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A New Modified Secant Condition for Non-linear Conjugate Gradient Methods with Global Convergence

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Abstract:
The Conjugate Gradient Methods(CGM) are well-recognized techniques for handling nonlinear optimization problems. Dai and Liao (2001) employ the secant condition approach, this study utilizes the modified secant condition proposed by Yabe-Takano (2004) and Zhang and Xu (2001), which is satisfied at each iteration through the implementation of the strong Wolf-line search condition. Additionally, please provide three novel categories of conjugate gradient algorithms of this nature. We examined 15 well-known test functions. This novel approach utilises the existing gradient and function value to accurately approximate the goal function with high-order precision. The worldwide convergence of our novel algorithms is demonstrated under certain conditions. Numerical results are provided, and the efficiency is proven by comparing it to other approaches.

Keywords: Conjugate Gradient technique, Un-constrained optimization, numerical studies, preconditioning, Sufficient descent condition, Convergence

1. Introduction

The unconstrained Problem for an optimization defended by:

Min $f\langle x\rangle \qquad x \in R^n$,
(1)

We have $f: R^n \to R$ is smooth and it's ∇f is available Conjugate gradient(CG) method For Solving (1)is Iterative methods of the from

$$x_{k+1} = x_k + \alpha_k d_k \tag{2}$$

Where $\alpha_k > 0$ is step size and d_k is search direction, the d_k is recursively known as:

$$d_k = \begin{cases} -g_k & for\ \ k = 1, \\ -g_k + \beta_k d_{k-1} & for\ \ k \geq 2, \end{cases} \tag{3}$$

g_k means $\nabla f\langle x\rangle$ and β_k is a parameter, if $f\langle x\rangle$ is a strictly convex quadratic function and α_k is an exact one-dimension min .(1)-(3) is knows (CG) method, Also, (1)-(2) is knows the nonlinear (CG) method. There are different general unconstrained optimization problems, Famous prescription for β are the (LS) [1]. (PR) [2] and (HS) [3] which are as:

$$\beta_k^{LS} = \frac{g_{k+1}^T y_k}{d_k^T g_k} \tag{4}$$

$$\beta_k^{PR} = \frac{g_k^T\ y_{k-1}}{\|g_{k-1}\|^2} \tag{5}$$

$$\beta_k^{HS} = \frac{g_k^T y_{k-1}}{d_{K-1}^T y_{k-1}} \quad (6)$$

To prove the convergence of this approach, it is often needed that the step-size α_k satisfies the strong Wolfe condition, where y_{k-1} is defined as g_k minus g_{k-1}, and $\|.\|$ represents the Euclidean norm.

$$f(x_k) - f(x_k + \alpha_k d_k) \geq \delta \alpha_k g_k^T d_k \qquad [4] \quad (7)$$

$$|g(x_k + \alpha_k d_k) d_k|^T \leq -\sigma d_k^T d_k \qquad [4] \quad (8)$$

Where $0 < \delta < 0.5 < \sigma < 1$

Each method comes with its own advantages and disadvantages. see[5], [3] The Polak-Ribiere and Hestenes-Stiefel (HS) methods have comparable theoretical properties. Both of these methods are favored over the Liu-Storey (LS) method in terms of numerical performance. This is because they both restart after encountering a bad direction. However, the Yabe-Takano (YT) method, derived by Zhang et al in 2004, stands out .[6] and Zhang[7]and Xu [8] proposed by Yabe-Takano (YT)[9] we propose a three news formulas for $\beta_k^{N1}$and $\beta_k^{N2}$and β_k^{N3} by exploiting the modified secant condition in this paper is organized as follows ,in section2 we state a conjugacy condition and the formulas In section3,the modified secant condition is described In Section 4, We suggest a fresh requirement for conjugacy and develop novel equations for β. In Section 5, we demonstrate the worldwide convergence of the latest conjugate gradient techniques under specific assumptions. Section 6 includes the presentation of a few numerical trials.

1.2 Yabe-Takano (YT)

Conjugate gradient algorithm generates a direction search such that the conjugacy condition holds, as,

$$d_i^T Q d_j = 0, \qquad \forall i \neq j \quad (9)$$

The matrix Q is positive definite for the quadratic objective function. For general nonlinear functions, the mean value theorem (M.V.T.) guarantees the existence of a value τ in the interval τ∈(0,1) such that

$$d_k^T y_{k-1} = \alpha_{k-1} d_k^T \nabla^2 f(x_{k-1} + \tau \alpha_{k-1} d_{k-1})$$

Hence, it is acceptable to substitute (9) with the subsequent condition:

$$d_k^T y_{k-1} = 0 \quad (10)$$

Recently, extension of the CG has been studied Yabe-Takano. [10] using the secant condition of Quasi-Newton (QN) methods,

$$H_k y_{k-1} = S_{k-1}. \quad (11)$$

where H_k is an inverse approximate for the Hessian and $S_{k-1} = x_k - x_{k-1}$. For quasi_Newton methods, the search direction d_k can be calculated by:

$$d_k = -H_k g_k \quad (12)$$

By (11) and (12). We have that

$$d_k^T y_{k-1} = -(H_k g_k)^T y_{k-1} = -g_k^T (H_k y_{k-1}) = -g_k^T s_{k-1} \quad (13)$$

By this relation, Dai and Liao replaced the conjugacy condition by the condition

$$d_k^T y_{k-1} = -t g_k^T s_{k-1} \quad (14)$$

Where $t \geq 0$ is a scalar. In the scenario where t=0, equation (11) simplifies to the standard conjugacy condition (10). Conversely, when t=1, equation (11) becomes equivalent to (10). In order to guarantee that the search direction d_k meets this requirement, we can substitute equation (3) into (14) to obtain:

$$-g_k^T y_{k-1} + \beta_k d_{k-1}^T y_{k-1} = -t g_k^T s_{k-1} \quad [10] \quad (15)$$

2.New formulas for β_k^{N1}, β_k^{N2}, β_k^{N3}

We are developed a new conjugate gradient (CG) method in this part based on the work of Yabe-Takano [10]. To achieve this work, we apply a modified secant condition (3.2) instead of the usual one (6). Let z_{k-1} be defined with a scalar parameter $\rho \geq 0$:

$$\begin{cases} z_{k-1} = y_{k-1} + \rho\left(\frac{\theta_{k-1}}{s_k^T u_{k-1}}\right) \\ \theta_{k-1} = 6(f_{k-1} - f_k) + 3(g_{k-1} + g_k)^T s_{k-1} \end{cases} \quad [11],[12] \quad (16)$$

Building upon the same reasoning as in part 2, we examine the adjusted secant condition involving with z_{k-1}

$$H_k z_{k-1} = s_{k-1} \quad (17)$$

when $\rho = 0$ and $\rho = 1$,[6] This situation aligns with the typical secant condition (6) and the revised secant condition (3.2) individually. It follows from (7) and (12) that:

$$d_k^T z_{k-1} = -(H_k g_k)^T z_{k-1} = -g_k^T (H_k z_{k-1}) = -g_k^T s_{k-1} \quad (18)$$

Considering this relationship, we substitute the conjugacy requirement with the updated condition.:

$$d_k^T z_{k-1} = -t g_k^T s_{k-1}\,, \quad (19)$$

$$d_k = -g_k + \beta_k d_{k-1}$$

$$H_K z_{k-1} = s_{k-1}$$

$$H_K y_{k-1} = s_{k-1} \quad (20)$$

$$d_k z_k = -g_k s_{k-1}$$

$$d_k z_k = -t g_k s_{k-1}$$

While $z_k = y_k + \frac{\theta_k}{\|s_k\|^2} s_k$ (21)

$\theta_k = 6(f_k - f_{k-1}) + 3(g_k - g_{k-1}) s_k$ for $t \geq 0$ (22)

$$d_k = -H_k g_k \quad (23)$$

$d_k^T y_k = 0$ (Perry), this condition is true for all liner functions

$$d_k y_k = -H_k g_k y_k \tag{24}$$

$d_k^T y_k = -g_{k-1}^T s_{k-1}$, this condition is true for all nonlinear functions

$$d_k y_k = -t g_k s_{k-1} \tag{25}$$

Multiply the equation(25)by z_KWe get

$$d_k z_k = -g_k z_k + \beta_k d_{k-1} z_k \tag{26}$$

$$-t g_k^T s_k = -g_k^T z_k + \beta_k d_{k-1}^T z_k \tag{27}$$

$$-t g_k^T s_k = -g_k^T z_k + \beta_k d_{k-1}^T y_k \tag{28}$$

$$(g_k^T z_k - t g_k^T s_k) = \beta_k d_k^T y_{k-1} \tag{29}$$

$$\beta_k^{N1} = \frac{g_k^T (z_k - t s_k)}{d_k^T y_{k-1}} \tag{30}$$

In the next section ,we proved the global convergence of the new methods (30) , following the Lio-story methods , we are get by using (30)

$$\beta_k^{N1} = Max\left\{\frac{g_k^T z_k}{d_{k-1}^T y_{k-1}}, 0\right\} - t\ \left\{\frac{g_k^T s_k}{d_k^T y_{k-1}}, 0\right\} \tag{31}$$

this case, if (ELS) then we have $\theta_{k-1} = 0, z_{k-1} = y_{k-1}$ the new parameters become$g_k^T y_{k-1} / d_k^T y_{k-1}$. Thus, our equation (30) simplifies to the Hestenes-Stiefel formula when considering linear conjugate gradient techniques. .

Similarly we have derive , β_k^{N2}, β_k^{N3}

$$\beta_k^{N2} = \frac{g_k^T (z_k - t s_k)}{\|g_{k-1}\|^2} \tag{32}$$

where $z_k = y_k + \frac{1}{3}\frac{\theta_k}{\|s_k\|^2} s_k$ [13] (33)

and

$$\theta_k = 2(f_k - f_{k+1}) + (g_{k+1} + g_k)^T s_k \tag{34}$$

and

$$\beta_k^{N3} = \frac{g_k^T (z_k - t s_k)}{-g_{k-1}^T d_{k-1}} \tag{35}$$

where $z_k = y_k + \frac{2}{3}\frac{\theta_k}{\|s_k\|^2} s_k$ [14] (36)

$\theta_k = 4(f_{k-1} - f_k) + 2(g_k + g_{k+1})^T s_{k-1}$ [14][15] (37)

2.1 Algorithm

Step1:Take $x_0 \in R^n$,and $0 < \delta \le \sigma < 1$,Calculate $f(x_0)$ and $g_0 = \nabla f(x_0)$, set $d_0 = -g_0$ for $k = 0$

Step2: Compute α_k Satisfying Strong "Wolfe Condition "(3.10) (3.11) and then Compute

$x_{k+1} = x_k + \alpha_k d_k$

Step3:If($\|g_k\|_\infty \leq 10^{-10}$ or ($|\alpha_k g_k d_k| \leq 10^{-10}|f_k|$) is Satisfy then stop.

Step4: Compute the new search direction :

$$d_k = -g_k + \beta_k d_{k-1}$$

If the restart criterion of Powell, such that $|g_k^T g_{k-1}| \geq 0.2\|g_k\|^2$ is Satisfied, then set $d_k = -g_k$; otherwise, define

Step5:Compute the new Parameters $\beta_k^{N1}, \beta_k^{N2}, \beta_k^{N3}$from(30,31,34)respectively

Step6: Set $K = K + 1$ and go to Step2.

3. Convergence analysis

3.1 Introductory

In this section we are position verify global convergence of the new methods .

3.1 Hypothesis *1: see* [10].

Dia et al. demonstrated that any CG algorithm utilizing the powerful Wolf line search yields the subsequent beneficial outcome [16],[17].

3.2 Lemma

From he hypothesis 1 holds .we have for any CG method in the form (2)-(3), where $d_k$satisfies the D.C. in (26) ,where α_k is gets by the strong Wolf line search (28)-(29). Let $\varphi \in [0,4]$ be given[18]. If

$$\sum_{k\geq 1} \frac{\|g_k\|^\varphi}{\|g_k\|^2} = \infty$$

Then the following holds

$$\lim_{k\to\infty} \|g_k\| = 0$$

by then. . . if $\theta = 0 \Rightarrow \lim_{k\to\infty} \|g_k\| = 0$

3.3 Theorem

Assuming assumption 1 is valid and f is a uniformly convex function, let's take a look at the conjugate gradient method using equation (15). In this method, the values of d_k and u_k must satisfy the descent condition (26) and condition (5.9) respectively, while α_k is determined through the strong Wolfe line search algorithm. If $L = \mu$ then our method with $\rho \geq 0$ satisfies $\lim_{k\to\infty} \|g_k\| = 0$. If L > μ, then our method $0 \leq \rho < \frac{L}{3(L-\mu)}$ satisfies $\lim_{k\to\infty} \|g_k\| = 0$.

Proof: from paper [10]we have that

From $\begin{cases} z_{k-1} = y_{k-1} + \rho\left(\frac{\theta_{k-1}}{s_k^T u_{k-1}}\right) \\ \theta_{k-1} = 6(f_{k-1} - f_k) + 3(\mathrm{g}_{k-1} + \mathrm{g}_k)^T s_{k-1} \end{cases}$

and $f_{k-1} - f_k \geq g_k^T s_{k-1} + \frac{\mu}{2}\|s_{k-1}\|^2$

and $\mu\|s_{k-1}\|^2 \leq s_{k-1}^T y_{k-1} \leq L\|s_{k-1}\|^2$

$$s_{k-1}^T z_{k-1} = s_{k-1}^T y_{k-1} + \rho\theta_{k-1}$$
$$= s_{k-1}^T y_{k-1} + 6\rho(f_{k-1} - f_k) + 3\rho(g_{k-1} + g_k)^T s_{k-1}$$
$$z_{k-1} = y_{k-1} + \rho\left(\frac{\theta_{k-1}}{s_{k-1}^\pi u_{k-1}} u_{k-1}\right).$$
$$\theta_{k-1} = 6(f_{k-1} - f_k) + 3(g_{k-1} + g_k)^\top s_{k-1}$$
$$H_k z_{k-1} = s_{k-1}$$
$$s_{k-1}^\top z_{k-1} = s_{k-1}^\top y_{k-1} + \rho\theta_{k-1}$$
$$= s_{k-1}^T y_{k-1} + 6\rho(f_{k-1} - f_k) + 3\rho(g_{k-1} + g_k)^T s_{k-1}$$
$$\geq s_{k-1}^\top y_{k-1} + 6\rho\left(-g_k^\top s_{k-1} + \frac{\mu}{2}\|s_{k-1}\|^2\right) + \rho 3(g_{k-1} + g_k)^T s_{k-1}$$
$$= s_{k-1}^T y_{k-1} - 3\rho g_k^T s_{k-1} + 3\rho g_{k-1}^T s_{k-1} + 3\rho\mu\|s_{k-1}\|^2$$
$$= s_{k-1}^T y_{k-1} - 3\rho s_{k-1}^T y_{k-1} + 3\rho\mu\|s_{k-1}\|^2$$
$$= (1 - 3\rho)s_{k-1}^T y_{k-1} + 3\rho\mu\|s_{k-1}\|^2$$
$$\geq (1 - 3\rho)s_{k-1}^T y_{k-1} + \frac{3\rho\mu}{L} s_{k-1}^T y_{k-1}$$
$$= \frac{L - 3\rho(L-\mu)}{L} S_{k-1}^T y_{k-1} \qquad (38)$$

We have $0 \leq \rho < \frac{L}{3(L-\mu)}$, and we have:

$$s_{k-1}^T z_{k-1} \geq \left\{\frac{L - 3\rho(L-\mu}{L}\right\} \mu\|s_{k-1}\|^2$$
$$\|d_k\| = \|-g_k + \beta_k d_{k-1}\|$$
$$= \left\|-g_k + \frac{g_k^\top(y_k - ts_{k-1})}{d_{k-1}^\top y_{k-1}} d_{k-1}\right\|$$
$$\leq \|g_k\| + \frac{\|g_k\|(\|y_{k-1}\| + t\|s_{k-1}\|}{\left|d_{k-1}^\tau y_{k-1}\right|}\|d_{k-1}\|$$
$$\leq \|g_k\| + \frac{\|g_k\|(F\|s_{k-1}\| + t\|s_{k-1}\|}{|s_{k-1}^T z_{k-1}|}\|s_{k-1}\|$$

$$\leq \|g_k\| + \frac{(F + \mathrm{t})\|g_k\| \, \|s_{k-1}\|^2}{f_1 \|s_{k-1}\|^2}$$

$$= \left(1 + \frac{F + t}{f_1}\right) \|g_k\|$$

$$\leq \frac{(f_1 + F + t)\bar{\gamma}}{f_1}$$

$$\sum \frac{1}{\| d_1 d|^2} \geq \left\{\frac{f_1}{(f_1 + F + t)\gamma}\right\}^2 \sum_{k\geq 1} 1 = \infty$$

By Theorem 3. If $\sigma = 0 \Rightarrow \lim_{k\to\infty} \|g_k\| = 0$

4. Numerical results

This section provides the numerical results that assessed the effectiveness of the conjugate gradient algorithms employing the Fletcher-Reeves (FR), Polak-Ribiere (PR), and Liu-Storey (LS) techniques. This information is also available in reference [19].

The program's requirements for stopping $\|g_{k+1}\| \leq 10^{-5}$ and written in (FORTRAN90). Table 1 shows the number of the function (NOF) and the number of the iteration (NOI) and confirms that the new methods are superior (NI) with dimension n=1000, 10000[19],[20]

Table (1) Comparison between the new $\beta N1$, $\beta N2, \beta N3$ methods against βHS , βPR , βLS methods for the Total of 30-problems with n=1000 , 10000

P.No. Fns	N	βLS		βPR		βHS		$\beta N3$		$\beta N2$		$\beta N1$	
		NOI	NOF	NOI	NOF	NOI	NOF	NOI	NOF	NOI	NOF	NOI	NOF
1	1000	52	102	53	103	52	103	27	77	26	76	23	71
	10000	51	132	52	127	49	124	29	80	32	82	29	80
2	1000	21	92	50	112	32	91	15	72	19	80	13	70
	10000	36	71	42	73	32	73	21	59	28	61	20	59
3	1000	36	82	37	91	37	81	21	68	23	73	21	67
	10000	47	113	61	122	47	122	35	101	43	106	35	98
4	1000	51	71	65	93	63	83	31	55	45	70	38	60
	10000	53	72	53	89	72	93	30	51	39	64	44	69
5	1000	33	54	36	60	39	51	20	41	23	47	18	38
	10000	23	61	40	66	34	60	21	49	28	54	20	45
6	1000	27	45	32	47	32	47	15	29	14	30	14	30
	10000	41	51	37	52	41	52	15	30	16	32	16	32
7	1000	41	142	43	161	37	153	14	115	15	115	13	102
	10000	26	109	28	121	26	104	14	95	14	109	12	90
8	1000	24	111	27	115	23	102	12	99	15	103	11	90
	10000	28	192	31	200	29	182	16	178	19	188	15	166
9	1000	41	142	42	152	37	82	15	120	17	129	19	60
	10000	52	302	52	302	42	226	19	199	19	200	18	195
10	1000	29	111	29	133	27	107	17	99	17	121	15	95
	10000	41	162	47	233	30	182	18	185	21	198	14	163
11	1000	51	142	62	209	52	143	35	108	39	112	32	105

	10000	61	172	61	182	61	145	45	139	45	145	40	133
12	1000	53	82	52	82	51	82	24	55	25	56	23	55
	10000	41	83	51	83	44	82	23	51	33	60	20	50
13	1000	37	72	42	78	37	64	25	60	30	66	25	52
	10000	50	184	54	203	43	173	38	170	41	191	31	160
14	1000	41	111	52	122	29	92	18	83	35	90	17	80
	10000	41	103	41	103	34	102	20	77	20	77	12	70
15	1000	30	101	41	92	31	83	18	83	35	90	17	80
	10000	81	120	83	126	70	122	60	102	70	106	58	100
16	1000	27	110	26	112	25	106	13	98	14	100	11	94
	10000	31	122	26	121	30	106	19	109	14	109	16	94
Total		1317	3720	1448	3965	1288	3416	743	2937	874	3140	710	2753

Clearly, we have from the Table (2) that New1algorithm beats (HS) algorithm in about (44%) NOI; (21%) NOF, also, we have the New2 algorithm beats (PR) algorithm in about (40%) NOI,(21%) NOF then we have the New3 algorithm beats (LS) algorithm in about (45%) NOI,(19%) NOF.

Table2: percentage modified of the new algorthims

	HS algorithm	$\beta N1$	PR algorithm	$\beta N2$	LS algorithm	$\beta N3$
NI	100%	56%	100%	60%	100%	55%
NF	100%	79%	100%	79%	100%	81%

The charts below (Fig(1), Fig(2)) show the comparison of the new algorithm with similar algorithms (HS, PR, LS) based on the number of iterations and the number of function calculations, respectively.

We used More,Dolan to compare the new methods with the classical methods, based on the number of iterations and the number of function calculations.

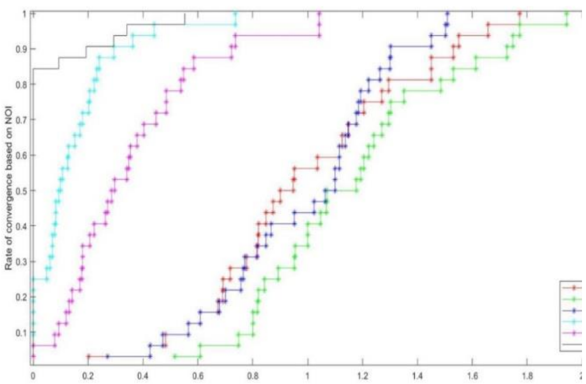

Figure 1: Number of iteration comparing bewteen New methods to standred methods

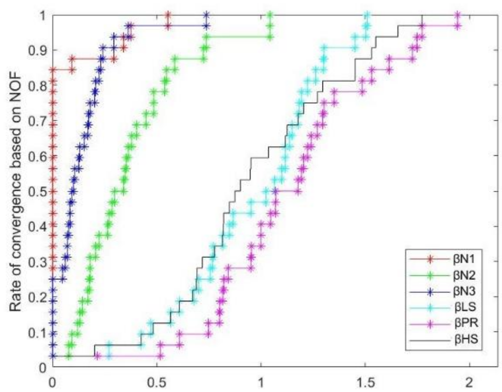

Figure 2: Number of function evaluation comparing bewteen New methods to standred method

5. Conclusion

1. The CG Methods are proposed for solving nonlinear Optimization Problems
2. Adequate decrease and worldwide convergence can be achieved under certain conditions.

   The numerical findings shown in the previously mentioned figure.

3.The new algorithms ($\beta^{N1}, \beta^{N2}, \beta^{N3}$)have prove its efficiency through results in table(1) and(2)

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Appendix

The Test Function For Unconstrained Optimization

No.	The Test Function
1	Beale
2	Trigonametric
3	Generalized Quadratic
4	Hager
5	Diagonal 1
6	Diagonal 2
7	EDENSCH
8	EDENSCHNB
9	FLETCHER
10	NONDIA
11	Extend Rosenbrock
12	Extend Powell
13	Extend Hiebert
14	Extend Wood
15	Extend Quadratic