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Optimizing Advertising and Pricing for Perishable Inventory with Freshness-Related Demand in a Hybrid Partial Prepayment and Trade Credit Supply Chain

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Abstract:

This paper investigates an inventory model addressing non-instantaneous deteriorating items with expiration concerns, incorporating a hybrid payment approach. The demand factors of this study are price-sensitive demand, product freshness considerations, and advertising frequency impact, which are critical elements in contemporary supply chain challenges. The proposed model investigates multiple prepayments and delayed payments to bolster business operations during financial emergencies. This study aims to enhance business flexibility, and the model reflects real-world complexities by introducing a time-dependent holding cost and accounting for partial backlogged shortages. The establishment of convexity ensures efficient optimization and numerical examples clearly illustrate how the proposed strategies influence overall enterprise profitability.

Furthermore, the utilization of MATLAB for graphical representation enhances result availability and understanding. Sensitivity analysis enriches decision-making by providing important managerial insights during the recovery phase. This research contributes a strong foundation for inventory optimization that can adapt to changing post-pandemic economic conditions.

Keywords: Price sensitive demand, Advertisement, Freshness of the goods, Trade Credit, Maximum lifetime, Hybrid payment.

1. Introduction

In response to the worldwide pandemic, scholars have concentrated on creating effective inventory administration techniques that maximize revenue and reduce expenses, by considering the various payment possibilities. A customized inventory model that considers the market's state has been made, underlining how pricing significantly influences demand, especially when it comes to perishable commodities and expiry dates. The impact of advanced and deferred payment plans in the post pandemic period, on sales and overall inventory performance is also examined.

Nowadays, consumers' awareness of food freshness and safety drives demand for products with expiration dates increased. They are underscoring the importance of effective advertising in creating awareness and meeting consumer needs. Well-crafted advertisements inform consumers about product attributes, evoke emotions, establish brand recognition, and positively influence purchasing decisions. This innovative inventory model integrates various dynamics, including non-instantaneous item deterioration, price-sensitive demand with advertising frequency and freshness of the products, and advanced payment strategies and discount rates, offering a comprehensive approach to inventory management that contributes significantly to the field's advancement.

The initial motivating force is derived from understanding and experience with hybrid payment systems in real-world business situations. Advanced payment and payment discounts help to reduce consumer burdens, increase sales, and improve the economics of the supply chain. The second source of inspiration is the challenges of deteriorating goods. Prices fall, and storage costs rise when item quality deteriorates over time, adversely affecting the system's efficiency and financial advantages. The following motivating forces are the promotion of freshness of goods, advertisement for current products, emphasizing sensory appeal, health advantages, and quality control to appeal to customer preferences. Research through surveys and analyzing successful campaigns helps advertisers understand which freshness aspects connect best with different demographics, guiding effective messaging strategies to drive purchasing decisions.

Another reason for this study is to address existing research gaps highlighted in the literature review.

2 Literature Review

The literature review emphasizes the necessity for inventory models that accommodate the non-instantaneous deterioration of perishable items, advertisement frequency, product freshness, time-varying holding costs, trade credit, and advanced payment schemes.

2.1 Non-instantaneous Expiration date-related deterioration

A rich and diverse body of research focuses on perishable inventory management, particularly considering the impact of expiration dates and deteriorating quality on supply chain dynamics. Managing a perishable inventory system with variable holding costs and non-instantaneous deterioration is essential in the retail, pharmaceutical, and agricultural sectors. Wu(2) defined this phenomenon as "**Non-instantaneous deterioration (NID)**". Hsu et al.([3]) introduced the idea of expiration within inventory models, highlighting the importance of managing perishable goods based on their maximum lifetime and deterioration rates. This phenomenon occurs frequently in everyday life, such as when some vegetables and fruits stay fresh for a short period with little or no rotting.

Priyan and Mala(6) developed pricing models considering quantity discounts and demand elasticity influenced by perishable item deterioration and expiration dates, emphasizing the need for dynamic pricing strategies. Sebatjane and Adetunji(12) investigated supply chain models for perishable items where demand is influenced by expiration dates, highlighting the interconnectedness of supply chain participants in managing perishable inventories effectively.

2.2 Inventory model for Time varying holding cost

Another notable aspect of inventory management is holding cost, which changes the rate of deterioration of products during storage. Still, holding costs may increase over time. The decision-maker may allocate additional resources to minimize the number of deteriorated items and decrease the associated losses incurred from these items. Researchers worldwide have developed several inventory models focusing on holding costs that vary over time. However, most researchers pursued their work in a typical environment rather than a hybrid partial prepayment and partial trade credit scheme, which is crucial for retail businesses during the post-pandemic. As a result, in our suggested study, we attempt to overcome this gap.

2.3 Inventory model with an advanced payment

In the post-pandemic situation, much research on hybrid payment schemes is needed. Changing company demands is necessary for improving inventory management systems since integrating advanced payment and trade credit mechanisms is critical for supply chain resilience. Prepaid service and (or) the discount percentage of the total purchase price for products by buyers to vendors under an inventory model with an advanced payment plan help stabilize markets in the face of erratic demand and supply. Scholars have studied prepayment systems with pioneering work by Zhang followed by innovative applications such as genetic algorithms for inventory models with advance payments and price discounts, as demonstrated by Gupta.

Advanced payments on inventory management systems have influenced non-instantaneous deterioration and partial backlogging in recent years. Studies by Duary(15), Khan(16), and Tavassoli(18) have formulated inventory models integrating advanced payments with considerations like deterioration and partial back ordering alongside delayed payment policies. Additionally, The fixed lifetime with advanced payment and the discount rate was developed by Mukunda(20).

2.4 Inventory model with trade credit

This research focuses on incorporating two payment schemes (Advanced payment and credit payment) with allowable shortages of items into an inventory model, representing an innovative approach within trade credit policies and inventory management. This approach adds complexity and realism to the model by considering different payment options and the possibility of item shortages, which can significantly impact supply chain dynamics and business operations. Furthermore, allowing shortages acknowledges the business's challenges in maintaining perfect inventory levels. It highlights the trade-offs between inventory costs, customer service levels, and financial constraints.

Over the last twenty years, many researchers have incorporated trade credit in an inventory model. Some studies have suggested delaying payments partly to encourage retailers to place larger orders. Trade credit can stimulate the Inter-organization's power in the supply chain system developed by Lin(5). Mahato(9) established a trade credit policy-based sustainable EOQ model adopting partial back ordering shortages. Choudhury(14) investigated an integrated inventory model with capacity constraints under order-size-dependent trade credit, all-unit discount, and partial backlogging. In their study, they considered order size-dependent trade credit along with all unit discounts. Chung(21) created a model for managing inventory that considers products with varying deterioration rates over time and expiration dates while also considering that the retailer provides a credit period to its customers. Momena et al.(24) have recently devised a trade credit inventory model incorporating a quantity discount policy within a dual-storage facility.

Previous studies predominantly explored inventory models incorporating trade credit payment schemes without simultaneously integrating multiple prepayment and trade credit policies alongside allowable shortages. In contrast, our research examines explicitly two payment schemes (advanced payment and credit payment) within an inventory model that realistically allows for item shortages, presenting a more comprehensive and practical approach to inventory management.

2.5 Price-dependent demand with advertisement

This study focusing on an inventory model with a demand function incorporating a linearly decreasing price dependence and a nonlinear increase that is influenced by advertising and various additional factors such as prepayment, delay payment, partial backlogging, and time-varying deterioration with expiration date represents a comprehensive and timely approach to addressing real-world challenges in supply chain management; businesses are adapting to changing market conditions and consumer behaviours, especially after the pandemic.

Market pricing is a critical aspect that is rigidly adhered to by market demand. Increasing the market price diminishes the market demand and vice versa. As a result, it plays a crucial role in inventory management. Connection with the Inventory model, where the selling price of the items governs the demand function, has been studied since then in the seminal work published by Whitin(1). Selling price-dependent demand with various conditions is a significantly trending research area for evidence in the last five years; Udayakumar et al.(13) developed delayed payment inventory models based on the assumption that non-instantaneous deterioration and demand is a deterministic function of selling price and advertising cost. This study highlights the complexities of perishable inventory management under extended supply chain dynamics.

Many researchers, such as Khara et al.(8), San-Jose et al.(11), Alshanbari et al.(7), and Khan et al.(16) have investigated optimal advertising policies and hybrid payment mechanisms that combine advance and cash payments to maximize retailer profitability during business operations. Subhash(22) formulated a green product inventory model to address price-dependent demand, employing a teaching and learning-based optimization algorithm for solution refinement.

Some of the related works are presented in Table 1.

Table 1 Comparison of the study with related works

Source	Demand rate	Deterioration	Holding cost	Advertisement	Payment Scheme		Discount on %	Shortage
					Pre-Payment	Delayed Payment		
[4] (2020)	P & SD	Constant	Constant	No	Yes	No	Yes	Constant & PB
[7] (2021)	PD	TV & IS	Constant	Yes	Yes	No	No	PB
[10] (2021)	P & SD	Constant	Constant	No	Yes	No	Yes	PB
[17] (2022)	P & TD	Constant	Constant	No	Yes	Yes	No	No
[16] (2022)	TD	TV & NIS	LTD	No	Yes	Yes	No	PB
[19] (2023)	PD with Freshness	TV with expiration	LTD	Yes	No	No	No	PB
[20] (2023)	PD	TV with expiration	LTD with PC	No	Yes	Yes	Yes	PB with WTD
[23] (2024)	PD	Constant & NIS	Constant	Yes	Yes	No	No	PB
This work	PD with Freshness	NIS & TVwith expiration	LTD with PC	Yes	Yes	Yes	Yes	PB with WTD

PD – Price Dependent, P & SD – Price and Stock Dependent, P & TD – Price and Time Dependent, TV – Time Varying, IS – Instantaneous, NIS – Non-Instantaneous, LTD – Linearly Time Dependent, PC – Purchase Cost, PB – Partial Backlogging, WTD – Waiting Time Dependent

Based on the literature review, a research gap in inventory management was identified and addressed by formulating a comprehensive mathematical model. Price-sensitive demand, focusing on advertising frequency and product freshness, prepayment, delay payment, partial backlogging, and time-varying deterioration, considering non-instantaneous deterioration, are innovatively integrated into this model. This optimization model aims to maximize profit and underscores the significance of this research in the following research questions:

2.6 Research Questions

1. How do prepayment and delay-in payment schemes impact inventory management strategies and profitability?
2. How can we optimize balance inventory levels and partial backlogging to maximize profit while meeting demand?
3. What is the impact of considering time-varying deterioration with expiration dates to improve inventory decision-making for perishable goods?
4. How does price-sensitive demand, influenced by advertising frequency and product freshness, affect inventory policies and profitability?
5. What are this business strategy's practical challenges, implications, and market segments to ensure sustained profitability?

This model fills a significant research gap and provides actionable outputs for businesses to optimize inventory practices, maximize profitability, and adapt to evolving market demands effectively.

3 Notations and assumptions

The following notations and assumptions create a mathematical model of the proposed problem.

3.1 Notations

The notations are as follows

K - Replenishment cost per order

c - Purchase cost per unit

s - Shortage cost per unit

d - Cost of deterioration per unit of time

l - Lost sale cost per unit

I_E - Interest earned in a year

L - Delivery lead-time

M - Permissible delay in the payment period

n - Number of installments defined for prepayments during the lead time

I_L - Interest rate on loan per year

I_C - Interest rate charges per year

t_d - Time at which the product remains fresh because of no deterioration

α - Discount rate for purchasing cost due to prepayments

S - Maximum stock per cycle

R - Maximum shortage per cycle

G - Expenditure per Advertisement

Q - Order quantity per replenishment cycle

Π_i - Total profit for i = 1, 2, 3

Decision Variables

t_1 - Time at which the stock reaches zero

p - Selling price per unit

T - Cycle Time

3.2 Assumptions

1. There is an expiration date on items that are deteriorating. The degrading rate approaches the maximum lifetime '*m*', at which point it approaches 1, which makes it practically relevant. To make this scenario possible, the rate of degradation is $\theta(t) = \frac{1}{1+m-t}, 0 \le t \le T \le m$

It may be noted that the product will not be sold after its maximum life of m. T, thus, needs to be less than or equal to m for the replenishing cycle time.

2. *D(p)* is the demand function that is contingent upon a tripartite relationship between advertising frequency, product freshness and selling price. Demand function expressed as $D(p) = A^{\eta}(a-bp)\left(\frac{m-t}{m}\right)$, where '*A*' for the advertisement frequency, $\eta \in [0, 1)$ is the elasticity of the advertising factor, '*a*' for the scaling parameter and '*b*' for the price sensitivity parameter and $a - bp > 0$.
3. Backorder has been used in this investigation. Given that some demand is backordered and this backorder rate is predicated on waiting time, it is naturally declining. The fraction is $B(T-t) = \frac{1}{1+\delta(T-t)}$, $T - t$ representing the waiting time until the next replenishing point is reached and δ is a backlogging parameter, $\delta > 0$.
4. The holding cost function can be expressed as: $H(t) = g + ht$, '*g*' is the constant part of the holding cost, '*h*' is the rate at which the holding cost increases per unit time of storage duration *t*.
5. The product is without deterioration between $[0, t_d]$; deterioration occurs at a variable rate $\theta(t)$ within the interval $[t_d, t_1]$.
6. There is no lead-time and hence an endless replenishing rate.

7. Backlogged demand is immediately satisfied from inventory ordered in the previous cycle at the start of the subsequent cycle.

8. Damaged products are immediately replaced, repaired, or removed from stock.

9. During the lead period, the supplier returns the total purchase cost as a portion L and collects the remaining purchasing cost upon batch delivery.

10. The retailer secures a loan from a third party to cover prepayments, repaying it with principal and interest upon receiving consumer payment.

11. n equal instalments the retailers can pay a portion σcQ of the purchase price over L months before delivery, receiving a discount from the supplier. Shorter installment terms result in higher discount rates (α), with a maximum discount of $J\%$ for full upfront payment.

12. Planning horizon infinite.

4 Mathematical Formulation

This research presents a mathematical model that integrates inventory management with payment strategies within a supply chain framework and focuses on hybrid prepayment and trade-credit policies, considering factors like shortages and gradual product deterioration. Specifically, the retailer pays a portion (σ) of the purchase price in n equal installments within a specified delivery lead time (L). The purchase price's remaining portion $(1-\sigma)$ is settled upon order receipt ($t = 0$). Additional discounts or rebates ($J\,\%$) to encourage the retailer's prepayment offers to the supplier, particularly in challenging circumstances such as during a pandemic.

The supplier delivers goods to the retailer based on a predetermined payment schedule in the supply chain management. Also, the retailer determines replenishment cycles and pricing strategies by considering the payment terms and market conditions. Ultimately, clients receive retail products as part of this supply chain ecosystem.

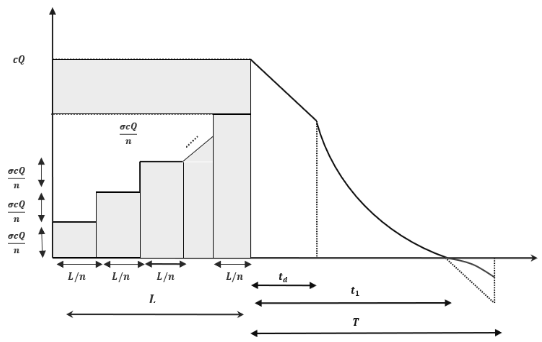


Figure 1. Graphical representation of the inventory system

The model integrates an Economic Order Quantity (EOQ) framework to optimize order quantities and maximize the total profit of the inventory system. The inventory level $I(t)$ is influenced by demand and deterioration, following a declining pattern over time. Specifically, the stock of this model decreases based on the demand rate from 0 to t_d. At $t = t_1$, continuous demand depletes the inventory to zero. Subsequently, the supply experiences partial backlogging due to shortages between t_1 and T.

Differential equations of the inventory system can be described as

$$\frac{dI_1(t)}{dt} = -A^{\eta}(a-bp),\ 0 \le \mathrm{t} \le \mathrm{t_d} \tag{1}$$

with $I_1(0) = I_{max}$. After solving (1), the inventory level $I_1(t)$ is obtained as

$$I_1(t) = I_{\max} - A^{\eta}(a-bp)t \tag{2}$$

During the period $[t_d, t_1]$, the inventory level drops due to inventory-reliance demand and degradation. As a result, the following differential equation is used to represent the inventory system at any given time t

$$\frac{dI_2(t)}{dt} + \frac{1}{1+m-t} I_2(t) = -A^{\eta}(a-bp)\left(\frac{m-t}{m}\right),\ \mathrm{t_d} \le \mathrm{t} \le \mathrm{t_1} \tag{3}$$

with $I_2(t_1) = 0$. The inventory level $I_2(t)$ is derived by solving (3).

$$I_2(t) = \frac{A^{\eta}(a-bp)}{m}\left[(1+m-t)(t_1-t) + (1+m-t)\,In\left(\frac{1+m-t_1}{1+m-t}\right)\right] \tag{4}$$

Considering the continuity of $I(t)$ at $t = t_d$, it follows from (2) and (4) that $I_1(t_d) = I_2(t_d)$ gives the maximum level of inventory for each cycle as

$$S = I_{\max} = A^{\eta}(a-bp)\left[t_d + \left(\frac{1+m-t_d}{m}\right)\left((t_1-t) + In\left(\frac{1+m-t_1}{1+m-t_d}\right)\right)\right] \tag{5}$$

The demand at time t is partially backlogged as a fraction $\frac{1}{1+\delta(T-t)}$ during the period $[t_1, T]$. As a result, the differential equation used to represent the inventory system at any given time t $(t_1 \le t \le T)$ is as follows:

$$\frac{dI_3(t)}{dt} = -\frac{A^{\eta}(a-bp)}{1+\delta(T-t)},\ \mathrm{t_1} \le \mathrm{t} \le T \tag{6}$$

with $I_3(t_1) = 0$. Solving (6), yields the inventory level $I_3(t)$ as:

$$I_3(t) = \frac{A^{\eta}(a-bp)}{\delta}\left[In\left(1+\delta(T-t)\right) - In\left(1+\delta(T-t_1)\right)\right] \tag{7}$$

The maximum amount of demand backlogged during each cycle can be computed as

$$R = -I_3(T) = \frac{A^{\eta}(a-bp)}{\delta}\left[In\left(1+\delta(T-t_1)\right)\right] \tag{8}$$

As a result, the ordered quantity Q can be determined using (5) and (8) as follows:

$$Q = S + R \tag{9}$$

The system consists of the following relevant costs:

1. Ordering costs are the fixed expenses associated with each retailer's purchase order to replenish inventory. Ordering cost plays a significant role in decision-making for supply chain management and inventory control strategies. The ordering cost can be expressed simply as: Ordering Cost (OC) = K

2. The total number of advertisements during each inventory cycle is A, and the cost per advertisement is G. Therefore, the total advertising cost during each inventory cycle would be: Total Advertising Cost (AC) = AG

3. The retailer's total inventory holding cost over the planning period $(0, t_1)$, which is divided into two parts $(0, t_d)$ and (t_d, t_1) with corresponding inventory levels $I_1(t)$ and $I_2(t)$, considering the inventory holding cost for each part separately and then combining them gives the total cost incurred by the retailer to maintain inventory throughout the specified period.

$$HC = c\left\{\int_0^{t_d}(g+ht)I_1(t)\,dt + \int_{t_d}^{t_1}(g+ht)I_2(t)\,dt\right\}$$

$$= c\Bigg\{\left[gI_{\max}t_d - \left(gA^\eta(a-bp) - hI_{\max}\right)\frac{t_d^2}{2} - hA^\eta(a-bp)\frac{t_d^3}{3}\right] + \frac{A^\eta(a-bp)}{m}\Bigg[g\Bigg(\left((m+1)t_1(t_1-t_d)\right.$$

$$\left. - (1+m-t_1)\left(\frac{t_1^2-t_d^2}{2}\right) + \left(\frac{t_1^3-t_d^3}{3}\right)\right) - \frac{g}{4}\left((1+m-t_1)^2 - (1+m-t_d)^2\left(2In\left(\frac{1+m-t_1}{1+m-t_d}\right)+1\right)\right)$$

$$+h\left((m+1)t_1\left(\frac{t_1^2-t_d^2}{2}\right) - (1+m-t_1)\left(\frac{t_1^3-t_d^3}{3}\right) + \left(\frac{t_1^4-t_d^4}{4}\right)\right) + h\Bigg(\frac{(m+1)^2}{6}(t_1-t_d)$$

$$+(m+1)\left(\frac{t_1^2-t_d^2}{12}\right) - \left(\frac{t_1^3-t_d^3}{9}\right) + In\left(\frac{1+m-t_1}{1+m-t_d}\right)\left(-\frac{t_d^2}{2} + \frac{(m+1)^2}{2} + \frac{t_d^3}{3} - \frac{(m+1)^3}{3}\right)\Bigg)\Bigg]\Bigg\} \quad (10)$$

4. The cost of deterioration over the interval $[t_d, t_1]$ for the retailer's inventory, where deterioration affects the inventory's value due to spoilage, wear-and-tear, obsolescence, and similar factors, formulate the expression based on the quantity of inventory that has deteriorated during this period.

$$DC = d\left(I_2(t_d) - \int_{t_d}^{t_1} A^\eta(a-bp)\left(\frac{m-t}{m}\right)dt\right) = \frac{dA^\eta(a-bp)}{m}\left[(1-t_d)(t_1-t_d) + (1+m-t_d)In\left(\frac{1+m-t_1}{1+m-t_d}\right) - \frac{(t_1^2-t_d^2)}{2}\right] \quad (11)$$

5. Shortage cost is calculated during the time interval $t_1 < t \le T$, where customers experience delays or unavailability of goods. Various components related to the consequences of inadequate inventory levels constitutes the shortage cost. The expression for shortage cost is thus:

$$SC = s\int_{t_1}^{T}(-I_3(t))\,dt = \frac{sA^\eta(a-bp)}{\delta}\left[(T-t_1) - \frac{1}{\delta}In\left(1+\delta(T-t_1)\right)\right] \quad (12)$$

6. The last sale cost (LSC) is calculated as the integral of unmet demand $D(p)$ over the time interval $t_1 < t \le T$, multiplied by the unit opportunity cost 'l'. This mathematical representation

assesses the economic implications of customers seeking alternatives due to inventory shortages. It guides inventory management to enhance sales performance and competitiveness.

$$LSC = l\int_{t_1}^{T} A^{\eta}\left(a-bp\right)\left(1-\frac{1}{1+\delta\left(T-t_1\right)}\right)dt = lA^{\eta}\left(a-bp\right)\left[\left(T-t_1\right)-\frac{1}{\delta}In\left(1+\delta\left(T-t_1\right)\right)\right] \quad (13)$$

7. The overall sales income for the entire cycle length, taking into account the item's selling price p and demand function $D(p)$ during the period of favourable stock (t_1), expresses the total sales revenue as:

$$SR = p\int_{0}^{t_d} A^{\eta}\left(a-bp\right)dt + p\int_{0}^{t_d} A^{\eta}\left(a-bp\right)\left(\frac{m-t}{m}\right)dt + pR = pA^{\eta}\left(a-bp\right)\left[t_1-\frac{1}{2m}\left(t_1^2-t_d^2\right)\right]+pR \quad (14)$$

8. The cyclic capital costs associated with prepayments made by the retailer, where a percentage 'σ' of the purchase price is borrowed and paid in 'n' installments, and the remaining balance '$1-\sigma$' is paid at the time of delivery, derive the expression for interest costs incurred during the prepayment period.

$$CP = \frac{c\sigma I_L Q\left(n+1\right)L}{2n} \quad (15)$$

9. The discount rate 'α' offered by the supplier on prepayment of the purchase cost, which varies depending on the number of payments 'n' and the maximum discount rate 'J', use the following mathematical formula:

$$DIS = \frac{c\sigma JQ}{n} \quad (16)$$

Analyzing the impact of payment terms 'M' across different inventory scenarios helps retailers make informed decisions to optimize revenue, manage costs, and enhance customer satisfaction. By understanding the financial implications of payment terms on business performance, retailers can implement effective strategies to drive growth and profitability.

Case I: $M \le t_d \le t_1 \le T$

The monthly interest rate IC charged to the retailer for maintaining inventory during the settlement period $[M, t_1]$ derives the formula for 'I_C' based on the interest rate applied to the value of inventory held over this period. Let's formulate the expression for IC_1 as the yearly interest rate:

$$IC_1 = cI_C\left\{\int_{M}^{t_d} I_1\left(t\right)dt + \int_{t_d}^{t_1} I_2\left(t\right)dt\right\}$$

$$= cI_C\left\{\left[I_{\max}\left(t_d-M\right)-A^{\eta}\left(a-bp\right)\left(\frac{t_d^2-M^2}{2}\right)\right]+\frac{A^{\eta}\left(a-bp\right)}{m}\left[\left(\left(m+1\right)t_1\left(t_1-t_d\right)-\left(1+m-t_1\right)\left(\frac{t_1^2-t_d^2}{2}\right)\right.\right.\right.$$

$$\left.\left.\left.+\left(\frac{t_1^3-t_d^3}{3}\right)\right)-\frac{1}{4}\left(\left(1+m-t_1\right)^2-\left(1+m-t_d\right)^2\left(2In\left(\frac{1+m-t_1}{1+m-t_d}\right)+1\right)\right)\right]\right\} \quad (17)$$

The total interest earned IE by the retailer based on customer demand and the interest rate, 'I_E'

during the period *[0, M]*, where the retailer earns interest on all items sold to customers, derives a formula that considers the sales revenue generated during this period. It is formulate the expression for *IE* as the total interest earned:

$$IE_1 = (1-\sigma)\left\{ pI_E \int_0^M \int_0^t D(p)\,du\,dt + pI_e D(p) \int_0^M In\left(1+\delta\left(T-t_1\right)\right)dt \right\}$$

$$= (1-\sigma)\, pI_E A^{\eta}(a-bp)\left[\frac{M^2}{2} - \frac{1}{2m}\left(\frac{M^3}{3} - t_d^2 M\right) + \frac{M}{\delta} In(1+\delta\left(T-t_1\right)\right] \tag{18}$$

Therefore, the inventory system for the total profit per unit of time for this scenario is defined as:

$\Pi_1(p, t_1, T) = (SR - OC - AC - DC - HC - SC - LSC - CP + IE_1 + DIS - IC_1) / T$ (19)

Case II: $t_d \leq M \leq t_1 \leq T$

The cost of interest charges for unsold stock over the time $[M, t_1]$ is illustrated as follows

$$IC_{12} = cI_C \int_M^{t_1} I_2(t)\,dt$$

$$= \frac{cI_C A^{\eta}(a-bp)}{m}\left[\left((m+1)t_1(t_1-M) - (1+m-t_1)\left(\frac{t_1^2-M^2}{2}\right) + \left(\frac{t_1^3-M^3}{3}\right)\right)\right.$$

$$\left. -\frac{1}{4}\left((1+m-t_1)^2 - (1+m-M)^2\left(2In\left(\frac{1+m-t_1}{1+m-M}\right)+1\right)\right)\right] \tag{20}$$

The interest earned for this scenario is

$$IE_2 = IE_1 = (1-\sigma)\left\{ pI_E \int_0^M \int_0^t D(p)\,du\,dt + pI_E D(p) \int_0^M In\left(1+\delta\left(T-t_1\right)\right)dt \right\}$$

$$= (1-\sigma)\, pI_E A^{\eta}(a-bp)\left[\frac{M^2}{2} - \frac{1}{2m}\left(\frac{M^3}{3} - t_d^2 M\right) + \frac{M}{\delta} In(1+\delta\left(T-t_1\right)\right] \tag{21}$$

Thus, the inventory system for the total profit per unit of time for this scenario is represented as:

$\Pi_2(p, t_1, T) = (SR - OC - AC - DC - HC - SC - LSC - CP + IE_2 + DIS - IC_2) / T$ (22)

Case III: $t_d \leq t_1 \leq M \leq T$

After *M*, there is no positive stock remaining; consequently, the interest charged as:

$IC_3 = 0$ (23)

The retailer accrues additional interest on inventory from *M* to *T*, and the interest received on sold products goes up to *M*.

$$IE_3 = (1-\sigma)\, pI_E A^{\eta}(a-bp)\left[\frac{t_1^2}{2} + (M-t_1)t_1 + \frac{MIn\left(1+\delta\left(T-t\right)\right)}{\delta}\right] \tag{24}$$

Hence, the inventory system for the total profit per unit of time for this scenario is given as:

$$\Pi_3(p, t_1, T) = (SR - OC - AC - DC - HC - SC - LSC - CP + IE_3 + DIS - IC_3) / T \tag{25}$$

The optimal total profit function for this inventory system is expressed as:

$$\Pi(p,t_1,T)=\begin{cases}\Pi_1(p,t_1,T), 0<M<t_d<t_1<T\\ \Pi_2(p,t_1,T), 0<t_d<M<t_1<T\\ \Pi_3(p,t_1,T), 0<t_d<t_1<M<T\end{cases} \tag{26}$$

4.1 Solution procedure

The main objective of the presented model in above section to maximize the total cost for the retailer; it is necessary to found out the value of p, t_1, and T, which maximizes the cost.

Directly obtaining the required optimal values of the total profit function is impossible because all the total profit functions are highly nonlinear. Initialize all the needed inputs for the total profit cost values $\Pi_i(p, t_1, T)$ for i = 1, 2, 3 without decision variables, Using MATLAB Solver to obtain the optimization of the total profit function $\Pi_i(p, t_1, T)$ for i = 1, 2, 3 with the constraints and to get optimal values of p, t_1 and T. to obtain the optimal values of the decision variables p, t_1 and T, also find optimal order quantity and optimal profit cost are found using obtained optimal values.

To prove that the total profit function $\Pi_i(p, t_1, T)$ for i = 1, 2, 3 is concave, we need to verify that the second derivatives of $\Pi_i(p, t_1, T)$ with respect to p, t_1, and T are all negative (or non-positive) within the above obtained optimal decision variables. It is demonstrated step-by-step for each $\Pi_i(p, t_1, T)$ as:

1. Define the Total Profit Functions $\Pi_i(p, t_1, T)$ for i = 1, 2, 3.
2. Compute the second partial derivatives of each profit function $\Pi_i(p, t_1, T)$ with respect to p, t_1, and T.
3. Verify Concavity:

For concavity in p, ensure that $\frac{\partial^2 \Pi_i(p,t_1,T)}{\partial p^2} \le 0$ for all p, t_1, and T.

For concavity in t_1, ensure that $\frac{\partial^2 \Pi_i(p,t_1,T)}{\partial t_1^2} \le 0$ for all p, t_1, and T.

For concavity in T, ensure that $\frac{\partial^2 \Pi_i(p,t_1,T)}{\partial T^2} \le 0$ for all p, t_1, and T.

4. Ensure the mixed partial derivatives $\frac{\partial^2 \Pi_i(p,t_1,T)}{\partial t_1 \partial p}, \frac{\partial^2 \Pi_i(p,t_1,T)}{\partial T \partial p}$ and $\frac{\partial^2 \Pi_i(p,t_1,T)}{\partial T \partial t_1}$ also satisfy conditions that indicate concavity.
5. If all the second derivatives satisfy the concavity conditions (i.e. non-positive) within the feasible parameter space, then the total profit function $\Pi_i(p, t_1, T)$ for i = 1, 2, 3 is concave.

6. Verify the condition

$$\begin{bmatrix} \left(\frac{\partial^2\Pi_i(p,t_1,T)}{\partial p^2}\right)\left(\left(\frac{\partial^2\Pi_i(p,t_1,T)}{\partial t_1^2}\right)\left(\frac{\partial^2\Pi_i(p,t_1,T)}{\partial T^2}\right)-\left(\frac{\partial^2\Pi_i(p,t_1,T)}{\partial T\partial t_1}\right)^2\right) \\ -\left(\frac{\partial^2\Pi_i(p,t_1,T)}{\partial t_1\partial p}\right)^2\left(\frac{\partial^2\Pi_i(p,t_1,T)}{\partial T^2}\right)-\left(\frac{\partial^2\Pi_i(p,t_1,T)}{\partial T\partial p}\right)\left(\frac{\partial^2\Pi_i(p,t_1,T)}{\partial t_1^2}\right) \\ +2\left(\frac{\partial^2\Pi_i(p,t_1,T)}{\partial T\partial t_1}\right)\left(\frac{\partial^2\Pi_i(p,t_1,T)}{\partial t_1\partial p}\right)\left(\frac{\partial^2\Pi_i(p,t_1,T)}{\partial T\partial p}\right) \end{bmatrix} < 0$$

which is the Hessian matrix analysis to examine the concavity properties of the total profit functions.

7. Analyzing the derivatives of the given profit functions, determine whether $\Pi_i(p, t_1, T)$ is concave for $i = 1, 2, 3$ under the specified conditions. This verification is crucial in understanding the optimization landscape and ensuring the validity of profit maximization strategies.

5. Numerical examples and Sensitivity Analysis

Example 1:

The data provided in this example demonstrates numerically, the outcomes of the model developed. Consider the values a = 90, b = 1.5, A = 4, K = Rs.620 per order, c = Rs.15 per unit, s = Rs.11per unit, d = Rs. 10 per unit, l = Rs. 12 per unit, h = Rs.0.05 per unit per month, g = Rs. 0.3 per unit per month, M = 0.4 month, t_d = 0.7 month, m = 3 month, A = 4, η = 0.6, G = 50, δ = 0.5, σ = 0.4, J = 1, n = 3, L = 0.3, I_C = 0.2, I_E = 0.07, I_L = 0.05. Optimal values for the above data are: Π_1 = Rs.2512.26, p = Rs.27.71, t_1 = 1.4991, T = 1.7656, S = 141.7648, R = 27.8385, Q = 169.6034 in appropriate units. The Hessian values $H_1 = -5.7217 < 0$, $H_2 = 10008.9540 > 0$ and $H_3 = -2291002.4621 < 0$ conform the concavity.

Example 2:

Other parameters are the same as in example 1 except M = 0.9. The optimal values for the data are: Π_2 = Rs.2588.10, p = Rs.27.51, t_1 = 1.4938, T = 1.7520, S = 142.2509, R = 27.1894, Q = 169.4403 in appropriate units, and the Hessian values $H_1 = -5.8008 < 0$, $H_2 = 10381.8905 > 0$ and $H_3 = -2446368.8423 < 0$ conform the concavity.

Example 3:

Other parameters are the same as example 1 except M = 2.3 and c = 14. Taking the above data, determine the optimal values as Π_3 = Rs.2706.21, p = Rs.25.77, t_1 = 2.2068, T = 2.4286, S = 198.9472, R = 24.8118, Q = 223.7590 in appropriate units, and the Hessian values are $H_1 = -5.1182 < 0$, $H_2 = 5594.1131 > 0$ and $H_3 = -52067.6911 < 0$ conform the concavity.

Example 3 clearly shows that when the credit facility increases, the total profit increases. The optimum values of this model are p*, t_1*, T*, S*, R*, Q* and Π* are obtained from example 3.

The study has examined the impact of changes made to well-known characteristics such as inventory cost, degradation, and demand. The results are tabulated, ranging from –50 % to +50% from the original values. Furthermore, changes are only considered for one parameter at a time, while the others remain unaltered. The results of this study are displayed in Table 2.

Table 2 Sensitivity analysis for some input parameters for case I

Parameter	% Change of parameter	p	t_1	T	S	R	Q	Π_1
c	–50%	27.90	1.5771	1.8356	146.5663	26.8925	173.4587	2443.59
	–25%	27.81	1.5374	1.8000	144.1161	27.3702	171.4863	2477.62
	+25%	27.60	1.4619	1.7320	139.4923	28.2878	167.7801	2547.51
	+50%	27.49	1.4256	1.6992	137.2425	28.7290	165.9715	2583.36
h	–50%	27.70	1.5001	1.7663	141.8822	27.8177	169.6999	2512.67
	–25%	27.71	1.4996	1.7660	141.8016	27.8287	169.6303	2512.47
	+25%	27.71	1.4987	1.7652	141.7354	27.8385	169.5740	2512.06
	+50%	27.71	1.4982	1.7649	141.6987	27.8582	169.5569	2511.85
g	–50%	27.65	1.5055	1.7693	142.4989	27.6245	170.1234	2518.52
	–25%	27.68	1.5023	1.7675	142.1317	27.7366	169.8683	2515.39
	+25%	27.74	1.4959	1.7638	141.3982	27.9500	169.3481	2509.14
	+50%	27.76	1.4927	1.7619	141.0755	28.0600	169.1355	2506.03
I_C	–50%	27.45	1.5536	1.8094	146.9123	27.0011	173.9134	2536.03
	–25%	27.58	1.5260	1.7872	144.3124	27.4275	171.7399	2524.03
	+25%	27.83	1.4730	1.7446	139.3212	28.2334	167.5547	2500.73
	+50%	27.95	1.4474	1.7241	136.9159	28.6235	165.5394	2489.43
I_E	–50%	27.65	1.5168	1.7773	143.3278	27.2993	170.6270	2504.66
	–25%	27.68	1.5080	1.7715	142.5501	27.5694	170.1195	2508.44
	+25%	27.74	1.4903	1.7597	140.9865	28.0970	169.0835	2516.11
	+50%	27.77	1.4814	1.7537	140.2006	28.3545	168.5551	2519.99
σ	–50%	28.21	1.5097	1.7832	140.3352	28.0831	168.4183	2424.99
	–25%	27.96	1.5044	1.7744	141.0533	27.9638	169.0170	2468.43
	+25%	27.46	1.4938	1.7569	142.4698	27.7174	170.1872	2556.49
	+50%	27.21	1.4885	1.7482	143.1682	27.5906	170.7587	2601.12
J	–50%	28.18	1.5224	1.7919	141.3828	27.7234	169.1062	2417.05
	–25%	27.95	1.5107	1.7787	141.5556	27.7778	169.3334	2464.44
	+25%	27.47	1.4876	1.7526	141.9662	27.8970	169.8632	2560.51
	+50%	27.24	1.4761	1.7396	142.1089	27.9447	170.0536	2609.18
δ	–50%	28.06	1.4319	1.8065	135.3023	39.4136	174.7159	2539.03
	–25%	27.86	1.4708	1.7821	139.0293	32.6104	171.6397	2523.38
	+25%	27.59	1.5207	1.7537	143.8797	24.2944	168.1741	2503.95
	+50%	27.50	1.5376	1.7447	145.5186	21.5606	167.0792	2497.50
η	–50%	26.70	1.9785	2.3134	118.6573	23.4431	142.1004	1517.28
	–25%	27.28	1.7147	2.0145	129.3204	25.5847	154.9051	1957.84
	+25%	28.06	1.3122	1.5478	155.5248	30.1814	185.7063	3208.50
	+50%	28.36	1.1459	1.3530	170.3567	32.5679	202.9247	4081.60

Table 3 Sensitivity analysis for some input parameters for case II

Parameter	% Change of parameter	p	t_1	T	S	R	Q	Π_2
c	–50%	27.92	1.5344	1.7944	143.4068	27.0225	170.4293	2497.32
	–25%	27.72	1.5136	1.7728	142.7837	27.1122	169.8959	2542.48
	+25%	27.31	1.4747	1.7320	141.7005	27.2670	168.9675	2634.18
	+50%	27.10	1.4563	1.7127	141.2215	27.3517	168.5732	2680.73
h	–50%	27.51	1.4947	1.7528	142.3175	27.1795	169.4970	2588.51

	–25%	27.51	1.4942	1.7524	142.2805	27.1894	169.4699	2588.31
	+25%	27.52	1.4933	1.7517	142.1701	27.2009	169.3710	2587.89
	+50%	27.52	1.4929	1.7513	142.1405	27.2009	169.3414	2587.68
g	–50%	27.46	1.5001	1.7557	142.9364	26.9729	169.9093	2594.42
	–25%	27.49	1.4970	1.7539	142.5753	27.071	169.6525	2591.26
	+25%	27.54	1.4906	1.7502	141.8828	27.3030	169.1858	2584.95
	+50%	27.57	1.4874	1.7483	141.5147	27.4063	168.9213	2581.81
I_C	–50%	27.47	1.5121	1.7692	143.7789	27.1137	170.8926	2590.24
	–25%	27.49	1.5028	1.7605	143.0042	27.1566	170.1608	2589.16
	+25%	27.53	1.4850	1.7438	141.5116	27.2322	168.7438	2587.06
	+50%	27.56	1.4766	1.7359	140.7579	27.2565	168.0144	2586.05
I_E	–50%	27.33	1.5481	1.7950	147.0509	26.2105	173.2614	2564.33
	–25%	27.42	1.5209	1.7736	144.6471	26.7172	171.3663	2576.10
	+25%	27.60	1,4665	1.7303	139.8345	27.6672	167.5017	2600.34
	+50%	27.69	1.4392	1.7083	137.4151	28.1111	165.5262	2612.82
σ	–50%	28.08	1.4783	1.7470	138.6258	27.7330	166.3588	2511.33
	–25%	27.80	1.4861	1.7496	140.4157	27.4670	167.8828	2549.87
	+25%	27.23	1.5012	1.7544	144.0287	26.9231	170.9518	2627.14
	+50%	26.94	1.5085	1.7567	145.8515	26.6552	172.5067	2666.62
J	–50%	27.98	1.5157	1.7770	141.7860	27.0989	168.8848	2492.23
	–25%	27.75	1.5047	1.7645	141.9998	27.1460	169.1458	2539.96
	+25%	27.28	1.4829	1.7397	142.4442	27.2420	169.6863	2636.66
	+50%	27.05	1.4721	1.7274	142.6313	27.2826	169.9139	2685.64
δ	–50%	27.88	1.4283	1.7929	135.7971	38.6224	174.4195	2614.08
	–25%	27.67	1.4663	1.7686	139.5176	31.9039	171.4215	2598.87
	+25%	27.39	1.5146	1.7401	144.3173	23.7071	168.0244	2580.07
	+50%	27.30	1.5310	1.7311	145.9278	21.0090	166.9368	2573.85
η	–50%	26.50	1.9610	2.2863	118.6259	22.9563	141.5822	1566.55
	–25%	27.08	1.7061	1.9970	129.6199	25.0265	154.6464	2019.03
	+25%	27.88	1.3082	1.5362	156.0246	29.4234	185.4480	3302.44
	+50%	28.21	1.1421	1.3423	170.7133	31.6825	202.3958	4197.97

Table 4 Sensitivity analysis for some input parameters for case III

Parameter	% Change of parameter	p	t_1	T	S	R	Q	Π_3
c	–50%	26.07	2.1972	2.4194	19.6556	24.6364	221.2920	2578.26
	–25%	26.11	2.2147	2.4415	197.4188	25.0904	222.5091	2624.73
	+25%	25.31	2.2291	2.4510	202.9076	25.1560	228.0636	2715.45
	+50%	25.05	2.2454	2.4685	205.3628	25.4746	230.8374	2762.30
h	–50%	25.51	2.2322	2.4521	201.9140	24.7969	226.7109	2671.40
	–25%	25.51	2.2314	2.4516	201.8686	24.8290	226.6976	2671.03
	+25%	25.52	2.2298	2.4505	201.7191	24.8753	226.5944	2670.28
	+50%	25.58	2.2158	2.4383	200.5693	25.0242	225.5935	2668.62
g	–50%	25.45	2.2369	2.4537	202.5322	24.5073	227.0394	2678.36
	–25%	25.49	2.2335	2.4522	202.1049	24.6827	226.7876	2674.49
	+25%	25.55	2.2284	2.4506	201.4640	25.0140	226.4780	2666.92
	+50%	25.58	2.2267	2.4505	201.1919	25.1630	226.3549	2663.22

I_E	–50%	25.54	2.2638	2.4993	203.5101	26.4385	229.9486	2598.58
	–25%	25.80	2.2527	2.4852	201.3618	25.9225	227.2843	2635.77
	+25%	25.59	2.2060	2.4226	199.9472	24.3866	224.3337	2703.33
	+50%	26.08	2.1800	2.4008	195.6069	24.4818	220.0888	2734.36
σ	–50%	26.43	2.1555	2.3800	192.1725	24.6144	216.7869	2618.74
	–25%	26.03	2.1967	2.4190	196.8587	24.6760	221.5347	2646.13
	+25%	25.64	2.2677	2.4993	203.1346	25.9483	229.0829	2694.03
	+50%	25.23	2.2661	2.5012	205.4693	26.6335	232.1027	2713.95
J	–50%	26.09	2.1952	2.4176	196.4251	24.6429	221.0680	2575.32
	–25%	26.10	2.2138	2.4405	197.4261	25.0873	222.5134	2623.27
	+25%	25.25	2.2453	2.4659	204.1820	25.0593	229.2413	2718.42
	+50%	25.04	2.2467	2.4699	205.4956	25.4927	230.9883	2765.24
δ	–50%	25.99	2.1856	2.5164	196.4251	24.6429	221.0680	2575.32
	–25%	25.69	2.2034	2.4681	197.4261	25.0873	222.5134	2623.27
	+25%	25.43	2.2408	2.4317	204.1820	25.0593	229.2413	2718.42
	+50%	25.48	2.2628	2.4320	205.4956	25.4927	230.9883	2765.24

Case I

Figure 2. Concavity of the total profit function with respect to t1 and T

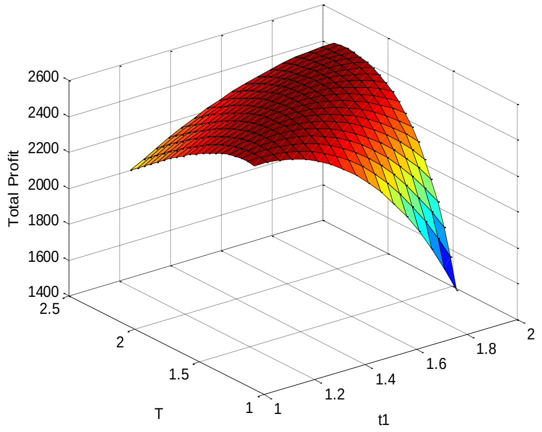
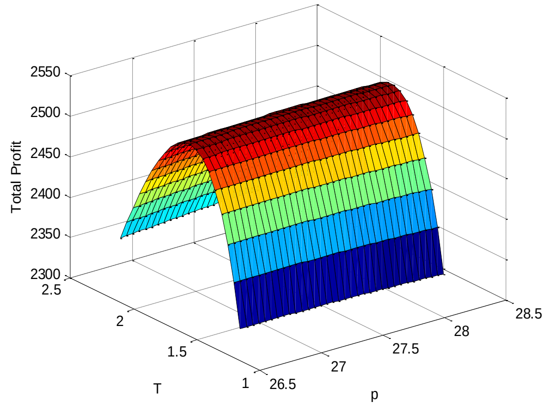


Figure 3. Concavity of the total profit function with respect to p and T



Case II

Figure 4. Concavity of the total profit function with respect to t1 and T

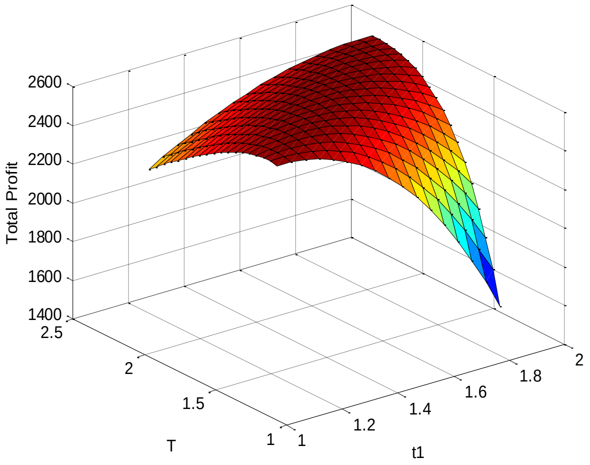
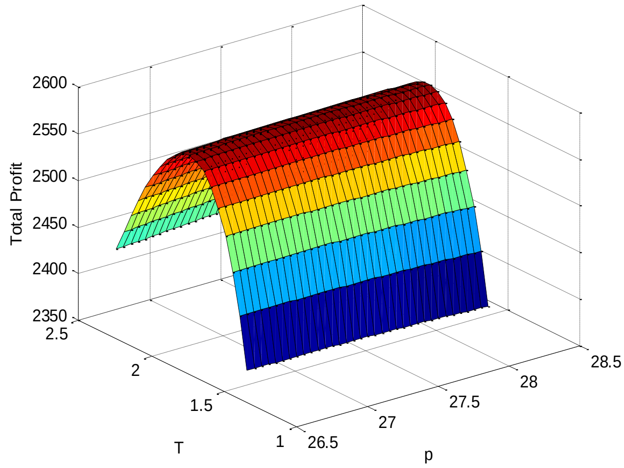


Figure 5. Concavity of the total profit function with respect to p and T



Case III

Figure 6. Concavity of the total profit function with respect to t1 and T

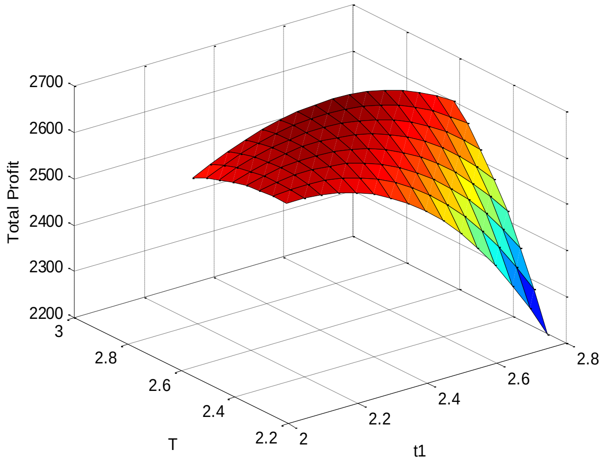
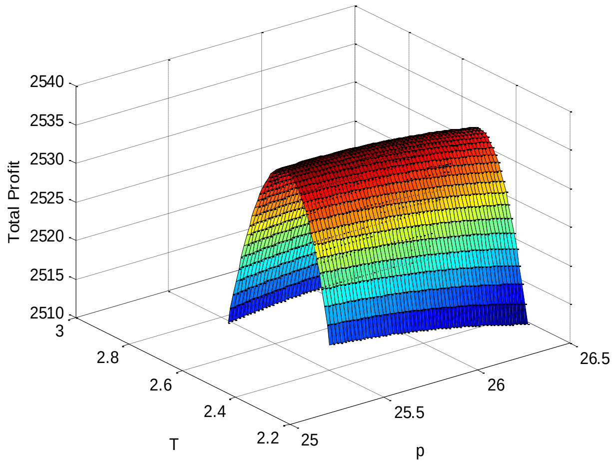


Figure 7. Concavity of the total profit function with respect to p and T



Observations of the table and graphs

Analysis of the observations of the study based on the changes in parameters and their effects on various decision variables and performance metrics, including total profit, at the provided parameter changes and resulting values are presented below. Graphs of all three cases visualize the concavity of the total profit function with various decision parameters. The changes in each parameter, the decision variables and total profit for all three cases are described below:

Table 2 highlights that decreasing purchase costs, interest rates, backlogging, increasing prepayments, and advertising elasticity enhance inventory management profitability-however, minimal impact parameters include holding costs, maximum discount, and interest earned. Strategic adjustments in these factors can significantly affect profitability, emphasizing the importance of informed decision-making in inventory management.

Table 3 highlights the significance of examining and refining financial terms, holding costs, and purchasing costs to reduce total expenses and increase profitability in inventory management. Lower financing costs, improve operational efficiency and boost cash flow by carefully modifying purchase prices, interest rates, and payment periods. Furthermore, balancing inventory levels and backlog criteria allows for efficient demand fulfillment while reducing expenses, ultimately boosting supply chain operation's profitability and competitiveness.

Table 4 outlines the strategic factors that decision-makers should take into account. It emphasizes the significance of financial management and cost optimization, as well as the awareness of how different parameters interact to affect profitability. It offers ways to increase profitability through financial planning, inventory control techniques, and successful negotiating tactics.

Case III emphasizes the critical role of cost optimization and financial management in maximizing profitability. It demonstrates that managing purchasing costs and optimizing financial parameters significantly impact profits. Among the cases, Case III is likely to yield the highest profit by refining inventory control techniques and economic strategies. This efficiency in inventory management leads to improved financial arrangements and market competitiveness for the retailer.

6 Conclusion

In this research article, the innovative attempt to provide a variety of payment options while operating an inventory which is subject to deterioration and sensitive demand has shown promising outcomes suitable for real-world situations. The effectiveness of inventory turnover in preserving product quality and boosting sales through dynamic controls and preservation techniques before deterioration is demonstrated numerically.

This research has examined the effects of advanced payments and trade credit laws on a deteriorating product inventory model, accounting for factors that influence demand, such as product freshness, price, and advertisement frequency. The solution for the model clearly indicates the optimal selling price, cycle length, and shortage incidence period in order to maximize overall profit. The research also addresses the gaps appropriately, as mentioned above.

It is evident that: Longer loan terms boost profitability, as the optimization process results in higher profits. The greatest revenue occurs when ($t_1 \leq M \leq T$). The significance of adopting a flexible approach to trade credit and degradation mitigation to maximize profit and expand the business through enhanced goodwill was underscored in this paper.

The research outcomes can be briefed as follows: Enhanced Operational Reliability on trade credit and payback strategies provide timely stock delivery, improved cash flow, and stable finances. Sensitivity analysis, when combined with dynamic systems and precise demand forecasts, balances inventory levels, reduces the possibility of backlogs, and guarantees efficient operations. By optimizing shelf life management and accounting for time-varying degradation, choices about perishable inventory may be made more profitably and competitively in the market. Achieving maximum profitability in agile demand management for price-sensitive products may minimize waste through focused advertising and safety stock adjustments, boost customer satisfaction, and decrease waste and stockouts. Effective business strategies to meet the practical challenges are to sense the futuristic changes in demand and appropriately implement product pricing and payment options to ensure sustained profitability.

The scope of the research encompasses the analysis of various real-world supply chain scenarios, specifically in the post-pandemic period, such as refurbishing items and handling products that are near the expiration date. Future research will explore various supply chain scenarios, including periodic price discount splits, product discounts, and stochastic demand.

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