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Some Applications via Coupled Fixed Point Theorems for (α, φ)-H-Contraction Mappings in Partial b- Metric Spaces

Kavvampalli Jyothirmayi Rani [1*], V .Naga Raju [2]

[1*] Research scholar, Department of mathematics, Osmania University, Hyderabad, Telangana, India. Mail.id : jyothirmai.ran2013@gmail.com

[2.] Professor, Department of mathematics, Osmania University, Hyderabad, Telangana, India.

Mail id : viswanag2007@gmail.com

*Corresponding Author

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Abstract:

This work establishes unique common coupled fixed point theorems for given mapping in complete partial b-metric spaces with the concept of (α, ϕ)-H-contraction in the context of partial b-metric spaces. (α, ϕ)-H-contraction

Furthermore, we show how the results may be used and present applications to integral equations and Homotopy theory.

Introduction In previous work, authors have discussed various fixed point theorems on partial b-metric spaces with (ψ, ϕ)-weakly contractive mappings, α−ψ-contractive type, Suzuki type contractions, rational contraction and H-weak contractions. In our work, with the help of (α, ϕ)-H-contraction, we investigated coupled fixed point theorems in partial b-metric spaces.

Objectives: Finding the unique common fixed points for a given mapping in partial b-metric spaces via (α, ϕ)-H-contraction

Methods with the help of α-admissible mapping, H-rational type, (α, ϕ)−H-contraction we have shown coupled fixed point findings in complete partial b-metric spaces

Results: We obtained unique common coupled fixed point results via (α, ϕ)−H-contraction type for the given mapping in complete partial b-metric spaces.

Conclusions: This present study uses contractive mappings of the H type in the reference of partial b-metric space to give some fixed point results, appropriate examples that illustrate the main findings, In addition, boundary value problems and homotopy applications are given.

Keywords: Partial b-metric space, ω-compatible, H-type rational Contraction, Coupled fixed point.

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1. Introduction

The principle of Banach contraction [1] holds significant importance in fixed point theory due to its widespread application across various mathematical and mathematical sciences fields. The concept of *b*-metric spaces was established by Czerwik ([2],[3]) in 1993, while Matthews [4] introduced the idea of partial metric spaces in the year 1994.In 2013 Shukla [5] combined the concepts of the idea of partial metric spaces and *b*-metric spaces . Mustafa [6] introduced a modified version of partial *b*-metric spaces that is dependent on *b*-metric spaces and demonstrated some common fixed point solutions for $(\psi,\ \varphi)$-weakly contractive mappings..

Samet et al. [7] (2011) proved fixed point theorems for such mappings in the complete metric spaces and introduced α- ψ-contractive type mappings to obtain a very general structure that combines several existing fixed point theorems .They also developed the concepts of α-contractive and α-admissible mappings. Karapinar and Samet [8] enhanced the findings in [7] by creating the idea of generalized α-ψ-contractive type mappings. In particular the theory proved by Jaggi [9] in 1977 meets a contractive criterion of rational type .A rational type contraction is a novel contractive condition that was created by Dass and Gupta [10] in 1975.

In 1987 Guo and Lakshmikantham [11]first introduced the idea of coupled fixed point Later employing a weak contractivity type assumption. Bhaskar and Lakshmikantham [12] created a novel fixed point theory for a mixed monotone mapping in a metric space driven by partial ordering. Refer to relevant references and study results in ([13]-[20]) for additional details on coupled fixed point outcomes.

This work proves common coupled fixed point theorem for two mappings satisfying (α,φ) - H-type contractive constraints in the partial b-metric space. We also examine at numerous boundary value problems and homotopy applications, withexamples.

In partial b-metric space, this work establishes a common coupled fixed point theorem for two mappings meeting (α,φ) - H-type contractive constraints. Along with examples, we also look at number of boundary value issues and homotopy applications.

Objectives

Finding the unique common fixed points for a given mapping in partial b-metric spaces via (α, ϕ)-H-contraction

Methods

With the help of α-admissible mapping, H-rational type, (α, ϕ)−H-contraction we have shown coupled fixed point findings in complete partial b-metric spaces

2. Preliminaries:

Definition 2.1. ([5]) Let $\mathsf{v}\geq 1$ be a given real number and $\Im$ be a nonempty set. A partial b-metric is defined as a function $\varsigma_b:\Im\times\Im\to[0,\infty)$ if the following criteria are met for each $æ_1,æ_2,æ_3\in\Im$.

$(\varsigma_b 1) æ_1 = æ_2$ *if and only* $\varsigma_b(æ_1,æ_1)=\varsigma_b(æ_1,æ_2)=\varsigma_b(æ_2,æ_2)$

$(\varsigma_b 2)$ $\varsigma_b(æ_1,æ_1)\leq\varsigma_b(æ_1,æ_2)$

$(\varsigma_b 3)$ $\varsigma_b(æ_1,æ_2)=\varsigma_b(æ_2,æ_1)$

$(\varsigma_b 4)\varsigma_b(æ_1,æ_2)\leq\mathsf{v}(\varsigma_b(æ_1,æ_3)+\varsigma_b(æ_3,æ_2)-\varsigma_b(æ_3,æ_3))$.

The pair $(\Im,\varsigma_b)$ is called a partial b−metric space.

Remark 2.2. The class of partial b-metric spaces $(\Im,\varsigma_b)$ is more extensive than the class of partial metric spaces because a partial metric space is a specific instance of a partial b-metric space $(\Im,\varsigma_b)$ when d = 1. In addition, the class of partial b- metric spaces, denoted as $(\Im,\varsigma_b)$, is larger than the class of b-metric spaces because a b-metric space is a specific case of a partial b-metric space $(\Im,\varsigma_b)$ where the self- distance $P(æ_1,æ_1)$ is equal to 0. The examples shown demonstrate that both a b-metric on $\Im$ and a partial b-metric on $\Im$ do not necessarily have to satisfy the conditions stated in ([5]-[6]).

Example 2.3. ([5])Assume that $\Im = [0,1)$. $\varsigma_b(z_1;z_2) = [\max\{z_1, z_2\}]^2 + |z_1 - z_2|^2$, is the formula to create a function. ς_b. For every $z_1, z_2 \in \Im$. the pair $(\Im,\varsigma_b)$ is called a partial b-metric space when $v = 2 > 1$. However, ς_b is neither a b- metric nor a partial metric on $\Im$.

Definition 2.4. ([6])Every partial b-metric ς_b defines a b-metric d_{ς_b}, where

$$d_{\varsigma_b}(z_1,z_2) = 2\varsigma_b(z_1,z_2) - \varsigma_b(z_1,z_1) - \varsigma_b(z_2,z_2), \quad \textit{for all} \quad z_1,z_2 \in \Im$$

Definition 2.5. ([6])In a partial b-metric space $(\Im,\varsigma_b)$, a sequence $\{æ_p\}$ is defined as follows

(i) The $\varsigma_b - \textit{convergent toward a target if } \lim_{p\to\infty}\varsigma_b(æ,æ_p) = \varsigma_b(æ,æ) \textit{ then } æ \in \Im$

(ii) In ς_b if $\lim_{p,q\to\infty}\varsigma_b(æ_p,æ_q)$ exists and is finite, then ς_b - Cauchy sequence

(iii) A $(\Im,\varsigma_b)$ partial b-metric space ς_b is said to be ς_b -complete if and only if, for each ς_b - Cauchy sequence $\{æ_p\}$ in $\Im$, converges to a point $æ \in \Im$ such that

$$\lim_{p,q\to\infty}\varsigma_b(æ_p,æ_q) = \lim_{p\to\infty}\varsigma_b(æ,æ_p) = \varsigma_b(æ,æ)$$

Lemma 2.6. ([6]) A sequence $\{æ_n\}$ is a ς_b-Cauchy sequence in a partial b-metric space $(\Im, \varsigma_b)$, if and only if it is a ς_b-Cauchy sequence in the b-metric space $(\Im, d_{\varsigma_b})$.

Lemma 2.7. ([6]) If and only if the b-metric space $(\Im, d_{\varsigma_b})$ is ς_b -complete, a partial

b-metric space $(\Im,\varsigma_b)$ qualifies as ς_b-complete. Additionally,

$$\lim_{p,q\to\infty} d_{\varsigma_b}(æ_p,æ_q) = 0 \quad \Leftrightarrow \lim_{p\to\infty}\varsigma_b(æ_p,æ) = \lim_{q\to\infty}\varsigma_b(æ_q,æ) = \varsigma_b(æ,æ).$$

Definition 2.8. ([12]) Let a nonempty set be $\Im$. If $æ = S(æ,œ)$ *and* $œ = S(œ,æ)$, Then An element $(æ,œ) \in \Im \times \Im$ is referred to as a coupled fixed point of the mapping $S: \Im\times\Im \to \Im$.

Definition 2.9. ([13])Suppose $S: \Im^2 \to \Im$ and $f: \Im \to \Im$ are two mappings. A point $(æ,œ)$ is a connected coincident point of S and f if

$$S(æ,œ) = fæ, S(œ,æ) = fœ.$$

Definition 2.10. ([13])Suppose $S:\Im^2 \to \Im$ and $f:\Im \to \Im$ are two mappings. A point $(æ, œ)$ is a coupled common point of S and f if

$S(æ, œ) = fæ = æ, S(œ, æ) = fœ = œ.$

Definition 2.11. ([13]) Let $(\Im, \varsigma_b)$ denote a partial b-metric space. Weakly compatible pairs are those where $f(S(æ, œ)) = S(fæ, fœ)$ whenever for all $æ, œ \in \Im$ such that

$S(æ, œ) = fæ, S(œ, æ) = fœ.$

Definition 2.12. ([7]) Consider $S:\Im^2 \to \Im$ and $\alpha:\Im^2 \to R^+$. If $y_1, y_2 \in \Im$, then S is α-admissible.

$\alpha(y_1, y_2) \geq 1$ *implies* $\alpha(S(y_1, y_2), S(y_2, y_1)) \geq 1$

Definition 2.13. ([7])Suppose $S:\Im^2 \to \Im, f:\Im \to \Im$ and $\alpha:\Im^2 \to R^+$ are mappings. If $y_1, y_2 \in \Im$, then S and f are α-admissible.

$\alpha(fy_1, fy_2) \geq 1$ *implies* $\alpha(S(y_1, y_2), S(y_2, y_1)) \geq 1$

Definition 2.14. ([19], [20]) A rational type contraction $S:\Im \to \Im$ in the complete metric space $(\Im, d)$ is referred to as H-rational type, If $0 \leq \delta + \beta + 2\gamma < 1$ for every $y_1, y_2 \in \Im$ then the following inequality holds

$$d(Sy_1, Sy_2) \leq \delta d(y, y_2) + \beta \frac{d(y_2, Sy_2)[1 + d(y_1, Sy_1)]}{1 + d(y_1, y_2)} + \gamma(d(y_1, Sy_1) + d(y_2, Sy_2))$$

Let Δ be a family of functions $\varphi:[0,\infty) \to [0,\infty)$ that meet the following requirements.

a) φ is non-decreasing;

b) $\varphi(s) < s \ \forall \ s > 0$ and $\varphi(s) = 0$ iff $s = 0$

We now prove our primary result.

3. Main Results & Discussion

Definition3.1. Consider $(\Im, \varsigma_b)$ as a partial b-metric space with coefficient $v \geq 1$ and $\alpha:\Im^2 \to R^+$. *Let* $S:\Im^2 \to \Im, f:\Im \to \Im$ be two mappings. If $0 \leq \delta + \lambda + 2\theta < 1$ and $\varphi \in \Delta$, for every $y_1, y_2, æ_1, æ_2 \in \Im$; then (α, φ) - H- contraction if

$$\alpha(fy_1, fæ_1)\varsigma_b(S(y_1, y_2), S(æ_1, æ_2)) \leq \delta\varphi\left(\max\begin{Bmatrix}\varsigma_b(fy_1, fæ_1), \\ \varsigma_b(fy_2, fæ_2)\end{Bmatrix}\right)$$

$$+\lambda\varphi\left(\max\left\{\begin{array}{c}\frac{\varsigma_b(fæ_1, S(æ_1,æ_2))[1+\varsigma_b(fy_1, S(y_1,y_2))]}{1+\varsigma_b(f\eta_1, fæ_1)},\\ \frac{\varsigma_b(fæ_2, S(æ_2,æ_1))[1+\varsigma_b(fy_2, S(y_2,y_1))]}{1+\varsigma_b(fy_2, fæ_2)}\end{array}\right\}\right)$$

$$+\theta\varphi\left(\max\left\{\begin{array}{c}\varsigma_b(fy_1, S(y_1,y_2)),\\ \varsigma_b(fy_2, S(y_2,y_1))\end{array}\right\}+\max\left\{\begin{array}{c}\varsigma_b(fæ_1, S(æ_1,æ_2)),\\ \varsigma_b(fæ_2, S(æ_2,æ_1))\end{array}\right\}\right) \tag{3.1}$$

Theorem 3.2. Consider $(\Im,\varsigma_b)$ be a Partial b- metric space with the coefficient $v\geq 1$ and $S:\Im^2\to\Im$ *and* $f:\Im\to\Im$ be two mappings satisfying $(\alpha,\ \varphi)-$H-contraction. Assume

(3.2.1) $S(\Im^2)\subseteq f(\Im)$ *and* $f(\Im)$ *is complete subspace of* $\Im$

(3.2.2) S and f are $\alpha-$admissible mappings

(3.2.3) $\exists\ y_0, æ_0\in\Im \ni \alpha(S(y_0,æ_0), S(fæ_0, fy_0))\geq 1,$

(3.2.4) (S,f) is weakly compatible pair.

Then S and f have a UCCFP (unique common coupled fixed point) in $\Im$.

Proof. Let $y_0, æ_0$ be arbitrary points in $\Im$. From (3.2.1), there exist sequences

$\{y_z\},\{æ_z\},\{œ_z\},\{B_z\}$ *in* $\Im$ *suchthat , for all* $z\geq 0$

$S(y_z,æ_z)=fy_{z+1}=œ_z \qquad S(æ_z,y_z)=fæ_{z+1}=B_z$

Case (i): If for some z_0, we have

$œ_{z_0}=œ_{z_{0+1}}=S(y_{z_0},æ_{z_0})=fy_{z_{0+1}} \qquad B_{z_0}=B_{z_{0+1}}=S(æ_{z_0},\eta_{z_0})=fæ_{z_{0+1}}$

then $(œ_{z_0},B_{z_0})$ is common coupled fixed point of S and f

Case (ii): Suppose that $œ_z\neq œ_{z+1}$ and $B_z\neq B_{z+1}$ for all $z\geq 0$.

Since S and f are α-admissible, we have

$\alpha(fy_0,fy_1)\geq 1\Rightarrow \alpha(S(y_0,æ_0),S(y_1,æ_1))=\alpha(fy_1,fy_2)\geq 1$

Recursively, we find that $\alpha(f\eta_z, f\eta_{z+1})\geq 1$, for all $z=0,1,...$

From (3.1), (3.2.2) and (3.2.3), we have that

$\varsigma_b(œ_z,œ_{z+1})=\varsigma_b(S(y_z,æ_z),S(y_{z+1},æ_{z+1}))$

$\quad\leq\alpha(fy_z,fy_{z+1})\varsigma_b(S(y_z,æ_z),S(y_{z+1},æ_{z+1}))$

$$\leq \delta\varphi\left(\max\begin{Bmatrix}\varsigma_b(\mathrm{f}y_z,\mathrm{f}y_{z+1}),\\ \varsigma_b(\mathrm{f}æ_z,\mathrm{f}æ_{z+1})\end{Bmatrix}\right)$$

$$+\lambda\varphi\left(\max\begin{Bmatrix}\dfrac{\varsigma_b(\mathrm{f}y_{z+1},S(y_{z+1},æ_{z+1}))[1+\varsigma_b(\mathrm{f}y_z,S(y_z,æ_z))]}{1+\varsigma_b(\mathrm{f}\eta_z,f\eta_{z+1})},\\ \dfrac{\varsigma_b(\mathrm{f}æ_{z+1},S(æ_{z+1},y_{z+1}))[1+\varsigma_b(\mathrm{f}æ_z,S(æ_z,y_z))]}{1+\varsigma_b(\mathrm{f}æ_z,\mathrm{f}æ_{z+1})}\end{Bmatrix}\right)$$

$$+\theta\varphi\left(\max\begin{Bmatrix}\varsigma_b(\mathrm{f}y_z,S(y_z,æ_z)),\\ \varsigma_b(\mathrm{f}æ_z,S(æ_z,y_z))\end{Bmatrix}+\max\begin{Bmatrix}\varsigma_b(\mathrm{f}y_{z+1},S(y_{z+1},æ_{z+1})),\\ \varsigma_b(\mathrm{f}æ_{z+1},S(æ_{z+1},y_{z+1}))\end{Bmatrix}\right)$$

$$\leq \delta\varphi\left(\left(\max\begin{Bmatrix}\varsigma_b(œ_{z-1},œ_z),\\ \varsigma_b(B_{z-1},B_z)\end{Bmatrix}\right)+\lambda\varphi\left(\max\begin{Bmatrix}\dfrac{\varsigma_b(œ_z,œ_{z+1})[1+\varsigma_b(œ_{z-1},œ_z)]}{1+\varsigma_b(œ_{z-1},œ_z)},\\ \dfrac{\varsigma_b(B_z,B_{z+1})[1+\varsigma_b(B_{z-1},B_z)]}{1+\varsigma_b(B_{z-1},B_z)}\end{Bmatrix}\right)\right)$$

$$+\theta\varphi\left(\max\begin{Bmatrix}\varsigma_b(œ_{z-1},œ_z),\\ \varsigma_b(B_{z-1},B_z)\end{Bmatrix}\right)+\left(\max\begin{Bmatrix}\varsigma_b(œ_z,œ_{z+1}),\\ \varsigma_b(B_z,B_{z+1})\end{Bmatrix}\right)$$

Since $\varphi(s)\leq s$ for all $s\geq 0$, then we obtain

$$\varsigma_b(œ_z,œ_{z+1})\leq \delta\max\begin{Bmatrix}\varsigma_b(œ_{z-1},œ_z),\\ \varsigma_b(B_{z-1},B_z)\end{Bmatrix}+\lambda\max\begin{Bmatrix}\varsigma_b(œ_z,œ_{z+1}),\\ \varsigma_b(B_z,B_{z+1})\end{Bmatrix}$$

$$+\theta\left(\max\begin{Bmatrix}\varsigma_b(œ_{z-1},œ_z),\\ \varsigma_b(B_{z-1},B_z)\end{Bmatrix}+\max\begin{Bmatrix}\varsigma_b(œ_z,œ_{z+1}),\\ \varsigma_b(B_z,B_{z+1})\end{Bmatrix}\right) \tag{3.2}$$

Similarly, we can prove that

$$\varsigma_b(B_z,B_{z+1})\leq \delta\max\begin{Bmatrix}\varsigma_b(œ_{z-1},œ_z),\\ \varsigma_b(B_{z-1},B_z)\end{Bmatrix}+\lambda\max\begin{Bmatrix}\varsigma_b(œ_z,œ_{z+1}),\\ \varsigma_b(B_z,B_{z+1})\end{Bmatrix}$$

$$+\theta\left(\max\begin{Bmatrix}\varsigma_b(œ_{z-1},œ_z),\\ \varsigma_b(B_{z-1},B_z)\end{Bmatrix}+\max\begin{Bmatrix}\varsigma_b(œ_z,œ_{z+1}),\\ \varsigma_b(B_z,B_{z+1})\end{Bmatrix}\right) \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$\max\begin{Bmatrix}\varsigma_b(œ_z,œ_{z+1}),\\ \varsigma_b(B_z,B_{z+1})\end{Bmatrix}\leq \delta\max\begin{Bmatrix}\varsigma_b(œ_{z-1},œ_z),\\ \varsigma_b(B_{z-1},B_z)\end{Bmatrix}+\lambda\max\begin{Bmatrix}\varsigma_b(œ_z,œ_{z+1}),\\ \varsigma_b(B_z,B_{z+1})\end{Bmatrix}$$

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$$+\theta\left(\max\left\{\begin{matrix}\varsigma_b(œ_{z-1},œ_z),\\ \varsigma_b(B_{z-1},B_z)\end{matrix}\right\}+\max\left\{\begin{matrix}\varsigma_b(œ_z,œ_{z+1}),\\ \varsigma_b(B_z,B_{z+1})\end{matrix}\right\}\right)$$

Implies that

$$\max\left\{\begin{matrix}\varsigma_b(œ_z,œ_{z+1}),\\ \varsigma_b(B_z,B_{z+1})\end{matrix}\right\}\le\frac{\delta+\theta}{1-\lambda-\theta}\max\left\{\begin{matrix}\varsigma_b(œ_{z-1},œ_z),\\ \varsigma_b(B_{z-1},B_z)\end{matrix}\right\}$$

$$\le\left(\frac{\delta+\theta}{1-\lambda-\theta}\right)^2\max\left\{\begin{matrix}\varsigma_b(œ_{z-2},œ_{z-1}),\\ \varsigma_b(B_{z-2},B_{z-1})\end{matrix}\right\}$$

$$\vdots$$

$$\le\left(\frac{\delta+\theta}{1-\lambda-\theta}\right)^n\max\left\{\begin{matrix}\varsigma_b(u_0,u_1),\\ \varsigma_b(B_0,\ B_1)\end{matrix}\right\}\to 0 \text{ as } n\to\infty$$

It follows that

$$\lim_{z\to\infty}\varsigma_b(œ_z,œ_{z+1})=\lim_{z\to\infty}\varsigma_b(B_z,B_{z+1})=0. \tag{3.4}$$

From (3.4) and ($\varsigma_b 2$), we have that

$$\lim_{z\to\infty}\varsigma_b(œ_z,œ_z)=\lim_{z\to\infty}\varsigma_b(B_z,B_z)=0. \tag{3.5}$$

From definition of d_{ς_b}, (3.4) and (3.5), we have that

$$\lim_{z\to\infty}d_b(œ_z,œ_{z+1})=\lim_{z\to\infty}d_b(B_z,B_{z+1})=0. \tag{3.6}$$

To show that $\{œ_z\}$ and $\{B_z\}$ are Cauchy sequences, we proceed by using triangleproperty.

$$\varsigma_b(œ_z,œ_w)\le v(\varsigma_b(œ_z,œ_{z+1})+\varsigma_b(œ_{z+1},œ_w))-\varsigma_b(œ_{z+1},œ_{z+1})$$

$$\le v\varsigma_b(œ_z,œ_{z+1})+v\varsigma_b(œ_{z+1},œ_w)$$

$$\le v\varsigma_b(œ_z,œ_{z+1})+v(v(\varsigma_b(œ_{z+1},œ_{z+2})+\varsigma_b(œ_{z+2},œ_w))-\varsigma_b(œ_{z+2},œ_{z+2}))$$

$$\le v\varsigma_b(œ_z,œ_{z+1})+v^2\varsigma_b(œ_{z+1},œ_{z+2})+v^2\varsigma_b(œ_{z+2},œ_w)$$

$$\le v\varsigma_b(œ_z,œ_{z+1})+v^2\varsigma_b(œ_{z+1},œ_{z+2})+v^3\varsigma_b(œ_{z+2},œ_{z+3})+\ldots+v^{w-z}\varsigma_b(œ_{w-1},œ_w)$$

$$\le v\left(\frac{\delta+\theta}{1-\lambda-\theta}\right)^z\max\left\{\begin{matrix}\varsigma_b(œ_0,œ_1),\\ \varsigma_b(B_0,B_1)\end{matrix}\right\}+v^2\left(\frac{\delta+\theta}{1-\lambda-\theta}\right)^{z+1}\max\left\{\begin{matrix}\varsigma_b(œ_0,œ_1),\\ \varsigma_b(B_0,B_1)\end{matrix}\right\}+$$

$$v^3\left(\frac{\delta+\theta}{1-\lambda-\theta}\right)^{z+2}\max\left\{\begin{matrix}\varsigma_b(œ_0,œ_1),\\ \varsigma_b(B_0,B_1)\end{matrix}\right\}+\ldots+$$

$$+v^{w-z}\left(\frac{\delta+\theta}{1-\lambda-\theta}\right)^{w-1}\max\left\{\begin{matrix}\varsigma_b(œ_0,œ_1),\\ \varsigma_b(B_0,B_1)\end{matrix}\right\}$$

$$\leq \mathrm{v}\left(\frac{\delta+\theta}{1-\lambda-\theta}\right)^{z}\left(1+\mathrm{v}\left(\frac{\delta+\theta}{1-\lambda-\theta}\right)+\mathrm{v}^{2}\left(\frac{\delta+\theta}{1-\lambda-\theta}\right)^{2}+\ldots\right)\max\left\{\begin{array}{c}\varsigma_{b}(œ_{0},œ_{1}),\\ \varsigma_{b}(\mathrm{B}_{0},\mathrm{B}_{1})\end{array}\right\}$$

$$\leq \frac{\mathrm{v}\left(\frac{\delta+\theta}{1-\lambda-\theta}\right)^{z}}{1-\mathrm{v}\left(\frac{\delta+\theta}{1-\lambda-\theta}\right)}\max\left\{\begin{array}{c}\varsigma_{b}(œ_{0},œ_{1}),\\ \varsigma_{b}(\mathrm{B}_{0},\mathrm{B}_{1})\end{array}\right\}\to 0 \ as\ z\to\infty$$

Since $0\leq\frac{\delta+\theta}{1-\lambda-\theta}<1$, that is $\lim_{z,w\to\infty}\varsigma_{b}(œ_{z},œ_{w})=0.$ Thus, $\{œ_{z}\}$ is a Cauchy sequence in $(\Im,\varsigma_{b})$.
From definition of d_{ς_b}, and Eq.(3.5), we have that

$$.\ \lim_{z,w\to\infty} d_{\varsigma_b}(œ_{z},œ_{w}) = 2\lim_{z,w\to\infty}\varsigma_{b}(œ_{z},œ_{w}) = 0$$

Therefore, $\{œ_{z}\}$ is a Cauchy sequence in $(\Im,\varsigma_{b})$. similarly, we can show that $\{\mathrm{B}_{z}\}$ is a Cauchy sequence in $(\Im, d_{\varsigma_b})$.

Suppose $\mathrm{f}(\Im)$ is complete subspace of $\Im$. Then $\{œ_{z}\}$ and $\{\mathrm{B}_{z}\}$ converges to $\mathbf{ö},\square$ *in* $(\mathrm{f}(\Im), d_{\varsigma_b})$, thus there exist $\mathrm{y},æ\in\mathrm{f}(\Im)$ such that

$$\lim_{z\to\infty}œ_{z}=\mathbf{ö}=\mathrm{fy} \text{ and } \lim_{z\to\infty}\mathrm{B}_{z}=\square=\mathrm{f}æ$$

That is $\lim_{z\to\infty} d_{\varsigma_b}(\mathrm{fy}_{z+1},\mathbf{ö})=0, \lim_{z\to\infty} d_{\varsigma_b}(\mathrm{f}æ_{z+1},\square)=0$ for some $\mathbf{ö}=\mathrm{fy},\square=\mathrm{f}æ$,

we have that

$$\varsigma_{b}(\mathbf{ö},\mathbf{ö})=\lim_{z,w\to\infty}\varsigma_{b}(\mathrm{fy}_{z},\mathrm{fy}_{w})=\lim_{z\to\infty}\varsigma_{b}(\mathrm{fy}_{z},\mathbf{ö})=\lim_{z\to\infty}\varsigma_{b}(\mathrm{fy}_{z+1},\mathbf{ö})=0. \qquad (3.7)$$

And

$$\varsigma_{b}(\square,\square)=\lim_{z,w\to\infty}\varsigma_{b}(\mathrm{f}æ_{z},\mathrm{f}æ_{w})=\lim_{z\to\infty}\varsigma_{b}(\mathrm{f}æ_{z},\square)=\lim_{z\to\infty}\varsigma_{b}(\mathrm{f}æ_{z+1},\square)=0. \qquad (3.8)$$

Assume that (S,f) is α-admissible mapping. Therefore, there is a sub sequence

$\{œ_{zk}\}$ and $\{\mathrm{B}_{z_k}\}$ of $\{œ_{z}\}$ and $\{\mathrm{B}_{z}\}$ respectively such that $\alpha(œ_{z},œ_{z+1})\geq 1$ and $\alpha(\mathrm{B}_{z},\mathrm{B}_{z+1})\geq 1$ for all $z\in N$ then $\alpha(œ_{z_k},\mathrm{fy})\geq 1$ and $\alpha(\mathrm{B}_{z_k},\mathrm{f}æ)\geq 1.$ Now we claim that $\mathrm{S}(\mathrm{y},æ)=\mathbf{ö}$ and $\mathrm{S}(æ,\mathrm{y})=\square$

From (3.1), we have

$$\varsigma_{b}(\mathrm{S}(\mathrm{y},æ),œ_{z})=\varsigma_{b}(\mathrm{S}(\mathrm{y},æ),\mathrm{S}(\mathrm{y}_{z},æ_{z}))$$

$$\leq \alpha(\mathrm{fy},\mathrm{fy}_{z})\varsigma_{b}(\mathrm{S}(\mathrm{y},æ),\mathrm{S}(\mathrm{y}_{Z},æ_{Z}))$$

$$\leq \delta\varphi\left(\max\begin{Bmatrix}\varsigma_b(\mathrm{fy},\mathrm{fy}_z),\\ \varsigma_b(\mathrm{fæ},\mathrm{fæ}_z)\end{Bmatrix}\right)+\lambda\varphi\left(\max\begin{Bmatrix}\dfrac{\varsigma_b(\mathrm{fy}_Z,S(y_Z,æ_Z))[1+\varsigma_b(\mathrm{fy},S(y,æ))]}{1+\varsigma_b(\mathrm{fy},\mathrm{fy}_z)},\\ \dfrac{\varsigma_b(\mathrm{fæ}_z,S(æ_z,y_z)[1+\varsigma_b(\mathrm{fæ},S(æ,y))]}{1+\varsigma_b(\mathrm{fæ},\mathrm{fæ}_z)}\end{Bmatrix}\right)$$

$$+\theta\varphi\left(\max\begin{Bmatrix}\varsigma_b(\mathrm{fy},S(y,æ)),\\ \varsigma_b(f æ,S(æ,y))\end{Bmatrix}+\max\begin{Bmatrix}\varsigma_b(\mathrm{fy}_Z,S(y_Z,æ_Z)),\\ \varsigma_b(\mathrm{fæ}_Z,S(æ_z,y_z))\end{Bmatrix}\right)$$

$$\leq \delta\varphi\left(\max\begin{Bmatrix}\varsigma_b(\mathbf{ö},\mathrm{fy}_Z),\\ \varsigma_b(\square,\mathrm{fæ}_Z)\end{Bmatrix}\right)+\lambda\varphi\left(\max\begin{Bmatrix}\dfrac{\varsigma_b(œ_{z-1},œ_z)[1+\varsigma_b(\mathbf{ö},S(y,æ))]}{1+\varsigma_b(\mathbf{ö},\mathrm{fy}_Z)}\\ \dfrac{\varsigma_b(B_{z-1},B_z)[1+\varsigma_b(\square,S(æ,y))]}{1+\varsigma_b(\square,\mathrm{fæ}_z)}\end{Bmatrix}\right)$$

$$+\theta\varphi\left(\max\begin{Bmatrix}\varsigma_b(\mathbf{ö},S(y,æ),\\ \varsigma_b(\square,S(æ,y))\end{Bmatrix}+\max\begin{Bmatrix}\varsigma_b(œ_{z-1},œ_z),\\ \varsigma_b(B_{z-1},B_z)\end{Bmatrix}\right)$$

Letting $z\to\infty$ in the previous inequality, we get that

$$\varsigma_b(S(y,æ),\mathbf{ö})\leq\ \theta\varphi\left(\max\begin{Bmatrix}\varsigma_b(\mathbf{ö},S(y,æ)),\\ \varsigma_b(\square,S(æ,y))\end{Bmatrix}\right)$$

$$\leq\theta\max\begin{Bmatrix}\varsigma_b(\mathbf{ö},S(y,æ)),\\ \varsigma_b(\square,S(æ,y)),\end{Bmatrix}$$

Similarly, we can prove

$$\varsigma_b(S(æ,y),\square)\ \leq\ \theta\max\begin{Bmatrix}\varsigma_b(\mathbf{ö},S(y,æ)),\\ \varsigma_b(\square,S(æ,y))\end{Bmatrix}$$

Therefore,

$$\max\begin{Bmatrix}\varsigma_b(S(y,æ),\mathbf{ö}),\\ \varsigma_b(S(æ,y),\square)\end{Bmatrix}\ \leq\ \theta\max\begin{Bmatrix}\varsigma_b(\mathbf{ö},S(y,æ)),\\ \varsigma_b(\square,S(æ,y))\end{Bmatrix}$$

$S(y,æ)=\mathbf{ö}=\mathrm{fy}$ and $S(æ,y)=\square=\mathrm{fæ}$ follow as $\theta<1$.Therefore a coupled coincidence point of S and f is $(y,æ)$. As a weakly compatible pair (S,f),

we have

$$\mathrm{f}\mathbf{ö}=f^2y=f(S(y,æ))=S(\mathrm{fy},\mathrm{fæ})=S(\mathbf{ö},\square)$$

$$\mathrm{f}\square=f^2æ=f((æ,y))=S(\mathrm{fæ},\mathrm{fy})=S(\square,\mathbf{ö})$$

We now establish that $\mathrm{f}\square=\square$ and $\mathrm{f}\mathbf{ö}=\mathbf{ö}$ we can see from (3.1) that

$$\varsigma_b(\mathrm{f}\partial,œ_z)=\varsigma_b(S(\partial,\square),S(y_z,æ_z))$$

$$\leq \alpha(f\partial, fy_z)\varsigma_b(S(\partial,\square), S(y_Z, æ_Z))$$

$$\leq \delta\varphi\left(\max\begin{Bmatrix}\varsigma_b(f\partial, fy_z),\\ \varsigma_b(f\square, fæ_z)\end{Bmatrix}\right) + \lambda\varphi\left(\max\begin{Bmatrix}\dfrac{\varsigma_b(fy_z, S(y_z, æ_z)[1+\varsigma_b(f\partial, S(\partial,\square))]}{1+\varsigma_b(f\partial, fy_z)},\\ \dfrac{\varsigma_b(fæ_z, S(æ_z, y_z)[1+\varsigma_b(f\square, S(\square,\partial))]}{1+\varsigma_b(f\square, fæ_z)}\end{Bmatrix}\right)$$

$$+\theta\varphi\left(\max\begin{Bmatrix}\varsigma_b(f\partial, S(\partial,\square)),\\ \varsigma_b(f\square, S(\square,\partial))\end{Bmatrix} + \max\begin{Bmatrix}\varsigma_b(fy_z, S(y_z, æ_z)),\\ \varsigma_b(fæ_z, S(æ_z, y_z))\end{Bmatrix}\right)$$

$$\leq \delta\varphi\left(\max\begin{Bmatrix}\varsigma_b(f\partial, œ_{z-1}),\\ \varsigma_b(f\square, B_{z-1})\end{Bmatrix}\right) + \lambda\varphi\left(\max\begin{Bmatrix}\dfrac{\varsigma_b(œ_{z-1}, œ_z)[1+\varsigma_b(f\partial, S(\partial,\square))]}{1+\varsigma_b(f\partial, œ_{z-1})}\\ \dfrac{\varsigma_b(B_{z-1}, B_z)[1+\varsigma_b(f\square, S(\square,\partial))]}{1+\varsigma_b(f\square, B_{z-1})}\end{Bmatrix}\right)$$

$$+\theta\varphi\left(\max\begin{Bmatrix}\varsigma_b(f\partial, S(\partial,\square)),\\ \varsigma_b(f\square, S(\square,\partial)\end{Bmatrix} + \max\begin{Bmatrix}\varsigma_b(œ_{z-1}, œ_z),\\ \varsigma_b(B_{z-1}, B_z)\end{Bmatrix}\right)$$

In the inequality above, letting $z \to \infty$, we have that

$$\varsigma_b(f\partial,\partial) \leq \delta\phi\left(\max\begin{Bmatrix}\varsigma_b(f\partial,\partial),\\ \varsigma_b(f\square,\square)\end{Bmatrix}\right)$$

$$\leq \delta\max\begin{Bmatrix}\varsigma_b(f\partial,\partial),\\ \varsigma_b(f\square,\square)\end{Bmatrix}$$

Similarly, we can prove that

$$\varsigma_b(f\square,\square) \leq \delta\ \max\begin{Bmatrix}\varsigma_b(f\partial,\partial),\\ \varsigma_b(f\square,\square)\end{Bmatrix}$$

Therefore,

$$\max\begin{Bmatrix}\varsigma_b(f\partial,\partial),\\ \varsigma_b(f\square,\square)\end{Bmatrix} \leq \delta\max\begin{Bmatrix}\varsigma_b(f\partial,\partial),\\ \varsigma_b(f\square,\square)\end{Bmatrix}$$

As $\delta < 1$. It follows that $S(\partial,\square) = \partial = f\partial$ and $S(\square,\partial) = \square = f\square$. Therefore $(\partial,\square)$ is CCFP (common coupled fixed point) of S and f for uniqueness let us suppose $(\partial^*,\square^*)$ be another CCFP of S and f such that $\partial \neq \partial^*$, and $\square \neq \square^*$ Now from (3.1), we have that

$$\varsigma_b(\partial,\partial^*) = \varsigma_b(S(\partial,\square), S(\partial^*,\square^*))$$

$$\leq \alpha(f\partial, f\partial^*)\varsigma_b(S(\partial,\square), S(\partial^*,\square^*))$$

$$\leq \delta\varphi\left(\max\begin{Bmatrix}\varsigma_b(\mathsf{f}\partial,\mathsf{f}\partial^*),\\ \varsigma_b(\mathsf{f}\square,\mathsf{f}\square^*)\end{Bmatrix}\right)+\lambda\varphi\left(\max\begin{Bmatrix}\dfrac{\varsigma_b(\mathsf{f}\partial^*,\mathsf{S}(\partial^*,\square^*)[1+\varsigma_b(\mathsf{f}\partial,\mathsf{S}(\partial,\square))]}{1+\varsigma_b(\mathsf{f}\partial,\mathsf{f}\partial^*)},\\ \dfrac{\varsigma_b(\mathsf{f}\square^*,\mathsf{S}(\square^*,\partial^*)[1+\varsigma_b(\mathsf{f}\square,\mathsf{S}(\square,\partial))]}{1+\varsigma_b(\mathsf{f}\square,\mathsf{f}\square^*)}\end{Bmatrix}\right)$$

$$+\theta\varphi\left(\max\begin{Bmatrix}\varsigma_b(\mathsf{f}\partial,\mathsf{S}(\partial,\square)),\\ \varsigma_b(\mathsf{f}\square,\mathsf{S}(\square,\partial))\end{Bmatrix}+\max\begin{Bmatrix}\varsigma_b((\mathsf{f}\partial^*,\mathsf{S}(\partial^*,\square^*)),\\ \varsigma_b((\mathsf{f}\square^*,\mathsf{S}(\square^*,\partial^*))\end{Bmatrix}\right)$$

$$\leq \delta\varphi\left(\max\begin{Bmatrix}\varsigma_b(\partial,\partial^*),\\ \varsigma_b(\square,\square^*)\end{Bmatrix}\right)+\lambda\varphi\left(\max\begin{Bmatrix}\varsigma_b(\partial^*,\partial^*),\\ \varsigma_b(\square^*,\square^*)\end{Bmatrix}\right)$$

$$+\theta\varphi\left(\max\begin{Bmatrix}\varsigma_b(\partial,\partial),\\ \varsigma_b(\square,\square)\end{Bmatrix}+\max\begin{Bmatrix}\varsigma_b(\partial^*,\partial^*),\\ \varsigma_b(\square^*,\square^*)\end{Bmatrix}\right)$$

$$\leq \delta\varphi\left(\max\begin{Bmatrix}\varsigma_b(\partial,\partial^*),\\ \varsigma_b(\square,\square^*)\end{Bmatrix}\right)+\lambda\varphi\left(\max\begin{Bmatrix}\varsigma_b(\partial^*,\partial),\\ \varsigma_b(\square^*,\square)\end{Bmatrix}\right)$$

$$+\theta\varphi\left(\max\begin{Bmatrix}\varsigma_b(\partial,\partial^*),\\ \varsigma_b(\square,\square^*)\end{Bmatrix}+\max\begin{Bmatrix}\varsigma_b(\partial^*,\partial),\\ \varsigma_b(\square^*,\square)\end{Bmatrix}\right)$$

Since $\varphi(s)\leq s$ for all $s\geq 0$, then we obtain

$$\varsigma_b(\partial,\partial^*)\ \leq(\delta+\lambda+2\theta)max\begin{Bmatrix}\varsigma_b(\partial^*,\partial),\\ \varsigma_b(\square^*,\square)\end{Bmatrix}$$

Since, $0\leq\delta+\lambda+2\theta<1$, we have

$$\varsigma_b(\partial,\partial^*)< max\begin{Bmatrix}\varsigma_b(\partial^*,\partial),\\ \varsigma_b(\square^*,\square)\end{Bmatrix}$$

Therefore,

$$\max\left\{\varsigma_b(\partial,\partial^*),\varsigma_b(\square,\square^*)\right\}<\lambda max\left\{\varsigma_b(\partial,\partial^*),\varsigma_b(\square,\square^*)\right\}$$

it is a contradiction.

Therefore the UCCFP (unique common coupled fixed point) of S and f *is* $(\partial,\square)$.

Corollary 3.3. Let $(\Im,\varsigma_b)$ be a Partial b- metric space with the coefficient $\mathsf{d}\geq 1$ and $\mathsf{S}:\Im^2\to\Im$ be a mapping satisfying H-contraction, for all $\mathsf{y}_1,\mathsf{y}_2,æ_1,æ_2\in\Im$

with positive real numbers δ,λ,θ such that $0\leq\delta+\lambda+2\theta<1$,

$$\varsigma_b(\mathrm{S}(y_1,y_2),\mathrm{S}(æ_1,æ_2) \leq \delta\max\begin{Bmatrix}\varsigma_b(y_1,æ_1),\\ \varsigma_b(y_2,æ_2)\end{Bmatrix}+\lambda\max\begin{Bmatrix}\dfrac{\varsigma_b(æ_1,\mathrm{S}(æ_1,æ_2)[1+\varsigma_b(y_1,\mathrm{S}(y_1,y_2))]}{1+\varsigma_b(y_1,æ_1)},\\ \dfrac{\varsigma_b(æ_2,\mathrm{S}(æ_2,æ_1)[1+\varsigma_b(y_2,\mathrm{S}(y_2,y_1))]}{1+\varsigma_b(y_2,æ_2)}\end{Bmatrix}$$

$$+\theta\left(\max\begin{Bmatrix}\varsigma_b(y_1,\mathrm{S}(y_1,y_2)),\\ \varsigma_b(y_2,\mathrm{S}(y_2,y_1))\end{Bmatrix}+\max\begin{Bmatrix}\varsigma_b(æ_1,\mathrm{S}(æ_1,æ_2)),\\ \varsigma_b(æ_2,\mathrm{S}(æ_2,æ_1))\end{Bmatrix}\right)$$

In $\Im$ there is a unique coupled fixed point for S .

Example 3.4. Let $\Im=\{1,2,3\}$ and $\varsigma_b:\Im^2\to[0,\infty)$ be defined as

$$\varsigma_b(\eta;æ)=\begin{cases}|\eta-æ|^2+\max\{\eta,æ\} & if \quad \eta\neq æ\\ \eta & for \quad \eta=æ\neq 1\\ 0 & for \quad \eta=æ=1\end{cases}$$

. Then $(\Im,\varsigma_b)$ is a complete partial b-metric space with coefficient $\mathrm{v}=4>1$

Define $\mathrm{S}:\Im^2\to\Im$ *be as* $\mathrm{S}(1,1)=1,\mathrm{S}(1,2)=1,\mathrm{S}(1,3)=2,\mathrm{S}(2,1)=1$,

$\mathrm{S}(2,2)=1,\mathrm{S}(2,3)=2,\mathrm{S}(3,1)=1,\mathrm{S}(3,2)=1,\mathrm{S}(3,3)=1$, and $\mathrm{f}:\Im\to\Im$ *by*

$\mathrm{f}1=1,\mathrm{f}2=3,\mathrm{f}3=2$. Also, define $\varphi:[0,\infty)\to[0,\infty)$ *as* $\varphi(t)=\dfrac{2t}{7}$ and

$$\alpha:\Im^2\to R^{+} as\ \alpha\ (\eta,æ)=\begin{cases}1 & for \quad \eta,æ\in\{1,2,3\}\\ 0 & for \quad otherwise\end{cases}$$

We show that S, f are α-admissible mappings. Let $\eta,æ\in\Im$, if $\alpha(\mathrm{f}\eta,\ \mathrm{f}æ)\geq 1$ then $\mathrm{f}\eta,\ \mathrm{f}æ\in\Im$ and so $\mathrm{S}(\mathrm{f}æ,\mathrm{f}\eta)\in\Im$ implies that $\alpha(\mathrm{S}(\eta,æ),\mathrm{S}(\mathrm{f}æ,\mathrm{f}\eta))\geq 1$.Therefore, the predication holds. Obviously, $\mathrm{S}(1,1)=1=\mathrm{f}1$ implies that (1, 1) is a coupled coincidence point of S and f. Moreover $\mathrm{f}(\Im)=\{1,2,3\}$ and $\mathrm{S}(\Im^2)=\{1,2\}$. Hence, $\mathrm{S}(\Im^2)\subseteq\mathrm{f}(\Im)$ and also $\mathrm{S}(1,1)=\mathrm{S}(\mathrm{f}1,\mathrm{f}1)=\mathrm{f}\mathrm{S}(1,1)=\mathrm{f}1=1$, then ($\mathrm{S}$, f) is ω-compatible. Then, S and f with $\delta=\dfrac{1}{3},\lambda=\dfrac{1}{4},\theta=\dfrac{1}{6}$, satisfy all the requirements of theorem 3.2. According to Theorem 3.2, S *and* f have a unique coupled fixed point which is (1, 1).

3.1 Application to BVP.

In this section, we investigate the existence of a unique solution to a boundary value problem as an application of Corollary 3.3.

Think about the boundary value problem.

$$\frac{d^2\eta}{d\delta^2}+\Gamma(\delta,\eta(\delta),\eta(\delta))=0,\delta\in I=[0,1],\eta(0)=\eta(1)=0 \tag{3.9}$$

The associated Greens function $\varsigma : I \times I \to I$ to Eq. 3.9, can be defined as follows

$$\varsigma(s,t) = \begin{cases} s(1-t) & if \quad 0 \le s \le t \le 1 \\ t(1-s) & if \quad 0 \le t \le s \le 1 \end{cases}$$

We have the following properties of the Greens function ς:

a) $\varsigma(s,t) \ge 0$ for all $s,t \in [0,1]$;

b) $\sup_{0\le t\le 1} \int_0^t \varsigma(s,\tau)d\tau = \frac{1}{4}$

Let $\Im = C(I)$ represents the set of continuous functions defined on I. Define the mapping

$d_{\varsigma_b} : \Im \times \Im \to [0,\infty)$ by $d_{\varsigma_b}(\mathrm{f},\mathrm{g}) = \|(\mathrm{f}-\mathrm{g})^2\| = \sup|\mathrm{f}(s)-\mathrm{g}(s)|^2 \ \forall \mathrm{f},\mathrm{g} \in \Im, s \in I.$

Then obviously, the pair $(\Im, d_{\varsigma b})$ is a complete with $\mathrm{v}=2$. The associated integral operator $\mathrm{S} : \Im^2 \to \Im$ to Eq. 3.9 is defined by

$$\mathrm{S}(\mathrm{f},\mathrm{g})(s) = \int_0^1 \varsigma(s,\tau)\Gamma(\tau,\mathrm{f}(\tau),\mathrm{g}(\tau))d\tau$$

It is noted that the operator S has a fixed point that solves Eq. 3.9. The condition under which the BVP has a solution is given by the following theorem.

Theorem 3.5. Let the function $\Gamma : IX\, C(I) XC(I) \to R$ is continuous and satisfies the following condition:

$$|\Gamma(s,\mathrm{f},\mathrm{g})-\Gamma(s,\mathrm{h},\mathrm{l})|^2 \le 16\left(\delta|\mathrm{f}(s)-\mathrm{h}(s)|^2+\lambda|\mathrm{h}(s)-\mathrm{S}(\mathrm{h},\mathrm{l})(s)|^2+\theta\begin{pmatrix}|\mathrm{g}(s)-\mathrm{S}(\mathrm{g},\mathrm{f})(s)|^2\\ +|\mathrm{l}(s)-\mathrm{S}(\mathrm{l},\mathrm{h})(s)|^2\end{pmatrix}\right) \text{for all}$$

$s \in I, \mathrm{f},\mathrm{g},\mathrm{h},\mathrm{l} \in C(I)$ and $\delta,\lambda,\theta \in (0,1)$ with $\delta+\lambda+2\theta<1$. Then the BVP Eq.3.9 has a solution.

Proof. To accomplish this proof, Corollary 3.3 will be used. The operator $\mathrm{S} : \Im^2 \to \Im$ defined above is continuous since the function Γ is continuous. We continue as follows to demonstrate that the mapping S forms a $H-$ contraction

$$|\mathrm{S}(\mathrm{f},\mathrm{g})(s)-\mathrm{S}(\mathrm{h},\mathrm{l})(s)|^2 = |\int_0^1 \varsigma(s,\tau)(\Gamma(\tau,\mathrm{f}(\tau),g(\tau))-\Gamma(\tau,\mathrm{h}(\tau),\mathrm{l}(\tau)))d\tau|^2$$

$$\le \left(\int_0^1 \varsigma(s,\tau)\sqrt{16\left(\begin{matrix}\delta|\mathrm{f}(s)-\mathrm{h}(s)|^2\\ +\lambda|\mathrm{h}(s)-\mathrm{S}(\mathrm{h},\mathrm{l})(s)|^2\end{matrix}+\theta\begin{pmatrix}|\mathrm{g}(s)-\mathrm{S}(\mathrm{g},\mathrm{f})(s)|^2\\ +|\mathrm{l}(s)-\mathrm{S}(\mathrm{l},\mathrm{h})(s)|^2\end{pmatrix}\right)}d\tau\right)^2$$

Since $(\sup\int_0^1 \varsigma(s,\tau)d\tau)^2 = \frac{1}{16}$ for all $s \in I$, thus, taking supremum on both sides of above inequality, we have

$$d_{\varsigma b}(\mathrm{S}(\mathrm{f},\mathrm{g}),\mathrm{S}(\mathrm{h},\mathrm{l})) \le \delta d_{\varsigma_b}(\mathrm{f},\mathrm{h}) + \lambda d_{\varsigma b}(\mathrm{h},\mathrm{S}(\mathrm{h},\mathrm{l})) + \theta\begin{pmatrix} d_{\varsigma_b}(\mathrm{l},\mathrm{S}(\mathrm{l},\mathrm{h})) \\ d_{\varsigma_b}(\mathrm{g},\mathrm{S}(\mathrm{g},\mathrm{f}))\end{pmatrix}$$

Now for any partial b-metric ς_b on $\Im$, we can have a b-metric $d_{\varsigma b}$ on $\Im$ by

$$d_{\varsigma_b}(\mathrm{f},\mathrm{g}) = \begin{cases} \varsigma_b(\mathrm{f},\mathrm{g}) & if \quad \mathrm{f} \ne \mathrm{g} \\ 0 & if \quad \mathrm{f} = \mathrm{g}\end{cases}$$

The last inequality can be written as:

$$\varsigma_b(\mathrm{S}(\mathrm{f},\mathrm{g}),\mathrm{S}(\mathrm{h},\mathrm{l})) \le \delta\varsigma_b(\mathrm{f},\mathrm{h}) + \lambda\varsigma_b(\mathrm{h},\mathrm{S}(\mathrm{h},\mathrm{l})) + \theta\begin{pmatrix} \varsigma_b(\mathrm{l},\mathrm{S}(\mathrm{l},\mathrm{h})) \\ +\varsigma_b(\mathrm{g},\mathrm{S}(\mathrm{g},\mathrm{f}))\end{pmatrix}$$

$$\le \delta\max\begin{Bmatrix} \varsigma_b(\mathrm{f},\mathrm{h}), \\ \varsigma_b(\mathrm{g},\mathrm{l})\end{Bmatrix} + \lambda\max\begin{Bmatrix} \dfrac{\varsigma_b(\mathrm{h},\mathrm{S}(\mathrm{h},\mathrm{l})[1+\varsigma_b(\mathrm{f},\mathrm{S}(\mathrm{f},\mathrm{g})]}{1+\varsigma_b(\mathrm{f},\mathrm{h})}, \\ \dfrac{\varsigma_b(\mathrm{l},\mathrm{S}(\mathrm{l},\mathrm{h})[1+\varsigma_b(\mathrm{g},\mathrm{S}(\mathrm{g},\mathrm{f})]}{1+\varsigma_b(\mathrm{g},\mathrm{l})}\end{Bmatrix}$$

$$+\theta\left(\max\begin{Bmatrix} \varsigma_b(\mathrm{f},\mathrm{S}(\mathrm{f},\mathrm{g})), \\ \varsigma_b(\mathrm{g},\mathrm{S}(\mathrm{g},\mathrm{f}))\end{Bmatrix} + \max\begin{Bmatrix} \varsigma_b(\mathrm{h},\mathrm{S}(\mathrm{h},\mathrm{l})), \\ \varsigma_b(\mathrm{l},\mathrm{S}(\mathrm{l},\mathrm{h}))\end{Bmatrix}\right)$$

Therefore the BVP (3.9) has a solution $\Im$ in according to Corollary 3.3

4. APPLICATION TO HOMOTOPY

In this section, we study the existence of a unique solution to homotopy theory.

Theorem 4.1. Let $(\Im,\varsigma_b)$ be complete partial b-metric space the coefficient $\mathrm{v} \ge 1$,

U and $\overline{\mathrm{U}}$ be an open and closed subset of $\Im$ such that $\mathrm{U} \subseteq \overline{\mathrm{U}}$. Suppose $\mathrm{A}_p : \overline{\mathrm{U}}^2 X[0,1] \to \Im$ be an operator with following conditions are satisfying, τ_0) $æ \ne \mathrm{A}_P(æ, œ, \lambda), œ \ne \mathrm{A}_P(œ, æ, \lambda)$, for each $æ, œ \in \overline{\mathrm{U}}$ and $\lambda \in [0,1]$ (here $\partial\mathrm{U}$ is boundary of U in $\Im$);

τ_1) *for all* $æ, œ, \mathrm{B}, \mathrm{x} \in \overline{\mathrm{U}}$, *and* $\lambda \in [0,1]$ *with* $0 \le \delta + \gamma + 2\theta < 1$ *such that*

$$\mathrm{v}\varsigma_b(\mathrm{A}_p(æ, œ, \lambda), \mathrm{A}_p(\mathrm{B}, \mathrm{x}, \lambda)) \le \delta\max\begin{Bmatrix} \varsigma_b(æ, \mathrm{B}), \\ \varsigma_b(œ, \mathrm{x})\end{Bmatrix}$$

$$+\lambda \max\left\{\begin{array}{c}\dfrac{\varsigma_b(\mathrm{B}, \mathsf{A}_p(\mathrm{B}, \mathsf{x}, \lambda))[1+\varsigma_b(æ, \mathsf{A}_p(æ, œ, \lambda))]}{1+\varsigma_b(æ, \mathrm{B})}, \\ \dfrac{\varsigma_b(\mathsf{x}, \mathsf{A}_p(\mathsf{x}, \mathrm{B}, \lambda))[1+\varsigma_b(œ, \mathsf{A}_p(œ, æ, \lambda))]}{1+\varsigma_b(œ, \mathsf{x})}\end{array}\right\}$$

$$+\theta\left(\max\left\{\begin{array}{l}\varsigma_b(æ, \mathsf{A}_p(æ.\mathrm{B}, \lambda)), \\ \varsigma_b(œ, \mathsf{A}_p(œ, æ, \lambda))\end{array}\right\}+\max\left\{\begin{array}{l}\varsigma_b(\mathrm{B}, \mathsf{A}_p(\mathrm{B}, \mathsf{x}, \lambda)), \\ \varsigma_b(\mathsf{x}, \mathsf{A}_p(\mathsf{x}, \mathrm{B}, \lambda))\end{array}\right\}\right)$$

τ_2) $\exists M \geq 0 \quad \ni \varsigma_b(\mathsf{A}_p(æ, œ, \lambda), \mathsf{A}_p(æ, œ, \zeta)) \leq M\,|\,\lambda-\zeta\,|$

for every $æ, œ \in \overline{\mathsf{U}}$ and $\lambda, \zeta \in [0,1]$.

Then $\mathsf{A}_p(.,0)$ has a coupled fixed point $\Longleftrightarrow \mathsf{A}_p(.,1)$ has a coupled fixed point.

Proof. Consider the set

$\mathrm{A} = \{\lambda \in [0,1] : æ = \mathsf{A}_p(æ, œ, \lambda), œ = \mathsf{A}_p(œ, æ, \lambda) \text{for some } æ, œ \in \mathsf{U}\}.$

We have that $(0,0) \in \mathrm{A}^2$ since $\mathsf{A}_p(.,0)$ has a coupled fixed point in U^2, proving that the set A is non-empty set. We will demonstrate that A is both open and closed in [0, 1]. Consequently, A = [0, 1] may be obtained by the connectedness. Consequently there is a coupled fixed point for $\mathsf{A}_p(.,1)$ in U^2. We first demonstrate the closure of A in [0, 1]. Let $\{\lambda_z\}_{z=1}^{\infty} \subseteq \mathrm{A}$ where $z \to \infty.\ \lambda_z \to \lambda \in [0,1]$ showing that $\lambda \in \mathrm{A}$ is necessary. Considering that $\lambda_z \in \mathrm{A}$

for $z = 1, 2, 3, \ldots,\ \exists\ æ_z, œ_z \in \mathsf{U}$ *and that* $æ_{z+1} = \mathsf{A}_p(æ_z, œ_z, \lambda_z), œ_{z+1} = \mathsf{A}_p(œ_z, æ_z, \lambda_z)$.

Think about

$$\varsigma_b(æ_z, æ_{z+1}) = \varsigma_b(\mathsf{A}_p(æ_{z-1}, œ_{z-1}, \lambda_{z-1}), \mathsf{A}_p(æ_z, œ_z, \lambda_z))$$

$$\leq \mathsf{v}\left\{\begin{array}{l}\varsigma_b(\mathsf{A}_p(æ_{z-1}, œ_{z-1}, \lambda_{z-1}), \mathsf{A}_p(æ_z, œ_z, \lambda_{z-1})) \\ +\varsigma_b(\mathsf{A}_p(æ_z, œ_z, \lambda_{z-1}), \mathsf{A}_p(æ_z, œ_z, \lambda_z)) \\ -\varsigma_b(\mathsf{A}_p(æ_z, œ_z, \lambda_{z-1}), \mathsf{A}_p(æ_z, œ_z, \lambda_{z-1}))\end{array}\right\}$$

$$\leq \mathsf{v}\varsigma_b(\mathsf{A}_p(æ_{z-1}, œ_{z-1}, \lambda_{z-1}), \mathsf{A}_p(æ_z, œ_z, \lambda_{z-1}) + \mathsf{v}M\left|\lambda_{z-1}-\lambda_z\right|$$

Letting $z \to \infty$, we get

$$\lim_{z\to\infty} \varsigma_b(æ_z, æ_{z+1}) \leq \lim_{z\to\infty} \mathsf{v}\varsigma_b(\mathsf{A}_p(æ_{z-1}, œ_{z-1}, \lambda_{z-1}), \mathsf{A}_p(æ_z, œ_z, \lambda_{z-1})) + 0.$$

From (τ_1) we obtain

$$\lim_{z\to\infty} \varsigma_b(æ_z, æ_{z+1}) \leq \lim_{z\to\infty} \delta \max\left\{\begin{array}{l}\varsigma_b(æ_{z-1}, æ_z), \\ \varsigma_b(œ_{z-1}, œ_z)\end{array}\right\}$$

$$+\lim_{z\to\infty}\gamma\max\left\{\begin{array}{c}\dfrac{\varsigma_b(æ_z,\mathrm{A}_p(æ_z,œ_z,\lambda_z))[1+\varsigma_b(æ_{z-1},\mathrm{A}_p(æ_{z-1},œ_{z-1},\lambda_{z-1}))]}{1+\varsigma_b(æ_{z-1},œ_z)},\\ \dfrac{\varsigma_b(œ_z,\mathrm{A}_p(œ_z,æ_z,\lambda_z))[1+\varsigma_b(œ_{z-1},\mathrm{A}_p(œ_{z-1},æ_{z-1},\lambda_{z-1}))]}{1+\varsigma_b(œ_{z-1},œ_z)}\end{array}\right\}$$

$$+\lim_{z\to\infty}\theta\max\left\{\begin{array}{c}\varsigma_b((æ_{z-1},\mathrm{A}_p(æ_{z-1},œ_{z-1},\lambda_{z-1})),\\ \varsigma_b(œ_{z-1},\mathrm{A}_p(œ_{z-1},æ_{z-1},\lambda_{z-1}))\end{array}\right\}$$

$$+\lim_{z\to\infty}\theta\max\left\{\begin{array}{c}\varsigma_b(æ_z,\mathrm{A}_p(æ_z,œ_z,\lambda_z)),\\ \varsigma_b(œ_z,\mathrm{A}_p(œ_z,æ_z,\lambda_z))\end{array}\right\}$$

$$\leq\lim_{z\to\infty}\delta\max\left\{\begin{array}{c}\varsigma_b(æ_{z-1},æ_z),\\ \varsigma_b(œ_{z-1},œ_z)\end{array}\right\}+\lim_{z\to\infty}\gamma\max\left\{\begin{array}{c}\varsigma_b(æ_z,æ_{z+1}),\\ \varsigma_b(œ_z,œ_{z+1})\end{array}\right\}$$

$$+\lim_{z\to\infty}\theta\left(\max\left\{\begin{array}{c}\varsigma_b(æ_{z-1},æ_z),\\ \varsigma_b(œ_{z-1},œ_z)\end{array}\right\}+\max\left\{\begin{array}{c}\varsigma_b(æ_z,æ_{z+1}),\\ \varsigma_b(œ_z,œ_{z+1})\end{array}\right\}\right)$$

Similarly

$$\lim_{z\to\infty}\varsigma_b(œ_z,œ_{z-1})\leq\lim_{z\to\infty}\delta\max\left\{\begin{array}{c}\varsigma_b(æ_{z-1},æ_z),\\ \varsigma_b(œ_{z-1},œ_z)\end{array}\right\}+\lim_{z\to\infty}\gamma\max\left\{\begin{array}{c}\varsigma_b(æ_z,æ_{z+1}),\\ \varsigma_b(œ_z,œ_{z+1})\end{array}\right\}$$

$$+\lim_{z\to\infty}\theta\left(\max\left\{\begin{array}{c}\varsigma_b(æ_{z-1},æ_z),\\ \varsigma_b(œ_{z-1},œ_z)\end{array}\right\}+\max\left\{\begin{array}{c}\varsigma_b(æ_z,æ_{z+1})\\ \varsigma_b(œ_z,œ_{z+1})\end{array}\right\}\right)$$

Therefore, we have

$$\lim_{z\to\infty}\max\left\{\begin{array}{c}\varsigma_b(æ_z,æ_{z+1})\\ \varsigma_b(œ_z,œ_{z+1})\end{array}\right\}\leq\lim_{z\to\infty}\delta\max\left\{\begin{array}{c}\varsigma_b(æ_{z-1},æ_z),\\ \varsigma_b(œ_{z-1},œ_z)\end{array}\right\}+\lim_{z\to\infty}\gamma\max\left\{\begin{array}{c}\varsigma_b(æ_z,æ_{z+1}),\\ \varsigma_b(œ_z,œ_{z+1})\end{array}\right\}$$

$$+\lim_{z\to\infty}\theta\left(\max\left\{\begin{array}{c}\varsigma_b(æ_{z-1},æ_z),\\ \varsigma_b(œ_{z-1},œ_z)\end{array}\right\}+\max\left\{\begin{array}{c}\varsigma_b(æ_z,æ_{z+1}),\\ \varsigma_b(œ_z,œ_{z+1})\end{array}\right\}\right)$$

It follows that

$$\lim_{z\to\infty}\max\left\{\begin{array}{c}\varsigma_b(æ_z,æ_{z+1})\\ \varsigma_b(œ_z,œ_{z+1})\end{array}\right\}\leq\lim_{z\to\infty}\frac{\delta+\theta}{1-\gamma-\theta}\max\left\{\begin{array}{c}\varsigma_b(æ_{z-1},æ_z),\\ \varsigma_b(œ_{z-1},œ_z)\end{array}\right\}$$

$$\leq\lim_{z\to\infty}\left(\frac{\delta+\theta}{1-\gamma-\theta}\right)^2\max\left\{\begin{array}{c}\varsigma_b(æ_{z-2},æ_{z-1}),\\ \varsigma_b(œ_{z-2},œ_{z-1})\end{array}\right\}$$

$$\vdots$$

$$\leq \lim_{z\to\infty}(\frac{\delta+\theta}{1-\gamma-\theta})^{z}\max\begin{Bmatrix}\varsigma_b(æ_0,æ_1),\\ \varsigma_b(œ_0,œ_1)\end{Bmatrix}$$

It follows that

$$\lim_{z\to\infty}\varsigma_b(æ_z,æ_{z+1})=0=\lim_{z\to\infty}\varsigma_b(œ_z,œ_{z+1}). \tag{4.1}$$

From $(\varsigma_b 2)$,

$$\lim_{z\to\infty}\varsigma_b(æ_z,æ_z)=0=\lim_{z\to\infty}\varsigma_b(œ_z,œ_z). \tag{4.2}$$

We now demonstrate that the Cauchy sequences in $(\Im,\ \varsigma_b)$ are $\{æ_z\}$ and $\{œ_z\}$. Conversely, let's say that neither $\{æ_z\}$ nor $\{œ_z\}$ is Cauchy. A monotonic rising series of natural numbers $\{w_k\}$ and $\{z_k\}$ with $\grave{o}>0$ exists such that $z_k>w_k$,

$$\varsigma_b(æ_{w_k},æ_{z_k})\geq\grave{o}\qquad \varsigma_b(œ_{w_k},œ_{z_k})\geq\grave{o} \tag{4.3}$$

and

$$\varsigma_b(æ_{w_k},æ_{z_k-1})<\grave{o}\qquad \varsigma_b(œ_{w_k},œ_{z_{k-1}})<\grave{o} \tag{4.4}$$

From (4.3) and (4.4) we obtain

$$\grave{o}\leq\varsigma_b(æ_{w_k},æ_{z_k})$$

$$\leq \mathsf{v}(\varsigma_b(æ_{w_k},æ_{w_{k+1}})+\varsigma_b(æ_{w_{k+1}},æ_{z_k})-\varsigma_b(æ_{w_{k+1}},æ_{w_{k+1}}))$$

Using $k\to\infty$ as the limit and working from (4.1) to (4.2), we obtain that

$$\grave{o}\leq\lim_{k\to\infty}\mathsf{v}\varsigma_b(æ_{w_{k+1}},æ_{z_k})=\lim_{k\to\infty}\mathsf{v}\varsigma_b(\mathsf{A}_p(æ_{w_k},œ_{w_k},\lambda_{w_k}),\mathsf{A}_p(æ_{z_{k-1}},œ_{z_{k-1,}}\lambda_{z_{k-1}}))$$

$$\leq\lim_{k\to\infty}\delta\max\begin{Bmatrix}\varsigma_b(æ_{w_k},æ_{z_{k-1}}),\\ \varsigma_b(œ_{w_k},œ_{z_{k-1,}})\end{Bmatrix}$$

$$+\lim_{z\to\infty}\gamma\max\begin{Bmatrix}\dfrac{\varsigma_b(æ_{z_{k-1}},\mathsf{A}_p(æ_{z_{k-1}},œ_{z_{k-1,}},\lambda_{z_{k-1}}))[1+\varsigma_b(æ_{w_k},\mathsf{A}_p(æ_{w_k},œ_{w_k},\lambda_{w_k}))]}{1+\varsigma_b(æ_{w_k},œ_{z_{k-1,}})},\\ \dfrac{\varsigma_b(œ_{z_{k-1,}},\mathsf{A}_p(œ_{z_{k-1,}},æ_{z_{k-1}},\lambda_{z_{k-1}}))[1+\varsigma_b(œ_{w_k},\mathsf{A}_p(œ_{w_k},æ_{w_k},\lambda_{w_k}))]}{1+\varsigma_b(œ_{w_k},œ_{z_{k-1}})}\end{Bmatrix}$$

$$+\lim_{z\to\infty}\theta\max\begin{Bmatrix}\varsigma_b(æ_{w_k},\mathsf{A}_p(æ_{w_k},œ_{w_k},\lambda_{w_k})),\\ \varsigma_b(œ_{w_k},\mathsf{A}_p(œ_{w_k},æ_{w_k},\lambda_{w_k}))\end{Bmatrix}$$

$$+\lim_{z\to\infty}\theta\max\begin{Bmatrix}\varsigma_b(æ_{z_{k-1}},\mathsf{A}_p(æ_{z_{k-1}},œ_{z_{k-1,}},\lambda_{z_{k-1}})),\\ \varsigma_b(œ_{z_{k-1,}},\mathsf{A}_p(œ_{z_{k-1,}},æ_{z_{k-1}},\lambda_{z_{k-1}}))\end{Bmatrix}$$

$$\leq \lim_{k\to\infty} \delta \max\begin{Bmatrix} \varsigma_b(æ_{w_k}, æ_{z_{k-1}}), \\ \varsigma_b(œ_{w_k}, œ_{z_{k-1},}) \end{Bmatrix} + \lim_{z\to\infty} \gamma \max\begin{Bmatrix} \varsigma_b(æ_{z_{k-1}}, æ_{z_k}), \\ \varsigma_b(œ_{z_{k-1},}, œ_{z_{k,}}) \end{Bmatrix}$$

$$+\lim_{z\to\infty} \theta\left(\max\begin{Bmatrix} \varsigma_b(æ_{w_k}, æ_{w_{k+1}}), \\ \varsigma_b(œ_{w_k}, œ_{w_{k+1},}) \end{Bmatrix} + \max\begin{Bmatrix} \varsigma_b(æ_{z_{k-1}}, æ_{z_k}), \\ \varsigma_b(œ_{z_{k-1},}, œ_{z_{k,}}) \end{Bmatrix}\right)$$

$$\leq \lim_{k\to\infty} \delta \max\begin{Bmatrix} \varsigma_b(æ_{w_k}, æ_{z_{k-1}}), \\ \varsigma_b(œ_{w_k}, œ_{z_{k-1}}) \end{Bmatrix}$$

$$\leq \lim_{k\to\infty} \delta^2 \max\begin{Bmatrix} \varsigma_b(æ_{w_{k-1}}, æ_{z_{k-2}}), \\ \varsigma_b(œ_{w_{k-1}}, œ_{z_{k-2},}) \end{Bmatrix}$$

$$\vdots$$

$$\leq \lim_{k\to\infty} \delta^k \max\begin{Bmatrix} \varsigma_b(æ_1, æ_0), \\ \varsigma_b(œ_1, œ_0) \end{Bmatrix}$$

$$\leq \ 0$$

Hence, $\delta < 1$, is in conflict with $\partial \leq 0$. Therefore in $(\Im, \varsigma_b)$, $\{æ_z\}$ is a Cauchy sequence and $\lim_{z,w\to\infty} \varsigma_b(æ_z, æ_w) = 0$. Similarly, we can demonstrate that $\{œ_z\}$ is a Cauchy sequence in $(\Im, \varsigma_b)$ and $\lim_{z,w\to\infty} \varsigma_b(œ_z, œ_w) = 0$. Given the completeness of $(\Im, \varsigma_b)$, $\mathrm{B}, \mathsf{X} \in \mathsf{U}$ exist.

$$\varsigma_b(\mathrm{B},\mathrm{B}) = \lim_{z\to\infty} \varsigma_b(æ_z, \mathrm{B}) = \lim_{z\to\infty} \varsigma_b(æ_{z+1}, \mathrm{B}) = \lim_{z,w\to\infty} \varsigma_b(æ_z, æ_w) = 0$$

$$\varsigma_b(\mathsf{X},\mathsf{X}) = \lim_{z\to\infty} \varsigma_b(æ_z, \mathsf{X}) = \lim_{z\to\infty} \varsigma_b(æ_{z+1}, \mathsf{X}) = \lim_{z,w\to\infty} \varsigma_b(œ_z, œ_w) = 0$$

From Lemma 2.7, we get $\lim_{z\to\infty} \varsigma_b(æ_z, \mathsf{A}_p(\mathrm{B}, \mathsf{X}, \lambda)) = \varsigma_b(\mathrm{B}, \mathsf{A}_p(\mathrm{B}, \mathsf{X}, \lambda))$.

Now,

$$\begin{aligned} \varsigma_b(æ_{z+1}, \mathsf{A}_p(\mathrm{B}, \mathsf{X}, \lambda)) &= \varsigma_b(\mathsf{A}_p(æ_z, œ_z, \lambda_z), \mathsf{A}_p(\mathrm{B}, \mathsf{X}, \lambda)) \\ &\leq \mathsf{v}\begin{cases} \varsigma_b(\mathsf{A}_p(æ_z, œ_z, \lambda_z), \mathsf{A}_p(æ_z, œ_z, \lambda)) + \varsigma_b(\mathsf{A}_p(æ_z, œ_z, \lambda), \mathsf{A}_p(\mathrm{B}, \mathsf{X}, \lambda)) \\ \qquad -\varsigma_b(\mathsf{A}_p(æ_z, œ_z, \lambda), \mathsf{A}_p(æ_z, œ_z, \lambda)) \end{cases} \\ &\leq \mathsf{v}M\left|\lambda_z - \lambda\right| + \mathsf{v}\varsigma_b(\mathsf{A}_p(æ_z, œ_z, \lambda), \mathsf{A}_p(\mathrm{B}, \mathsf{X}, \lambda)). \end{aligned}$$

Letting $z \to \infty$, we obtain

$$\begin{aligned} \varsigma_b(\mathrm{B}, \mathsf{A}_p(\mathrm{B}, \mathsf{X}, \lambda)) &\leq \lim_{z\to\infty} \mathsf{v}\varsigma_b(\mathsf{A}_p(æ_z, œ_z, \lambda), \mathsf{A}_p(\mathrm{B}, \mathsf{X}, \lambda)) \\ &\leq \lim_{z\to\infty} \delta \max\begin{Bmatrix} \varsigma_b(æ_z, \mathrm{B}), \\ \varsigma_b(œ_z, \mathsf{X}) \end{Bmatrix} \end{aligned}$$

$$+\lim_{z\to\infty}\gamma\max\left\{\begin{array}{c}\dfrac{\varsigma_b(B,A_p(B,x,\lambda))[1+\varsigma_b(æ_z,A_p(æ_z,œ_z,\lambda_z))],}{1+\varsigma_b(æ_z,B)}\\ \dfrac{\varsigma_b(x,A_p(x,B,\lambda))[1+\varsigma_b(œ_z,A_p(œ_z,æ_z,\lambda_z))]}{1+\varsigma_b(œ_z,x)}\end{array}\right\}$$

$$+\lim_{z\to\infty}\theta\left(\max\left\{\begin{array}{c}\varsigma_b(æ_z,A_p(æ_z,œ_z,\lambda_z)),\\ \varsigma_b(œ_z,A_p(œ_z,æ_z,\lambda_z))\end{array}\right\}+\max\left\{\begin{array}{c}\varsigma_b(B,A_p(B,x,\lambda)),\\ \varsigma_b(x,A_p(x,B,\lambda))\end{array}\right\}\right)$$

$$\leq(\gamma+\theta)\max\left\{\begin{array}{c}\varsigma_b(B,A_p(B,x,\lambda)),\\ \varsigma_b(x,A_p(x,B,\lambda))\end{array}\right\}$$

It follows that

$$(1-\gamma-\theta)\max\left\{\begin{array}{c}\varsigma_b(B,A_p(B,x,\lambda)),\\ \varsigma_b(x,A_p(x,B,\lambda))\end{array}\right\}\leq 0$$

This suggests that both $\varsigma_b(B,A_p(B,x,\lambda))=0$ and $\varsigma_b(x,A_p(x,B,\lambda))=0$. In order for $A_p(B,x,\lambda)=B$ and $A_p(x,B,\lambda)=x$ *to* be equal . Therefore $\lambda\in A$. Thus in $[0,1]$

A is closed. Let A contain λ_0. When $æ_0 = A_p(æ_0,œ_0,\lambda_0)$ then $\exists$ $æ_0,œ_0\in U$ as well as $œ_0=A_p(œ_0,æ_0,\lambda_0)$. $B_{\varsigma b}(æ_0,r)\subseteq U$ and $B_{\varsigma b}(œ_0,r)\subseteq U$ Since U is open

Choose $\lambda\in(\lambda_0-\grave{o},\lambda_0+\grave{o})$ such that $|\lambda-\lambda_0|\leq\dfrac{1}{M^z}<\in$

Then for $æ\in\overline{B_{\varsigma_b}(æ_0,r)}=\{æ\in\Im/\varsigma_b(æ,æ_0)\leq r+\varsigma_b(æ_0,æ_0)\}$ and $œ\in\overline{B_{\varsigma_b}(œ_0,r)}=\{œ\in\Im/\varsigma_b(œ,œ_0)\leq r+\varsigma_b(œ_0,œ_0)\}$. Now we have

$$\varsigma_b(A_p(æ,œ,\lambda),æ_0)=\varsigma_b(A_p(æ,œ,\lambda),A_p(æ_0,œ_0,\lambda_0))$$

$$\leq v\left\{\begin{array}{c}\varsigma_b(A_p(æ,œ,\lambda),A_p(æ,œ,\lambda_0))+\varsigma_b(A_p(æ,œ,\lambda_0),A_p(æ_0,œ_0,\lambda_0))\\ -\varsigma_b(A_p(æ,œ,\lambda_0),A_p(æ,œ,\lambda_0))\end{array}\right\}$$

$$\leq vM|\lambda-\lambda_0|+v\varsigma_b(A_p(æ,œ,\lambda_0),A_p(æ_0,œ_0,\lambda_0)).$$

$$\leq v\frac{1}{M^{z-1}}+v\varsigma_b(A_p(æ,œ,\lambda_0),A_p(æ_0,œ_0,\lambda_0)).$$

Letting $z\to\infty$, we obtain

$$\varsigma_b(A_p(æ,œ,\lambda),æ_0)\leq v\varsigma_b(A_p(æ,œ,\lambda_0),A_p(æ_0,œ_0,\lambda_0))$$

$$\leq\delta\max\left\{\begin{array}{c}\varsigma_b(æ,æ_0),\\ \varsigma_b(œ,œ_0)\end{array}\right\}$$

$$+\gamma \max\left\{\begin{array}{l}\dfrac{\varsigma_b(æ_0, \mathsf{A}_p(æ_0, œ_0, \lambda_0))[1+\varsigma_b(æ, \mathsf{A}_p(æ, œ, \lambda))]}{1+\varsigma_b(æ, æ_0)},\\ \dfrac{\varsigma_b(œ_0, \mathsf{A}_p(œ_0, æ_0, \lambda_0)[1+\varsigma_b(œ, \mathsf{A}_p(œ, æ, \lambda))]}{1+\varsigma_b(œ, œ_0)}\end{array}\right\}$$

$$+\theta\left(\max\left\{\begin{array}{l}\varsigma_b(æ, \mathsf{A}_p(æ, œ, \lambda)),\\ \varsigma_b(œ, \mathsf{A}_p(æ, œ, \lambda))\end{array}\right\}+\max\left\{\begin{array}{l}\varsigma_b(æ_0, \mathsf{A}_p(æ_0, œ_0, \lambda_0)),\\ \varsigma_b(œ_0, \mathsf{A}_p(œ_0, æ_0, \lambda_0)\end{array}\right\}\right)$$

$$\leq (\delta+\gamma+2\theta)\max\ \{\varsigma_b(æ, æ_0), \varsigma_b(œ, œ_0)\}$$

$$< \max\{\varsigma_b(æ, æ_0), \varsigma_b(œ, œ_0)\}$$

Similarly

$\varsigma_b(\mathsf{A}_p(œ, æ, \lambda), œ_0) < \max\{\varsigma_b(æ, æ_0), \varsigma_b(œ, œ_0)\}.$

Thus

$$\max\{\varsigma_b(\mathsf{A}_p(æ, œ, \lambda), æ_0), \varsigma_b(\mathsf{A}_p(œ, æ, \lambda), œ_0)\} \quad < \max\{\varsigma_b(æ, æ_0), \varsigma_b(œ, œ_0)\}$$

$$\leq \max\{r+\varsigma_b(æ_0, æ_0), r+\varsigma_b(œ_0, œ_0)\}$$

For every constant $\lambda \in (\lambda_0 - \grave{o},\ \lambda_0 + \grave{o})$, *this means that* $\mathsf{U}_p(.,\ \lambda): \overline{B_{\varsigma_b}(æ_0,\ r)} \to \overline{B_{\varsigma_b}(æ_0,\ r)}$ and $\mathsf{A}_p(.,\ \lambda): \overline{B_{\varsigma_b}(œ_0,\ r)} \to \overline{B_{\varsigma_b}(œ_0,\ r)}$. Since (τ_1) also holds, Theorem 4.1 is satisfied in all of its conditions. From this, we infer that there is a coupled fixed point for $\mathsf{A}_p(., \lambda)$ in $\overline{\mathsf{U}}^2$. However, since (τ_0) holds, this coupled fixed point has to reside in U^2. For any $\lambda \in (\lambda_0 - \grave{o}, \lambda_0 + \grave{o})$. $\lambda \in \mathrm{A}$. Therefore $(\lambda_0 - \grave{o}, \lambda_0 + \grave{o}) \subseteq \mathrm{A}$

Hence, in $[0, 1]$. A is open.

We employ the identical method for the opposite inference.

Conclusion

This paper presents several fixed point results and appropriate examples that demonstrate the major findings in the context of partial b-metric space using contractive mappings of the $(\alpha,\ \varphi)$ –H type. Applications to homotopy and boundary value problems are also provided.

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