Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 5s (2024)

The Fractional Calculus of Product of Special Functions

Karuna Laddha[1], Deepak Kumar Kabra[2], Seema Kabra[3]
[1,2,3]Department of Mathematics, Sangam University
[1]karunaladdha9@gmail.com, [2]dkabra20@gmail.com, [3]kabraseema@rediffmail.com

Article History:
Received: *17-05-2024*
Revised: *27-06-2024*
Accepted: *12-07-2024*

Abstract:
Introduction: In this paper, we aim to establish a closed form for the Pathway fractional integral operator and Marichev-Saigo-Maeda fractional integral and differential operators involving the product of Special G function and Generalized Mittag – Leffler function. The obtained results are evaluated in terms of generalized Wright hyper geometric function.

Keywords: Pathway fractional integral operator, Marichev-Saigo-Maeda operator generalized hyper geometric function , special G function , generalized Mittag- Leffler function.

1. Introduction

Throughout this paper, R and C denote the sets of real and complex numbers, respectively. Also $R^+ = (0,\infty)$, $N_0 = \{0, 1, \ldots\}$ and $Z^- = \{-1, -2, \ldots\}$.

Pathway Fractional Integral Operator :

In 2005 Mathai [1] presented the technique of evaluation and interpretation of special function and integral transform and its applications in statistics and physical sciences. It was further extended by Mathai and Hauhold [2,3], see also [14].

In 2009 Nair [4] derived a Pathway fractional integral operator as,

Let $f(x) \in L(a,b), \eta \in C, \mathrm{Re}(\eta) > 0, a > 0$ *and* $\alpha > 0$ *and* α is taken as pathway parameter such that $\alpha < 1$. Then the pathway fractional integration operator is defined and represented as follows:

$$(P_{0+}^{(\eta,\alpha,a)})\ (x) = x^{\eta} \int_0^{\frac{x}{a(1-\alpha)}} \left[\frac{a(1-\alpha)}{x}\right]^{\frac{\eta}{(1-\alpha)}} f(t)dt \tag{1}$$

where (a, b) is the set of Lebsgue measurable function defined on (a, b).

The pathway model is introduced by Mathai and studied further by Mathai and Hauboldm.

2. Objectives

Result Required:

The following result is required here

$$(P_{0+}^{(\eta,\alpha,a)})\{t^{\beta-1}\} = \frac{t^{\eta+\beta}}{[a(1-\alpha)]^{\beta}} \frac{\Gamma(\beta)\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\Gamma\left(\frac{\eta}{1-\alpha}+\beta+1\right)} \tag{2}$$

where $\alpha < 1$; $\operatorname{Re}(\eta) > 0$; $\operatorname{Re}(\beta) > 0$.

Marichev-Saigo-Maeda fractional Operators

The following MSM integral operators are required here [15], see also [16] to obtain the required results

Let $\nu, \nu', \tau, \tau', \varepsilon, \sigma \in C$ such that $\operatorname{Re}(\nu) > 0$

(a) If $\operatorname{Re}(\sigma) > \max\{0, \operatorname{Re}(\nu' - \tau'), \operatorname{Re}(\nu + \nu' + \tau - \varepsilon)\}$, then

$$(I_{0+}^{\nu,\nu',\tau,\tau',\varepsilon} t^{\sigma-1})(t) = \frac{\Gamma(\sigma)\Gamma(-\nu' + \tau' + \sigma)}{\Gamma(\tau' + \sigma)\Gamma(-\nu - \nu' + \varepsilon + \sigma)} \times \frac{\Gamma(-\nu - \nu' - \tau + \varepsilon + \sigma)}{\Gamma(-\nu' - \tau + \varepsilon + \sigma)} t^{-\nu-\nu'+\varepsilon+\sigma-1} \tag{3}$$

(b) If $\operatorname{Re}(\sigma) > \max\{\operatorname{Re}(\tau), \operatorname{Re}(-\nu - \nu' + \varepsilon), \operatorname{Re}(-\nu - \tau' + \varepsilon)\}$, then

$$(I_{-}^{\nu,\nu',\tau,\tau',\varepsilon} t^{-\sigma})(t) = \frac{\Gamma(-\tau + \sigma)\Gamma(\nu + \nu' - \varepsilon + \sigma)}{\Gamma(\sigma)\Gamma(\nu - \tau + \sigma)} \times \frac{\Gamma(\nu + \tau' - \varepsilon + \sigma)}{\Gamma(\nu + \nu' + \tau' - \varepsilon + \sigma)} t^{-\nu-\nu'+\varepsilon-\sigma} \tag{4}$$

(c) If $\operatorname{Re}(\sigma) > \max\{0, \operatorname{Re}(-\nu + \tau), \operatorname{Re}(-\nu - \nu' - \tau' - \varepsilon)\}$, then

$$(D_{0+}^{\nu,\nu',\tau,\tau',\varepsilon} t^{\sigma-1})(t) = \frac{\Gamma(\sigma)\Gamma(-\tau + \nu + \sigma)}{\Gamma(-\tau + \sigma)\Gamma(\nu + \nu' - \varepsilon + \sigma)} \times \frac{\Gamma(\nu + \nu' + \tau' - \varepsilon + \sigma)}{\Gamma(\nu + \tau' - \varepsilon + \sigma)} t^{\nu+\nu'-\varepsilon+\sigma-1} \tag{5}$$

(d) If $\operatorname{Re}(\sigma) > \max\{\operatorname{Re}(-\tau'), \operatorname{Re}(\nu' + \tau - \varepsilon), \operatorname{Re}(\nu + \nu' - \varepsilon)\}$, then

$$(D_{-}^{\nu,\nu',\tau,\tau',\varepsilon} t^{-\sigma})(t) = \frac{\Gamma(\tau' + \sigma)\Gamma(-\nu - \nu' + \varepsilon + \sigma)}{\Gamma(\sigma)\Gamma(-\nu' + \tau' + \sigma)} \times \frac{\Gamma(-\nu' - \tau + \varepsilon + \sigma)}{\Gamma(-\nu - \nu' - \tau + \varepsilon + \sigma)} t^{\nu+\nu'-\varepsilon-\sigma} \tag{6}$$

Special G function :

The special $G_{\rho,\eta,\gamma}[a, z]$ is defined by [5,6] as

$$G_{\rho,\eta,\gamma}[a, z] = z^{\gamma\rho-\eta-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n (az^{\rho})^n}{\Gamma(n\rho + \rho\gamma - \eta) n!} \tag{7}$$

Generalized Mittag- Leffler Function

Gosta Mittag – Leffler the Swedish mathematician introduced the term Gosta Mittag – Leffler function i.e. , Mittag – Leffler function is defined [7] as

$$E_{\gamma}(d) = \sum_{n=0}^{\infty} \frac{(d)^n}{\Gamma(\gamma n + 1)} \qquad (d \in C; R(\gamma) > 0)$$

where is a gamma function , after this Wiman generalized the Mittag – Leffler function as follows,

$$E_{\gamma,\nu}(d) = \sum_{n=0}^{\infty} \frac{(d)^n}{\Gamma(\gamma n + \nu)} \qquad (d \in C; \min(R(\gamma) R(\nu)) > 0)$$

there are number of ways in which Mittag- Leffler function

$$E_{\lambda,\beta}^{\phi}[z]=\sum_{m=0}^{\infty}\frac{(\phi)_m}{\Gamma(\lambda m+\beta)}\frac{z^m}{m!} \tag{8}$$

where $\lambda,\beta,\phi\in C \quad (R(\lambda)>0)$

Product of G function and Mittag Leffler function

$$G_{\rho,\eta,\gamma}[a,z]\times E_{\lambda,\beta}^{\phi}[z]=z^{\gamma\rho-\eta-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n(az^{\rho})^n}{\Gamma(n\rho+\rho\gamma-\eta)n!}\sum_{m=0}^{\infty}\frac{(\phi)_m}{\Gamma(\lambda m+\beta)}\frac{z^m}{m!}$$

let m = n = k then

$$G_{\rho,\eta,\gamma}[a,z]\times E_{\lambda,\beta}^{\phi}[z]=z^{\gamma\rho-\eta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k(az^{\rho})^k}{\Gamma(k\rho+\rho\gamma-\eta)k!}\frac{(\phi)_k}{\Gamma(\lambda k+\beta)}\frac{z^k}{k!} \tag{9}$$

Fox – Wright Generalized Hypergeometric Function

In 1933, E.M. Wright defined a more interesting generalized hypergeometric function of one variable[8] and further generalizations of the series ${}_pF_q$ were given by Fox[9] and Wright [10,11,12];

$${}_p\psi_q(z)={}_p\psi_q\left[\begin{matrix}(\alpha_1,A_1),\ldots,\ldots(\alpha_p,A_p)\\(\beta_1,B_1),\ldots,\ldots(\beta_q,B_q)\end{matrix}\middle| z\right]$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+A_1n)\Gamma(\alpha_2+A_2n)\ldots\ldots\Gamma(\alpha_p+A_pn)}{\Gamma(\beta_1+B_1n)\Gamma(\beta_2+B_2n)\ldots\ldots\Gamma(\beta_q+B_qn)}\frac{z^n}{n!} \tag{10}$$

where the coefficients $A_1,\ldots\ldots,A_p\in R^+$ and $B_1,\ldots\ldots,B_q\in R^+$ such that

$$1+\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i\geq 0$$

for suitably bounded values of $|z|$. $\alpha_1,\alpha_2,\ldots,\alpha_p,\beta_1,\beta_2,\ldots,\beta_q$ are complex parameters.

The Fox- Wright function is a special case of the Fox – H function as [13]

$${}_p\psi_q\left[\begin{matrix}(\alpha_1,A_1),\ldots,\ldots(\alpha_p,A_p)\\(\beta_1,B_1),\ldots,\ldots(\beta_q,B_q)\end{matrix}\middle| z\right]=H_{p,q+1}^{1,p}\left[-z\middle|\begin{matrix}(1-\alpha_1,A_1),\ldots\ldots(1-\alpha_p,A_p)\\(1-\beta_1,B_1),\ldots\ldots(1-\beta_q,B_q)\end{matrix}\right] \tag{11}$$

3. Methods

Theorem 1

Let $\xi<1$, $\rho,\eta,\gamma,a,\sigma,\lambda,b\in C$ then for $\mathrm{Re}(\mu),\mathrm{Re}(\beta)$, $\mathrm{Re}(\rho),\mathrm{Re}(\eta),\mathrm{Re}(\xi)>0$

$$P_{0+}^{\mu,\xi,b}\left[z^{\sigma-1}G_{\rho,\eta,\gamma}[a,z]\times E_{\lambda,\beta}^{\phi}[z]\right]$$

$$=\frac{z^{\mu+\sigma+\gamma\rho-\eta-1}\Gamma\left(1+\frac{\mu}{1-\xi}\right)}{\left(b(1-\xi)\right)^{\sigma+\gamma\rho-\eta-1}}{}_3\psi_6\left[\begin{matrix}(\gamma,1),(\phi,1),(\sigma+\gamma\rho-\eta-1,\rho+1)\\(\rho\gamma-\eta,\rho),(0,1),(0,1),(1,1),(\beta,\lambda),\left(\frac{\mu}{1-\xi}+\sigma+\gamma\rho-\eta,(\rho+1)k\right)\end{matrix}\middle|\frac{az^{(\rho+1)}}{\left(b(1-\xi)\right)^{\rho+1}}\right] \tag{12}$$

Proof : Let I be the left hand side of (12) and on applying (9) we get

$$I = P_{0+}^{\mu,\xi,b}\left[z^{\sigma-1}\times\sum_{k=0}^{\infty}\frac{(\gamma)_k(\phi)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!k!\Gamma(\lambda k+\beta)}z^{\rho k+k+\gamma\rho-\eta-1}\right]$$

$$= \sum_{k=0}^{\infty}\frac{(\gamma)_k(\phi)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!k!\Gamma(\lambda k+\beta)}\times P_{0+}^{\mu,\xi,b}\{z^{\sigma+\rho k+k+\gamma\rho-\eta-1-1}\}$$

$$= \sum_{k=0}^{\infty}\frac{(\gamma)_k(\phi)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!k!\Gamma(\lambda k+\beta)}\times\frac{\Gamma(\sigma+\rho k+k+\gamma\rho-\eta-1)\Gamma\left(1+\frac{\mu}{1-\xi}\right)}{\Gamma\left(\frac{\mu}{1-\xi}+\sigma+\rho k+k+\gamma\rho-\eta-1+1\right)\left[b(1-\xi)\right]^{\sigma+\rho k+k+\gamma\rho-\eta-1}}z^{\mu+\sigma+\rho k+k+\gamma\rho-\eta-1}$$

after using equation (10) we get the right hand side of (12).

Corollary 1. The result of (12) can also be represented as Fox H function in the following manner.

$$P_{0+}^{\mu,\xi,b}\left[z^{\sigma-1}G_{\rho,\eta,\gamma}[a,z]\times E_{\lambda,\beta}^{\phi}[z]\right]$$

$$=\frac{z^{\mu+\sigma+\gamma\rho-\eta-1}\Gamma\left(1+\frac{\mu}{1-\xi}\right)}{\left(b(1-\xi)\right)^{\sigma+\gamma\rho-\eta-1}}$$

$$H_{3,7}^{1,3}\left[\frac{-az^{(\rho+1)}}{\{b(1-\xi)\}^{\rho+1}}\middle|\begin{matrix}(1-\gamma,1),(1-\phi,1),(\sigma+\gamma\rho-\eta,\rho+1)\\(0,1),(1-\rho\gamma-\eta,\rho),(1,1),(1,1),(0,1),(1-\beta,\lambda)\left(1-\frac{\mu}{1-\xi}-\sigma-\gamma\rho+\eta,(\rho+1)k\right)\end{matrix}\right] \tag{13}$$

Corollary 2. On taking $\phi=\beta=0$, the Generalized Mittag Leffler function reduces to the classical Mittag Leffler function and (12) becomes

$$P_{0+}^{\mu,\xi,b}\left[z^{\sigma-1}G_{\rho,\eta,\gamma}[a,z]\times E_{\lambda}[z]\right]$$

$$=P_{0+}^{\mu,\xi,b}\left[z^{\sigma-1}\times\sum_{k=0}^{\infty}\frac{(\gamma)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!\Gamma(\lambda k+1)}z^{\rho k+k+\gamma\rho-\eta-1}\right]$$

$$= \sum_{k=0}^{\infty}\frac{(\gamma)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!\Gamma(\lambda k+1)}\times P_{0+}^{\mu,\xi,b}\{z^{\sigma+\rho k+k+\gamma\rho-\eta-1-1}\}$$

$$= \sum_{k=0}^{\infty}\frac{(\gamma)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!\Gamma(\lambda k+1)}\times\frac{\Gamma(\sigma+\rho k+k+\gamma\rho-\eta-1)\Gamma\left(1+\frac{\mu}{1-\xi}\right)}{\Gamma\left(\frac{\mu}{1-\xi}+\sigma+\rho k+k+\gamma\rho-\eta-1+1\right)[b(1-\xi)]^{\sigma+\rho k+k+\gamma\rho-\eta-1}} z^{\mu+\sigma+\rho k+k+\gamma\rho-\eta-1}$$

$$=\frac{z^{\mu+\sigma+\gamma\rho-\eta-1}\Gamma\left(1+\frac{\mu}{1-\xi}\right)}{(b(1-\xi))^{\sigma+\gamma\rho-\eta-1}}\,{}_2\psi_4\left[\begin{matrix}(\gamma,1),(\sigma+\gamma\rho-\eta-1,\rho+1)\\(\rho\gamma-\eta,\rho),(0,1),(1,\lambda),\left(\frac{\mu}{1-\xi}+\sigma+\gamma\rho-\eta,(\rho+1)k\right)\end{matrix}\middle|\frac{az^{(\rho+1)}}{(b(1-\xi))^{\rho+1}}\right]$$

Theorem 2

Let $\nu,\nu',\tau,\tau',\varepsilon,\rho,\eta,\gamma,a,\sigma,\lambda,b\in C$ such that $\mathrm{Re}(\nu)>0,\rho,\eta,\gamma,a,\sigma,\lambda,b\in C$ then for $\mathrm{Re}(\mu),\mathrm{Re}(\beta),\mathrm{Re}(\rho),\mathrm{Re}(\eta)>0$

$$I_{0+}^{\nu,\nu',\tau,\tau',\varepsilon}\left[z^{\sigma-1}G_{\rho,\eta,\gamma}[a,z]\times E_{\lambda,\beta}^{\phi}[z]\right]$$

$$= z^{-\nu-\nu'+\varepsilon+\sigma+\gamma\rho-\eta-2}\,{}_5\psi_8\left[\begin{matrix}(\gamma,1),(\phi,1),(\gamma\rho+\sigma-\eta-1,\rho+1),(-\nu'+\tau'+\sigma+\gamma\rho-\eta-1,\rho+1),\\(-\nu'-\nu-\tau+\varepsilon+\sigma+\gamma\rho-\eta-1,\rho+1)\\(0,1),(0,1),(\rho\gamma-\eta,\rho),(\beta,\lambda),(1,1),(\tau'+\sigma+\gamma\rho-\eta-1,\rho+1),\\(-\nu-\nu'+\varepsilon+\sigma+\gamma\rho-\eta-1,\rho+1),(-\nu'-\tau+\varepsilon+\sigma+\gamma\rho-\eta-1,\rho+1)\end{matrix}\middle| az^{(\rho+1)}\right] \tag{14}$$

Proof : Let I be the left hand side of (14) and on applying (9) we get

$$I = I_{0+}^{\nu,\nu',\tau,\tau,\varepsilon}\left[z^{\sigma-1}\times\sum_{k=0}^{\infty}\frac{(\gamma)_k(\phi)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!k!\Gamma(\lambda k+\beta)}z^{\rho k+k+\gamma\rho-\eta-1}\right]$$

$$= \sum_{k=0}^{\infty}\frac{(\gamma)_k(\phi)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!k!\Gamma(\lambda k+\beta)}\times I_{0+}^{\nu,\nu',\tau,\tau,\varepsilon}\{z^{\sigma+\rho k+k+\gamma\rho-\eta-1-1}\}$$

After using (3) we get

$$= \sum_{k=0}^{\infty}\frac{(\gamma)_k(\phi)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!k!\Gamma(\lambda k+\beta)}\times\frac{\Gamma(\sigma+\rho k+k+\gamma\rho-\eta-1)\Gamma(-\nu'+\tau'+\sigma+\rho k+k+\gamma\rho-\eta-1)}{\Gamma(\tau'+\sigma+\rho k+k+\gamma\rho-\eta-1)\Gamma(-\nu-\nu'+\varepsilon+\sigma+\rho k+k+\gamma\rho-\eta-1)}\times$$
$$\frac{\Gamma(-\nu-\nu'-\tau+\varepsilon+\sigma+\rho k+k+\gamma\rho-\eta-1)}{\Gamma(-\nu'-\tau+\varepsilon+\sigma+\rho k+k+\gamma\rho-\eta-1)}z^{-\nu-\nu'+\varepsilon+\sigma+\rho k+k+\gamma\rho-\eta-1-1}$$

after using equation (10) we get the right hand side of (14).

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 5s (2024)

Theorem 3

Let $\nu,\nu^{'},\tau,\tau^{'},\varepsilon,\rho,\eta,\gamma,a,\sigma,\lambda,b\in C$ such that $\operatorname{Re}(\nu)>0,\rho,\eta,\gamma,a,\sigma,\lambda,b\in C$ then for $\operatorname{Re}(\mu),\operatorname{Re}(\beta),\operatorname{Re}(\rho),\operatorname{Re}(\eta)>0$

$$I_{-}^{\nu,\nu^{'},\tau,\tau^{'},\varepsilon}\left[z^{-\sigma}G_{\rho,\eta,\gamma}[a,z]\times E_{\lambda,\beta}^{\phi}[z]\right]$$

$$=z^{-\nu-\nu\;\;+\varepsilon+\sigma-\gamma\rho+\eta+1}{}_{5}\psi_{8}\left[\begin{matrix}(\gamma,1),(\phi,1),(-\tau-\gamma\rho+\sigma+\eta+1,-\rho-1),(\nu+\nu'-\varepsilon+\sigma-\gamma\rho+\eta+1,-\rho-1),\\(\nu+\tau^{'}-\varepsilon+\sigma-\gamma\rho+\eta+1,-\rho-1)\\(0,1),(0,1),(\rho\gamma-\eta,\rho),(\beta,\lambda),(1,1),(\sigma-\gamma\rho-\eta+1,-\rho-1),\\(\nu-\tau+\sigma-\gamma\rho+\eta+1,-\rho-1),(\nu+\nu^{'}+\tau^{'}-\varepsilon+\sigma-\gamma\rho+\eta+1,-\rho-1)\end{matrix}\middle|az^{-(\rho+1)}\right] \tag{15}$$

Proof : Let I be the left hand side of (15) and on applying (9) we get

$$I\quad=I_{-}^{\nu,\nu^{'},\tau,\tau^{'},\varepsilon}\left[z^{-\sigma}\times\sum_{k=0}^{\infty}\frac{(\gamma)_{k}(\phi)_{k}a^{k}}{\Gamma(k\rho+\rho\gamma-\eta)k!k!\Gamma(\lambda k+\beta)}z^{\rho k+k+\gamma\rho-\eta-1}\right]$$

$$=\sum_{k=0}^{\infty}\frac{(\gamma)_{k}(\phi)_{k}a^{k}}{\Gamma(k\rho+\rho\gamma-\eta)k!k!\Gamma(\lambda k+\beta)}\times I_{-}^{\nu,\nu^{'},\tau,\tau,\varepsilon}\{z^{-(\sigma-\rho k-k-\gamma\rho+\eta+1}\}$$

After using (4) we get

$$=\sum_{k=0}^{\infty}\frac{(\gamma)_{k}(\phi)_{k}a^{k}}{\Gamma(k\rho+\rho\gamma-\eta)k!k!\Gamma(\lambda k+\beta)}\times\frac{\Gamma(\tau^{'}+\sigma-\rho k-k-\gamma\rho+\eta+1)\Gamma(-\nu-\nu^{'}+\varepsilon+\sigma-\rho k-k-\gamma\rho+\eta+1)}{\Gamma(\sigma-\rho k-k-\gamma\rho+\eta+1)\Gamma(\nu-\tau+\sigma-\rho k-k-\gamma\rho+\eta+1)}\times$$
$$\frac{\Gamma(\nu+\tau^{'}-\varepsilon+\sigma-\rho k-k-\gamma\rho+\eta+1)}{\Gamma(\nu+\nu^{'}+\tau^{'}-\varepsilon+\sigma-\rho k-k-\gamma\rho+\eta+1)}z^{-\nu-\nu^{'}+\varepsilon+\sigma-\rho k-k-\gamma\rho+\eta+1}$$

after using equation (10) we get the right hand side of (17).

Theorem 4

Let $\nu,\nu^{'},\tau,\tau^{'},\varepsilon,\rho,\eta,\gamma,a,\sigma,\lambda,b\in C$ such that $\operatorname{Re}(\nu)>0,\rho,\eta,\gamma,a,\sigma,\lambda,b\in C$ then for $\operatorname{Re}(\mu),\operatorname{Re}(\beta),\operatorname{Re}(\rho),\operatorname{Re}(\eta)>0$

$$D_{0+}^{\nu,\nu^{'},\tau,\tau^{'},\varepsilon}\left[z^{\sigma-1}G_{\rho,\eta,\gamma}[a,z]\times E_{\lambda,\beta}^{\phi}[z]\right]$$

$$=z^{\nu+\nu\;\;-\varepsilon+\sigma+\gamma\rho-\eta-1}{}_{5}\psi_{8}\left[\begin{matrix}(\gamma,1),(\phi,1),(\gamma\rho+\sigma-\eta-1,\rho+1),(\nu-\tau+\sigma+\gamma\rho-\eta-1,\rho+1),\\(\nu+\nu^{'}+\tau^{'}-\varepsilon+\sigma+\gamma\rho-\eta-1,\rho+1)\\(0,1),(0,1),(\rho\gamma-\eta,\rho),(\beta,\lambda),(1,1),(-\tau+\sigma+\gamma\rho-\eta-1,\rho+1),\\(\nu+\nu^{'}-\varepsilon+\sigma+\gamma\rho-\eta-1,\rho+1),(\nu+\tau^{'}-\varepsilon+\sigma+\gamma\rho-\eta-1,\rho+1)\end{matrix}\middle|az^{(\rho+1)}\right] \tag{16}$$

Proof : Let I be the left hand side of (16) and on applying (9) we get

$$I \quad = D_{0+}^{\nu,\nu',\tau,\tau,\varepsilon}\left[z^{\sigma-1} \times \sum_{k=0}^{\infty} \frac{(\gamma)_k (\phi)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!k!\Gamma(\lambda k+\beta)} z^{\rho k+k+\gamma\rho-\eta-1} \right]$$

$$= \quad \sum_{k=0}^{\infty} \frac{(\gamma)_k (\phi)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!k!\Gamma(\lambda k+\beta)} \times D_{0+}^{\nu,\nu',\tau,\tau,\varepsilon}\{z^{\sigma+\rho k+k+\gamma\rho-\eta-1-1}\}$$

After using (5) we get

$$= \quad \sum_{k=0}^{\infty} \frac{(\gamma)_k (\phi)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!k!\Gamma(\lambda k+\beta)} \times \frac{\Gamma(\sigma+\rho k+k+\gamma\rho-\eta-1)\Gamma(\nu-\tau+\sigma+\rho k+k+\gamma\rho-\eta-1)}{\Gamma(-\tau+\sigma+\rho k+k+\gamma\rho-\eta-1)\Gamma(\nu+\nu'-\varepsilon+\sigma+\rho k+k+\gamma\rho-\eta-1)} \times$$
$$\frac{\Gamma(\nu+\nu'+\tau'-\varepsilon+\sigma+\rho k+k+\gamma\rho-\eta-1)}{\Gamma(\nu+\tau'-\varepsilon+\sigma+\rho k+k+\gamma\rho-\eta-1)} z^{\nu+\nu'-\varepsilon+\sigma+\rho k+k+\gamma\rho-\eta-1}$$

after using equation (10) we get the right hand side of (16).

Theorem 5 :

Let $\nu,\nu',\tau,\tau',\varepsilon,\rho,\eta,\gamma,a,\sigma,\lambda,b \in C$ such that $\mathrm{Re}(\nu)>0,\rho,\eta,\gamma,a,\sigma,\lambda,b\in C$ then for $\mathrm{Re}(\mu),\mathrm{Re}(\beta),\mathrm{Re}(\rho),\mathrm{Re}(\eta)>0$

$$D_{-}^{\nu,\nu',\tau,\tau',\varepsilon}\left[z^{-\sigma}G_{\rho,\eta,\gamma}[a,z]\times E_{\lambda,\beta}^{\phi}[z]\right]$$

$$= z^{\nu+\nu\ -\varepsilon+\sigma-\gamma\rho+\eta+1}\,{}_5\psi_8\left[\left.\begin{matrix}(\gamma,1),(\phi,1),(\tau'-\gamma\rho+\sigma+\eta+1,-\rho-1),(-\nu-\nu'+\varepsilon+\sigma-\gamma\rho+\eta+1,-\rho-1),\\(-\nu'-\tau+\varepsilon+\sigma-\gamma\rho+\eta+1,-\rho-1)\\(0,1),(0,1),(\rho\gamma-\eta,\rho),(\beta,\lambda),(1,1),(\sigma-\gamma\rho+\eta+1,-\rho-1),\\(-\nu'+\tau'+\sigma-\gamma\rho+\eta+1,-\rho-1),(-\nu-\nu'-\tau-\varepsilon+\sigma-\gamma\rho+\eta+1,-\rho-1)\end{matrix}\right| az^{-(\rho+1)}\right] \tag{17}$$

Proof : Let I be the left hand side of (17) and on applying (9) we get

$$I \quad = D_{-}^{\nu,\nu',\tau,\tau',\varepsilon}\left[z^{-\sigma} \times \sum_{k=0}^{\infty} \frac{(\gamma)_k (\phi)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!k!\Gamma(\lambda k+\beta)} z^{\rho k+k+\gamma\rho-\eta-1} \right]$$

$$= \quad \sum_{k=0}^{\infty} \frac{(\gamma)_k (\phi)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!k!\Gamma(\lambda k+\beta)} \times D_{-}^{\nu,\nu',\tau,\tau,\varepsilon}\{z^{-(\sigma-\rho k-k-\gamma\rho+\eta+1}\}$$

After using (6) we get

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_k (\phi)_k a^k}{\Gamma(k\rho + \rho\gamma - \eta) k! k! \Gamma(\lambda k + \beta)} \times \frac{\Gamma(-\tau + \sigma - \rho k - k - \gamma\rho + \eta + 1)\Gamma(\nu + \nu' - \varepsilon + \sigma - \rho k - k - \gamma\rho + \eta + 1)}{\Gamma(\sigma - \rho k - k - \gamma\rho + \eta + 1)\Gamma(-\nu' + \tau' + \sigma - \rho k - k - \gamma\rho + \eta + 1)} \times$$
$$\frac{\Gamma(-\nu' - \tau + \varepsilon + \sigma - \rho k - k - \gamma\rho + \eta + 1)}{\Gamma(-\nu - \nu' - \tau' + \varepsilon + \sigma - \rho k - k - \gamma\rho + \eta + 1)} z^{\nu + \nu' - \varepsilon + \sigma - \rho k - k - \gamma\rho + \eta + 1}$$

after using equation (10) we get the right hand side of (17).

4. Conclusion:

The Pathway fractional integral operator and MSM fractional operators can be used to construct multiple integral formulas by applying them to the product of special G function and Mittag Leffler function. Some corollaries are also derived from the main results as particular cases. This work also uses generalized Wright hypergeometric function to express the derived results.

Refrences

[1] Mathai, A. M.; “A pathway to matrix variate gamma and normal densities” *Linear Algebra and Its Applications*, 2005; 396, 317-328.

[2] Mathai, A. M. and Hauhold, H. J.; “On generalized distributions and pathways” *Physics Letters*, 2008; 372, 2109-2113.

[3] Mathai, A. M. and Hauhold, H. J.; “Pathway model , Superstatistics, T sallies statistics and a generalized measure of entropy”, *Physics A*, 2007; 375,110-122.

[4] Nair, S. S.; “Pathway fractional integration operator” *Fractional Calculus Applied Analysis*, 2009; 12(3), 237-252.

[5] Lorenzo, C. F. and Hartley, T. T.; “Initialized fractional calculus”, *International Journal of Applied Mathematics*, 2000; 3, 249-265.

[6] Lorenzo, C. F. and Hartley, T. T.; “Generalized functions for the fractional calculus”, *NASA , Tech Publication*, 1999; 209424, 1-17.

[7] Prabhakar, T. R.; “A Singular Integral Equation with a Generalized Mittag- Leffler function in the kernel”, *Yokohama Math. J,* 1971; 19, 171-183.

[8] Srivastava, H. M. and Karlson, P. W.; *Multiple Gaussian Hypergeometric Series, Chichester , Brisbone and Toronto, New York :* Halsted Press (Ellis Horwood Limited , Chichester), John Wiley and Sons, 1985.

[9] Fox, C.; “The Asymptotic Expansion of Generalized Hypergeometric Functions”, *Proc. London Math.* Soc., 1928; 27(2), 389-400.

[10] Wright, E. M.; “The Asymptotic Expansion of Integral Function Defined By Taylor Series”, *Philos. Trans. Roy. Soc. London,* ser. A, 1940; 238, 423-451.

[11] Wright, E. M.; “The Asymptotic Expansion of The Generalized Hypergeometric Function II”, *Proc. London Math. Soc.,* 1940; 46(2), 389-408.

[12] Wright, E. M.; “The Asymptotic Expansion of The Generalized Hypergeometric Function”, *J. London Math. Soc.,*1935; 10, 286-293.

[13] Mathai, A. M., Saxena, R. K. and Hauhold, H. J.; *The H-Function Theory and Application,* London; Springer New York Dordrecht, Heilberg, 2010.

[14] Kabra, S. and Nagar, H.; Composition of Pathway Integral operator on Generalized k-Wright function, *International Journal of Scientific Research and Review*, vol 8(6), (2019), 44-51.

[15] Saigo, M. and Maeda, N.; *More generalization of fractional calculus*, In: P. Rusev, I. Dimovski and Kiryakova V. (Eds.) Transform Methods and Special Functions , pp. 386-400, IMI-BAS, Sofia, Bulgaria, 1998.

[16] Kabra, S. and Nagar, H.; The p-k Extended Mittag-Leffler Function and Marichev-Saigo-Maeda Fractional Operators, *International Journal of Research and Innovation in Applied Science (IJRIAS)*, Volume IV , 2019.