mathcal{P}(\mathfrak{g}\nu_{\wp}, \mathfrak{S}\lambda_{2\eta})]\}$

$$= \max\{\mathcal{P}(u_{\wp}, u_{\wp}), \mathcal{P}(u_{\wp}, \mathfrak{g}\nu_{\wp}), \mathcal{P}(u_{\wp}, u_{\wp}), \tfrac{1}{2}[\mathcal{P}(u_{\wp}, u_{\wp}) + \mathcal{P}(u_{\wp}, \mathfrak{g}\nu_{\wp})]\}$$

$\tau + \mathcal{F}(\mathcal{P}(u_{\wp}, \mathfrak{g}\nu_{\wp})) \leq \mathcal{F}\left(\mathcal{P}(u_{\wp}, \mathfrak{g}\nu_{\wp})\right)$ this a contradiction with $\tau > 0$.

Thus $\mathfrak{g}\nu_{\wp}$=$u_{\wp}$. ……(3.1.15)

Therefore using (3.1.14) and (3.1.15) we obtain $\mathfrak{T}\nu_{\wp} = \mathfrak{g}\nu_{\wp} = u_{\wp}$ .Since $\mathfrak{g}$ and $\mathfrak{T}$ are WC mappings then, $\mathfrak{g}u_{\wp} = \mathfrak{g}\mathfrak{T}\nu_{\wp} = \mathfrak{T}\mathfrak{g}\nu_{\wp} = \mathfrak{T}u_{\wp}$. …………(3.1.16)

Now we show that $\mathfrak{g}u_{\wp} = \nu_{\wp}$.

On contrary, let $\mathfrak{g}u_{\wp} \neq \nu_{\wp}$ and use $\lambda = \lambda_{2\eta}, \mu = \nu_{\wp}$ in (3.1.1) then

$\tau + \mathcal{F}(\mathcal{P}(\mathfrak{f}\lambda_{2\eta}, \mathfrak{g}\nu_{\wp})) \leq \mathcal{F}\left(\mathcal{M}(\lambda_{2\eta} u_{\wp})\right)$

$$\mathcal{M}(\lambda_{2\eta}, u_{\wp}) = max\{\mathcal{P}(\mathfrak{S}\lambda_{2\eta}, \mathfrak{T}u_{\wp}), \mathcal{P}(\mathfrak{f}x\lambda_{2n}, \mathfrak{S}\lambda_{2\eta}), \mathcal{P}(\mathfrak{g}u_{\wp}, \mathfrak{T}u_{\wp}), \frac{1}{2}[\mathcal{P}(\mathfrak{f}\lambda_{2\eta}, \mathfrak{T}u_{\wp}) + \mathcal{P}(\mathfrak{g}u_{\wp}, \mathfrak{S}\lambda_{2n})]\}$$

$$= \mathcal{P}(u_{\wp}, \mathfrak{g}u_{\wp}).$$

By permitting, continuity of $\mathcal{F}$ and applying the limit as $\eta \to \infty$,we have

$\tau + \mathcal{F}(\mathcal{P}(u_{\wp}, \mathfrak{g}u_{\wp})) \leq \mathcal{F}\left(\mathcal{M}(u_{\wp}, \mathfrak{g}u_{\wp})\right)$ which is contradiction.

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Therefore $\mathcal{P}\left(u_{p}, \mathfrak{g}u_{p}\right) = 0$, this yields

$\mathfrak{T}u_{p} = \mathfrak{g}u_{p} = u_{p}$ ....(3.1.17)

Now we show that u_{p} is a fixed pint of the mappings $\mathfrak{f}$ and $\mathfrak{S}$. Since $\mathfrak{g}\,(\mathcal{X}) \subseteq \mathfrak{S}(\mathcal{X})$ $\exists$ a point $\mathfrak{z}_{p} \in \mathcal{X}$ such that $\mathfrak{g}u_{p} = \mathfrak{S}\mathfrak{z}_{p.}$

Suppose that $\mathfrak{f}\mathfrak{z}_{p} \neq \mathfrak{S}\mathfrak{z}_{p}$, then

$$\tau + \mathcal{F}(\mathcal{P}\left(\mathfrak{f}\mathfrak{z}_{p.}, \mathfrak{g}u_{p}\right)) \leq \mathcal{F}\left(\mathcal{M}\left(\mathfrak{z}_{p.}, u_{p}\right)\right)$$

$$\mathcal{M}\left(\mathfrak{z}_{p.}, u_{p}\right) = max\{\mathcal{P}\left(\mathfrak{S}\mathfrak{z}_{p.}, \mathfrak{T}u_{p}\right), \mathcal{P}\left(\mathfrak{f}\mathfrak{z}_{p.}, \mathfrak{S}\mathfrak{z}_{p.}\right), \mathcal{P}\left(\mathfrak{g}u_{p}, \mathfrak{T}u_{p}\right), \frac{1}{2}\left[\mathcal{P}\left(\mathfrak{f}\mathfrak{z}_{p.}, \mathfrak{T}u_{p}\right) + \mathcal{P}\left(\mathfrak{g}u_{p}, \mathfrak{S}\mathfrak{z}_{p.}\right)\right]\}$$

$$= max\{\mathcal{P}\left(\mathfrak{g}u_{p}, \mathfrak{g}u_{p}\right), \mathcal{P}\left(\mathfrak{g}u_{p}, \mathfrak{g}u_{p}\right), \mathcal{P}\left(\mathfrak{g}u_{p}, \mathfrak{f}\mathfrak{z}_{p.}\right), \frac{1}{2}\left[\mathcal{P}\left(\mathfrak{f}\mathfrak{z}_{p.}, \mathfrak{g}u_{p}\right) + \mathcal{P}\left(\mathfrak{g}u_{p}, \mathfrak{g}u_{p}\right)\right]\}$$

$$= \mathcal{P}\left(\mathfrak{g}u_{p}, \mathfrak{f}\mathfrak{z}_{p.}\right)$$

$\tau + \mathcal{F}(\mathcal{P}\left(\mathfrak{f}\mathfrak{z}_{p.}, \mathfrak{g}u_{p}\right)) \leq \mathcal{P}\left(\mathfrak{g}u_{p}, \mathfrak{f}\mathfrak{z}_{p}\right)$ this a contradiction with $\tau > 0$.

Thus $\mathfrak{g}u_{p} = \mathfrak{f}\mathfrak{z}_{p} = \mathfrak{S}\mathfrak{z}_{p}$ …. (3.1.18).

By using weakly compatible nature of $\mathfrak{f}$ and $\mathfrak{S}$, we get

$\mathfrak{S}u_{p} = \mathfrak{f}\,\mathfrak{S}\mathfrak{z}_{p} = \mathfrak{S}\mathfrak{f}\mathfrak{z}_{p} = \mathfrak{f}u_{p}$. …………(3.1.19).

Finally we show that $\mathfrak{f}u_{p} = u_{p}$ .Assume $\mathfrak{f}u_{p} \neq u_{p}$ and put $\lambda = \mu = u_{p}$ in (3.1.1) we get

$$\tau + \mathcal{F}(\mathcal{P}\left(\mathfrak{f}u_{p}, u_{p}\right)) \leq \mathcal{F}(\mathcal{P}\left(\mathfrak{f}u_{p}, \mathfrak{g}u_{p}\right)) \leq \mathcal{F}\left(\mathcal{M}\left(u_{p}, u_{p}\right)\right)$$

$$\mathcal{M}\left(u_{p}, u_{p}\right) = max\{\mathcal{P}\left(\mathfrak{S}u_{p}, \mathfrak{T}u_{p}\right), \mathcal{P}\left(\mathfrak{f}u_{p}, \mathfrak{S}u_{p}\right), \mathcal{P}\left(\mathfrak{g}u_{p}, \mathfrak{T}u_{p}\right), \frac{1}{2}\left[\mathcal{P}\left(\mathfrak{f}u_{p}, \mathfrak{T}u_{p}\right) + \mathcal{P}\left(\mathfrak{g}u_{p}, \mathfrak{S}u_{p}\right)\right]\}$$

$$= \mathcal{P}\left(u_{p}, \mathfrak{f}u_{p}\right)$$

$\tau + \mathcal{F}(\mathcal{P}\left(\mathfrak{f}u_{p}, u_{p}\right)) \leq \mathcal{P}\left(\mathfrak{f}u_{p}, u_{p}\right)$ this a contradiction with $\tau > 0$.

Thus $\mathfrak{f}u_{p} = \mathfrak{S}u_{p} = u_{p}$…..(3.1.20).

Using (3.1.17) and (3.1.20) u_{p} is fixed point of $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{S}$ and $\mathfrak{T}$.

Step-IV:

Let $\omega_{p}(\neq u_{p})$ be the another fixed point of $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{S}$ and $\mathfrak{T}$.

Put $\lambda = u_{p}$, $\mu = \omega_{p}$ in the contraction (3.1.1), we get

$$\tau + \mathcal{F}\left(\mathcal{P}\left(\mathfrak{f}u_{p}, \omega_{p}\right)\right) = \tau + \mathcal{F}\left(\mathcal{P}\left(\mathfrak{f}u_{p}, \mathfrak{g}\omega_{p}\right)\right) \leq \mathcal{F}\left(\mathcal{M}\left(u_{p}, \omega_{p}\right)\right)$$

$$\mathcal{M}(u_{\wp},,\omega_{\wp}) = max\{\mathcal{P}(\mathfrak{S}u_{\wp},\mathfrak{T}\omega_{\wp}),\mathcal{P}(\mathfrak{f}u_{\wp},\mathfrak{S}\omega_{\wp}),\mathcal{P}(\mathfrak{g}\omega_{\wp},\mathfrak{T}\omega_{\wp}),\frac{1}{2}[\mathcal{P}(\mathfrak{f}u_{\wp},\mathfrak{T}\omega_{\wp}) + \mathcal{P}(\mathfrak{g}\omega_{\wp},\mathfrak{S}u_{\wp})]\}$$
$$= \mathcal{P}(u_{\wp},\omega_{\wp})$$

$\tau + \mathcal{F}\left(\mathcal{P}(\mathfrak{f}u_{\wp},\omega_{\wp})\right) \leq \mathcal{P}(u_{\wp},\omega_{\wp})$ this a contradiction with $\tau > 0$. Thus $u_{\wp} = \omega_{\wp}$.

In conclusion, these four mappings have a unique common fixed point.

Example 3.2: Let $(\mathfrak{X},\mathcal{P})$ with $\mathfrak{X} = [0,1]$ is a complete PMS and $\mathcal{P}\ (\lambda,\mu)$=max{ λ,μ } $\forall\ \lambda,\mu \in \mathfrak{X}$.

Define mappings $\mathcal{F}\colon \mathbb{R}^+ \longrightarrow \mathbb{R}$ and $\mathcal{F}(\alpha) = \log_e \alpha$.

Let $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{S}$ and $\mathfrak{T}\colon \mathfrak{X} \to \mathfrak{X}$ be defined as $\mathfrak{f}(\lambda) = \frac{\lambda}{4}, \mathfrak{g}\ (\lambda) = 0, \mathfrak{S}(\lambda) = \frac{3\lambda}{2}, \mathfrak{T}(\lambda) = \lambda,$ then

$\mathfrak{f}(\mathfrak{X}) = \left[0,\frac{1}{4}\right]$, $\mathfrak{g}\ (\mathfrak{X}) = \{0\}$, $\mathfrak{S}(\mathfrak{X}) = \left[0,\frac{3}{4}\right]$, and $\mathfrak{T}(\mathfrak{X}) = [0,1]$ these ranges of mappings satisfying the inclusion inequalities (i) of theorem 3.1.

Now clearly $\mathfrak{f}(0) = \mathfrak{S}(0) = 0$ gives $\mathfrak{f}\mathfrak{S}(0) = \mathfrak{S}\mathfrak{f}(0)$ which gives the pair ($\mathfrak{f}$, $\mathfrak{S}$) is weakly compatible and $\mathfrak{g}(0) = \mathfrak{T}(0) = 0$ gives $\mathfrak{g}\mathfrak{T}(0) = \mathfrak{T}\mathfrak{g}(0)$ this implies the pair ($\mathfrak{g}$, $\mathfrak{T}$) is weakly compatible at the coincident point zero.

We discuss the existence of contraction condition (3.1.1) under some conditions

Now $\mathcal{P}(\mathfrak{f}\lambda,\mathfrak{g}\mu) = \max\left\{\frac{\lambda}{4},0\right\}$,$\mathcal{P}(\mathfrak{S}\lambda,\mathfrak{T}\mu) = max\left\{\frac{3\lambda}{2},\mu\right\}$,$\mathcal{P}(\mathfrak{f}\lambda,\mathfrak{S}\lambda) = \max\left\{\frac{\lambda}{4},\frac{3\lambda}{2}\right\}$,

$\mathcal{P}(\mathfrak{g}\mu,\mathfrak{T}\mu) = \max\{0,\mu\}$, $\mathcal{P}(\mathfrak{f}\lambda,\mathfrak{T}\mu) = \max\left\{\frac{\lambda}{4},\mu\right\}$ and $\mathcal{P}(\mathfrak{g}\mu,\mathfrak{S}\lambda) = \max\left\{0,\frac{3\lambda}{2}\right\}$.

Case I: If $\frac{3\lambda}{2} > \mu$ then

$$\mathcal{M}(\lambda,\mu) = max\left\{\frac{3\lambda}{2},\frac{3\lambda}{2},\mu,\frac{1}{2}\left[\frac{\lambda}{4}+\frac{3\lambda}{2}\right]\right\} = \frac{3\lambda}{2}$$

$$\mathcal{P}(\mathfrak{f}\lambda,\mathfrak{g}\mu) = max\left\{\frac{\lambda}{4},0\right\} = \frac{\lambda}{4} > 0 \implies \tau + \mathcal{F}\left(\frac{\lambda}{4}\right) \leq \mathcal{F}\left(\frac{3\lambda}{2}\right) \Longrightarrow \tau + \log_e\left(\frac{\lambda}{4}\right) \leq \log_e\left(\frac{6\lambda}{4}\right).$$

Case II: If $\frac{3\lambda}{2} < \mu$ then

$$\mathcal{M}(\lambda,\mu) = max\left\{\mu,\frac{3\lambda}{2},\mu,\frac{1}{2}\left[\mu+\frac{3}{4}\right]\right\} = \mu$$

$$\mathcal{P}(\mathfrak{f}\lambda,\mathfrak{g}\mu) = max\left\{\frac{\lambda}{4},0\right\} = \frac{\lambda}{4} > 0 \implies \tau + F\left(\frac{\lambda}{4}\right) \leq F(\mu) \Longrightarrow \tau + \log_e\left(\frac{\lambda}{4}\right) \leq \log_e(\mu).$$

This gives $\tau + \log_e \frac{1}{6}\left(\frac{3\lambda}{2}\right) \leq \log_e(\mu)$.

In both the cases, inequality (3.1.1) is satified for all λ,μ .

In this illustration, it is observed that zero is the unique common fixed point.

4.Conclusion: In this research article, we have proved a fixed point result for $\mathcal{F}-$contraction in PMS via weakly compatible mappings. Finally, we have provided one example to support our main result with unique common fixed point.

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