

A Widespread Study of Inequalities Instigating Through Discrete Information Models

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Abstract:

The philosophy of inequalities has profoundly been established for explaining numerous optimizational problems encountered in mathematical sciences. The contemporary developments in computational mathematics have made it conceivable to compute enormous entities articulated in expressions of inequalities. Inequalities in information theory have been determined by the aspiration to elucidate communication theoretic problems. To disentangle such problems, the algebra of information was established and chain rules for entropy and mutual information were framed. The field of information theory participates with a critical protagonist in accepting and enumerating the communication of information in innumerable systems. Inequalities in information theory have appeared as influential implements to investigate and illustrate the restrictions and opportunities in information dispensation. The contemporary communiqué is an accurate step in the construction of information inequalities for the discrete probability distribution. We have prepared abundant inequalities concerning finite sequences of positive real numbers. The exceptional cases of these inequalities are definitely advantageous especially, in connection with innumerable measures of entropies and inaccuracy surviving in the literature of information theory.

Keywords: Entropy, Inaccuracy, Probability distribution, Concavity, Divergence model, Monte-Carlo simulation, Shannon's lemma, Increasing function.

1. Introduction

The well-accredited and prominent truthfulness about the Coding theory delivers the exploration of combination of codes through discrete probabilistic entropic models and makes dialogues in the direction of demonstrations in predictable disciplines. Shannon [33] well-thought-out the conjectural background upon bestowing the decisive establishment of entropy involved with the disconnected probability spaces. The predominantly well-acknowledged observation of probabilistic entropy planned by Shannon [33] amplified the literature of coding theory with the expedition of abundant entropic models. This entrenched progression prearranged the stone of discrete entropic model with agreeable properties.

It has been observed that in an experimentation dealing with the proclamation about probabilities of dissimilar events, two varieties of errors are plausible, explicitly one because of the nonappearance of adequate data or indistinctness in test results and other from erroneous data. Shannon's [33] entropic model can be second-handed to enlighten the error because of ambiguity only whereas the both types of errors can be explained by using a measure identified as measure of inaccuracy which ascertains applications in statistical inference and a concept anticipated by Kerridge [14].

We have the understanding that $\Gamma_n = \left\{ (p_1, p_2, \dots, p_n) : p_i \geq 0; i = 1, \dots, n; \sum_{i=1}^n p_i = 1 \right\}$ represent the assemblage of all disconnected possibility distributions with nonnegative elements and full support on a set with cardinality n and $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$. A possibility distribution $p_i \in \Gamma_n$ which is not degenerate is believed to be a nondegenerate probability distribution. In numerous circumstances, one has to deliver transactions with discrete probability distributions in which each element is a positive real number. Consequently, we prerequisite the subsequent sets:

$$\Gamma_n^* = \left\{ (p_1, p_2, \dots, p_n) : p_i > 0; i = 1, \dots, n; \sum_{i=1}^n p_i = 1 \right\}.$$

For any probability distribution $p_i \in \Gamma_n$, we indicate below some existing discrete entropic models:

The Shannon [33] entropy:

$$H(P) = - \sum_{i=1}^n p_i \log_2 p_i \quad (1.1)$$

The Renyi [30] entropy:

$$H_\alpha(P) = (1 - \alpha)^{-1} \log_2 \left(\sum_{i=1}^n p_i^\alpha \right), \quad \alpha > 0, \alpha \neq 1 \quad (1.2)$$

The Havrda-Charvat [8] entropy:

$$H^\alpha(P) = (1 - 2^{1-\alpha})^{-1} \left(1 - \sum_{i=1}^n p_i^\alpha \right), \quad \alpha > 0, \alpha \neq 1 \quad (1.3)$$

To make available the augmentation in the collected works of discrete entropic models, Parkash and Kakkar [23, 24] structured the investigations of abundant entropic models for the discrete probability spaces from demonstration point of observation and consequently enriched the texts of entropy models by the development of the succeeding manifestations of quantitative entropic models:

$${}_B S(P) = \frac{\sum_{i=1}^n p_i \beta^{\log_D p_i} - 1}{1 - \beta}, \quad \beta > 1 \quad (1.4)$$

$$S_{\beta}(P) = \frac{\beta^{\sum_{i=1}^n p_i \ln p_i} - 1}{1 - \beta}, \quad \beta > 1 \quad (1.5)$$

$$S_a^b(p) = \frac{a^{\frac{1}{b-1} \ln \left(\sum_{i=1}^n p_i^b \right)} - 1}{1 - a}, \quad a > 1, \quad 0 < b < 1 \quad (1.6)$$

There survives an enormous assemblage of entropic models but still expectedness ascends to communicate amplification in their text. Furthermore, there happens to be perceptible astonishingly strong connotation networking entropy and Chi-square distribution. To undertake this target, Parkash, Sharma and Singh [28] sketched a new ground-breaking discrete entropic model by the subsequent appearance:

$$H_{\alpha, \beta}(P) = \frac{1}{\beta - \alpha} \sum_{i=1}^n \left[1 - p_i^{(\beta - \alpha) p_i} \right]; \alpha \neq \beta, \beta - \alpha > 0 \quad (1.7)$$

By providing work for the newly created model (1.7), the authors enhanced the application area of *maximum entropy principle* subsequent to the knowledge of contingency tables. Recently, Parkash and Kumar [25] investigated and twisted a new-fangled entropic model and reflected its solicitations to abundant disciplines comprising probability theory and queueing theory. Additionally, the authors reflected a wide-ranging study of their innovative discrete entropic model along with its presentations to queueing theory.

Additionally, Huang and Zhang [11] conveyed an unanticipated clarification with orientation to Shannon's [33] mutual information and stressed that it has comprehensively been second-handed its functioning computation. Furthermore, the authors carried out numerical replication and acknowledged that their projected modus operandi were surprisingly wonderful with burgeoning convenience to numerous realistic and hypothetical problems. This is supplementary additional that the discrete entropy models discover marvelous applications in abundant many disciplines. Lenormand et al. [15] delivered the presentations of entropy grounded models in urban atmosphere and commented that describing and enumerating longitudinal inequalities through the urban background remains an assorted and secretive task which has been accelerated by the cumulative accessibility of enormous geolocated successions. The outcomes of their research results provided illustration that the attractiveness of a specified locality measured by entropy is a domineering descriptor of the socioeconomic position of the locality and can consequently be second-handed as a demonstration for multifarious socioeconomic indicators.

Saraiva, P. [31] made accessible temporary and unstructured summary to Shannon's [33] entropy comprising of particular belongings and provided the solicitations of the model in two divergent outlooks from what was in its commencement: biological diversity and a pioneering learning on student migration. Manzoor et al. [17] delivered the solicitations of entropy model in the persuasion of chemistry and mentioned that through the provocation of Shannon's [33] entropy, the graph entropies with topological indices have been fetching the information-theoretic magnitudes for quantifying the operative information of chemical graphs and multifaceted structures. Elgawad et al

[5] delivered the presentations of Shannon's [33] entropy in the arena of statistics comprising of order statistics and for some documented disseminations. Bulinski and Kozhevin [3] delivered the presentations of entropy function to procure concerns which can be made practical to the feature selection problems. Some additional pioneers who have publicized their concentration to study the discrete entropic models are Parkash and Mukesh [26, 27], Yuan et al. [39], Sholehkerdar et al. [35], Lu et al. [16], Gui et al. [7], Zhang and Shi [40], Hojjati et al. [10], Schwartz and LeCun [36], Stoyanov et al. [37] etc.

This fundamental perception of inaccuracy has been explained subsequently:

Assume that an experimenter states that the probability of the i^{th} outcome of the random experiment is q_i while the exact probability is p_i . Then, taking some convinced postulates, Kerridge [14] proved that the inaccuracy of the above declaration is given by the subsequent numerical appearance:

$$I(P; Q) = -\sum_{i=1}^n p_i \log_2 q_i. \quad (1.8)$$

If $q_i = 0$, $p_i = 0$ for some index i , then we adopt the convention $0 \log_2 0 := 0$. On the other hand, if $q_i = 0$ but $p_i > 0$ for some index i , then $-p_i \log_2 q_i = +\infty$. Consequently, the right hand side of (1.8) is no longer a nonnegative real number. The inaccuracy $I(p_i; q_i) = 0$ iff $p_i = q_i = 1$ for exactly one i , so that $p_j = q_j = 0$ for all j , $1 \leq i \neq j \leq n$ if such j 's exist. In order to ensure that the right hand side of (1.8) is a nonnegative real number, one way is to consider only those $p_i \in \Gamma_n$, $q_i \in \Gamma_n$ which have the property that $p_i = 0$ whenever $q_i = 0$. For instance, consider $(p_1, p_2, \dots, p_5) \in \Gamma_5$ and $(q_1, q_2, \dots, q_5) \in \Gamma_5$ where

$$p_1 = \frac{1}{5}, p_2 = 0, p_3 = 0, p_4 = \frac{4}{5}, p_5 = 0 \text{ and } q_1 = \frac{2}{3}, q_2 = 0, q_3 = 0, q_4 = \frac{1}{6}, q_5 = \frac{1}{6}.$$

Notice that, here, $p_i = 0$ whenever $q_i = 0$, $i = 2, 3$. Since $q_5 = \frac{1}{6}$, $p_5 = 0$, it follows that

$$I\left(\frac{2}{3}, 0, 0, \frac{1}{6}, \frac{1}{6}; \frac{1}{5}, 0, 0, \frac{4}{5}, 0\right) = +\infty \text{ whereas } I\left(\frac{1}{5}, 0, 0, \frac{4}{5}, 0; \frac{2}{3}, 0, 0, \frac{1}{6}, \frac{1}{6}\right) \text{ is a positive real number.}$$

If we consider the probability distributions $p_i \in \Gamma_n^*$, $q_i \in \Gamma_n^*$, then both $I(p_i; q_i)$ and $I(q_i; p_i)$ are nonnegative real numbers, not necessarily equal. However, if $p_i \in \Gamma_n$ but $q_i \in \Gamma_n^*$, then $I(p_i; q_i)$ is a nonnegative real number but $I(q_i; p_i)$ may not be a nonnegative real number. However, if $p_i \in \Gamma_n^*$, $q_i \in \Gamma_n^*$, then $I(p_i; q_i)$ is always a nonnegative real number and we write (1.8) as

$$I(p_i; q_i) = \sum_{i=1}^n p_i \log_2 \left(\frac{1}{q_i} \right). \quad (1.9)$$

The inaccuracy model (1.8) reduces to Shannon's [33] entropic model for $P = Q$, that is, $p_i = q_i$. From Kerridge's [14] innovative description of a measure of inaccuracy, three deep-seated properties of an inaccuracy model are:

- (i) $I(P : P)$ should be a measure of entropy of P
- (ii) $I(P : Q) \geq I(P : P)$ and equivalence insignia clutches only when $P = Q$
- (iii) $I(P : Q) - I(P : P)$ should exemplify some divergence measure

Kerridge's [14] measure of inaccuracy (1.8) can be viewed as a generalization of the thought of entropy. It has broadly been employed as a practical and constructive instrument for the measurement of error in experimental results and accordingly discovers applications in statistical inference. Different authors have anticipated innovative inaccuracy models for the reason that their applicability in statistics, coding theory and other associated fields are imminent. Some of these models are:

$$I_R(P : Q) = \frac{1}{\alpha - 1} \log \frac{\sum_{i=1}^n p_i^\alpha q_i^{1-\alpha}}{\sum_{i=1}^n p_i^\alpha}, \alpha \neq 1, \alpha > 0 \quad (1.10)$$

which is Renyi's [30] inaccuracy model.

$$I_{HC}(P : Q) = \frac{1}{\alpha - 1} \sum_{i=1}^n p_i^\alpha (q_i^{1-\alpha} - 1), \alpha \neq 1, \alpha > 0 \quad (1.11)$$

which is Havrda-Charvat's [8] inaccuracy model.

$$I^{\alpha, \beta}(P : Q) = \frac{1}{\alpha - \beta} \left[\sum_{i=1}^n p_i^\alpha (q_i^{1-\alpha} - 1) - \sum_{i=1}^n p_i^\beta (q_i^{1-\beta} - 1) \right], \alpha \neq \beta \quad (1.12)$$

which represents Sharma and Taneja's [34] inaccuracy model.

Furthermore, the subsequent well known practically functional inaccuracy models have been developed by Kapur [12]:

$$I_{K1}(P : Q) = \frac{1}{\beta - \alpha} \log \left(\frac{\sum_{i=1}^n p_i^\beta q_i^{1-\beta}}{\sum_{i=1}^n p_i^\beta} \cdot \frac{\sum_{i=1}^n p_i^\alpha}{\sum_{i=1}^n p_i^\alpha q_i^{1-\alpha}} \right), \alpha \leq 1, \beta \geq 1 \text{ or } \alpha \geq 1, \beta \leq 1 \quad (1.13)$$

$$I_{K2}(P : Q) = \sum_{i=1}^n p_i \ln \frac{p_i}{q_i} - \frac{1}{a} \sum_{i=1}^n (q_i + ap_i) \ln \frac{1 + ap_i / q_i}{1 + a} - \sum_{i=1}^n p_i \log p_i \\ + \frac{1}{a} \sum_{i=1}^n (1 + ap_i) \ln (1 + ap_i) - \frac{1}{a} (1 + a) \log (1 + a) \quad (1.14)$$

Sathar et al. [32] made investigations about the past inaccuracy model and consequently recommended nonparametric estimators for these models. The authors made rigorous study of the asymptotic properties of these estimators under convinced appropriate and reliability conditions. Additionally, the authors made comparisons for the performance of the projected estimators by employing Monte-Carlo simulation technique.

Different instigators anticipated new-fangled inaccuracy models for the reasons that of their applicability in statistics, coding theory and supplementary associated fields. Some pioneers who have made efforts for the characterizations and applications of inaccuracy models are Parkash and Taneja [29], Kapur [13], Molloy and Ford [18], Thapliyal and Taneja [38], Eskandarzadeh et al. [6], da Costa Bueno and Balakrishnan [4] etc.

2. Development of Inequalities Via Discrete Entropy and Inaccuracy Models

In this segment, we prerequisite some knowledge about real-valued concave functions demarcated on numerous intervals in R .

Definition 2.1. A function $\varphi:]a, b[\rightarrow R$, is supposed to be a twice differentiable concave function if it is twice differentiable in $]a, b[$ and

$$\varphi''(x) \leq 0 \text{ for all } x \in]a, b[. \quad (2.1)$$

Lemma 2.2. If a function $\varphi:]a, b[\rightarrow R$ is twice differentiable in $]a, b[$ and $\varphi''(x) \leq 0$ for all $x \in]a, b[$, then the subsequent inequality holds:

$$\varphi\left(\sum_{i=1}^n \lambda_i t_i\right) \geq \sum_{i=1}^n \lambda_i \varphi(t_i) \text{ for all } t_i \in]a, b[, \text{ and all } (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Gamma_n, n = 2, 3, \dots \quad (2.2)$$

Definition 2.3. A function $\varphi:]a, b[\rightarrow R$, is supposed to be a twice differentiable strictly concave function if it is twice differentiable in $]a, b[$ and

$$\varphi''(x) < 0 \text{ for all } x \in]a, b[. \quad (2.3)$$

Lemma 2.4. If a function $\varphi:]a, b[\rightarrow R$ is twice differentiable in $]a, b[$ and $\varphi''(x) < 0$ for all $x \in]a, b[$, then for all $t_i \in]a, b[$, and all $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \Gamma_n$, the succeeding inequality holds:

$$\varphi\left(\sum_{i=1}^n \lambda_i t_i\right) > \sum_{i=1}^n \lambda_i \varphi(t_i) \text{ unless } t_1 = t_2 = \dots = t_n. \quad (2.4)$$

Lemma 2.5. If a real-valued function φ is demarcated on $[a, b]$, $a \in R$, $b \in R$, $a < b$; and is (i) twice differentiable in $]a, b[$ (ii) $\varphi''(x) \leq 0$ for all $x \in]a, b[$ (iii) continuous from the right at a and from the left at b ; then (2.2) holds for all $t_i \in [a, b]$, and all $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \Gamma_n$.

The inequality (2.2) is acknowledged as the Jensen inequality for real-valued twice differentiable concave functions with domain $]a, b[$.

For definitions 2.1, 2.3 and Lemmas 2.2, 2.4 and 2.5, see Aczel and Daroczy [1]; Hardy, Littlewood and Polya [9].

The real-valued function $x \mapsto \log_2 x$, $x > 0$, is a twice differentiable strictly concave function demarcated on $]0, \infty[= \{x \in \mathbb{R} : 0 < x < \infty\}$.

In the forthcoming subsections of the paper, unless otherwise revealed, we shall suppose that $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, a_1, a_2, \dots, a_n$ and b_1, b_2, \dots, b_n are positive real numbers.

Result 2.6 ([22], p. 88).

With the above declared assumptions, the subsequent inequality grips:

$$\left(\sum_{i=1}^n x_i \right) \log_2 \left(\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n y_i} \right) \leq \sum_{i=1}^n x_i \log_2 \left(\frac{x_i}{y_i} \right) \quad (2.5)$$

for all integers $n \geq 1$. If $n=1$, then (2.5) holds only as an equality.

In the sequel, we have presented innumerable inequalities originating through discrete inaccuracy measures.

Theorem 2.7. With the above declared assumptions and $n \geq 1$ a specified integer, the succeeding inequality is permanently accurate:

$$\frac{\sum_{i=1}^n x_i \log_2 y_i}{\sum_{i=1}^n x_i} \leq \log_2 \left(\frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i} \right) \quad (2.6)$$

If $n=1$, then (2.6) holds as an equality. If $n \geq 2$, then the sign of equality in (2.6) holds only for the equivalence of y_i .

Proof. If $n=1$, then the sign of equality holds in (2.6) as each side of it equals $\log_2 y_1$. Now suppose $n \geq 2$. Then by the inequality (2.5), we acquire the subsequent communication:

$$\left(\sum_{i=1}^n x_i \right) \log_2 \left(\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i y_i} \right) \leq \sum_{i=1}^n x_i \log_2 \frac{x_i}{x_i y_i} \quad (2.7)$$

which, upon simplification, contributes with the subsequent manifestation:

$$\log_2 \left(\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i y_i} \right) \leq \frac{\sum_{i=1}^n x_i \log_2 \frac{1}{y_i}}{\sum_{i=1}^n x_i} \quad (2.8)$$

from which (2.6) follows immediately. The emblem of equivalence in (2.7) clutches only for the equivalence of y_i . Consequently, the sign of equality in (2.6) holds only for the equivalence of y_i .

The inequality (2.6) remains effective if x_i and y_i are nonnegative real numbers such that $\sum_{i=1}^n x_i > 0$,

$\sum_{i=1}^n y_i > 0$; and $x_i = 0$ for all those indices i for which $y_i = 0$ (if any). The reason is that for such indices i , $x_i y_i = 0$ and $x_i \log_2 y_i = 0 \log_2 0 := 0$ and, thus, both sides of the inequality (2.6) remain unchanged.

The inequality (2.6) remains valid if x_i and y_i are nonnegative real numbers such that $\sum_{i=1}^n x_i > 0$. In

this case, in addition to the assumptions mentioned in the above paragraph, one needs to assume

$$x(-\infty) = \begin{cases} 0 & \text{if } x = 0 \\ -\infty & \text{if } x > 0. \end{cases}$$

Now, we point out the usefulness of (2.6) in information theory.

Remarks. We assume that at least two elements among y_i are unequal.

(i) If $y_i = \frac{1}{x_i}$, $x_i > 0$, $n \geq 2$ an integer, then (2.6) provides the subsequent appearance:

$$\frac{\sum_{i=1}^n x_i \log_2 \left(\frac{1}{x_i} \right)}{\sum_{i=1}^n x_i} < \log_2 \left(\frac{n}{\sum_{i=1}^n x_i} \right) \quad (2.9)$$

Note that the left hand side of the inequality (2.9) may not be a nonnegative real number.

(ii) If $y_i = x_i$, $n \geq 2$ an integer, then (2.6) reduces to the succeeding inequality:

$$\frac{\sum_{i=1}^n x_i \log_2 x_i}{\sum_{i=1}^n x_i} < \log_2 \left(\frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i} \right) \quad (2.10)$$

In particular, if $x_i = p_i$, such that $p_i \in \Gamma_n^*$, then equations (2.10), (1.1) and (1.2) provide the subsequent manifestation:

$$H_1(P) > H_2(P) \quad (2.11)$$

where $H_2(P)$ denotes the Renyi's [30] entropy of order 2, $H_1(P)$ the Shannon's [33] entropy and (2.11) holds except for the equivalence of p_i .

Definition 2.8. A real number k' is said to be conjugate to a real number k , $k \neq 0$, $k \neq 1$ if $\frac{1}{k} + \frac{1}{k'} = 1$.

Result 2.9 (Holder's inequality [9]). Suppose a real number k' is conjugate to a real number k , $k \neq 0$, $k \neq 1$. With the above declared assumptions and for $n \geq 2$ integers, the succeeding inequalities are permanently accurate:

$$\sum_{i=1}^n a_i b_i < \left(\sum_{i=1}^n a_i^k \right)^{\frac{1}{k}} \left(\sum_{i=1}^n b_i^{k'} \right)^{\frac{1}{k'}} \quad (k > 1) \quad (2.12)$$

$$\sum_{i=1}^n a_i b_i > \left(\sum_{i=1}^n a_i^k \right)^{\frac{1}{k}} \left(\sum_{i=1}^n b_i^{k'} \right)^{\frac{1}{k'}} \quad (k < 1) \quad (2.13)$$

except when $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots = \frac{a_n}{b_n}$.

Theorem 3.0. Let $\alpha > 0$, $\alpha \neq 1$ be a given real constant. With the above declared assumptions and for $n \geq 2$ an integer, the following conclusions hold:

(I) If $\alpha > 1$, then the successive dissimilarities are perpetually correct:

$$\left(\sum_{i=1}^n x_i y_i^{\alpha-1} \right) \left(\sum_{i=1}^n y_i x_i^{\alpha-1} \right) < \left(\sum_{i=1}^n x_i^\alpha \right) \left(\sum_{i=1}^n y_i^\alpha \right) \quad (2.14)$$

except for the equivalence of $\frac{x_i}{y_i}$.

(II) If $0 < \alpha < 1$, then the succeeding inequalities are forever accurate:

$$\left(\sum_{i=1}^n x_i y_i^{\alpha-1} \right) \left(\sum_{i=1}^n y_i x_i^{\alpha-1} \right) > \left(\sum_{i=1}^n x_i^\alpha \right) \left(\sum_{i=1}^n y_i^\alpha \right) \quad (2.15)$$

except for the equivalence of $\frac{x_i}{y_i}$..

Proof. Let $\alpha > 1$. By Result 2.9, we acquire the succeeding communication:

$$\left. \begin{aligned} \text{(i)} \quad & \sum_{i=1}^n x_i y_i^{\alpha-1} < \left(\sum_{i=1}^n x_i^\alpha \right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^n y_i^\alpha \right)^{\frac{\alpha-1}{\alpha}} \\ \text{(ii)} \quad & \sum_{i=1}^n y_i x_i^{\alpha-1} < \left(\sum_{i=1}^n y_i^\alpha \right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^n x_i^\alpha \right)^{\frac{\alpha-1}{\alpha}} \end{aligned} \right\} \quad (2.16)$$

except for the equivalence of $\frac{x_i}{y_i}$. The inequality (2.14) follows from (2.16) (i) and (ii).

Now, consider the case when $0 < \alpha < 1$. In this case, $\alpha - 1 < 0$. Now, by Result (2.9)

$$\left. \begin{aligned} \text{(i)} \quad & \sum_{i=1}^n x_i y_i^{\alpha-1} > \left(\sum_{i=1}^n x_i^\alpha \right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^n y_i^\alpha \right)^{\frac{\alpha-1}{\alpha}} \\ \text{(ii)} \quad & \sum_{i=1}^n y_i x_i^{\alpha-1} > \left(\sum_{i=1}^n y_i^\alpha \right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^n x_i^\alpha \right)^{\frac{\alpha-1}{\alpha}} \end{aligned} \right\} \quad (2.17)$$

except for the equivalence of $\frac{x_i}{y_i}$. The inequality (2.15) follows from (2.17) (i) and (ii).

Lemma 3.1. Let $\alpha > 0$, $\alpha \neq 1$ be a given real constant. With the above acknowledged conventions and for $n \geq 2$, if

$$\sum_{i=1}^n y_i^\alpha < \sum_{i=1}^n x_i^\alpha, \quad (2.18)$$

then the following conclusions hold good:

(I) If $\alpha > 1$, then the subsequent inequalities are forever correct:

$$\sum_{i=1}^n x_i y_i^{\alpha-1} < \sum_{i=1}^n x_i^\alpha \quad (2.19)$$

(II) If $0 < \alpha < 1$, then the succeeding inequalities are persistently accurate:

$$\sum_{i=1}^n x_i y_i^{\alpha-1} > \sum_{i=1}^n x_i^\alpha. \quad (2.20)$$

Proof. If $\alpha > 1$, then (2.19) follows from (2.16) (i) and (2.18). If $0 < \alpha < 1$, then follows from (2.17) (i) and (2.18) by means of the fact that $\alpha - 1 < 0$. (2.20)

If (2.18) is replaced by the subsequent inequality

$$\sum_{i=1}^n x_i^\alpha < \sum_{i=1}^n y_i^\alpha \quad (2.21)$$

and $\alpha > 0$, $\alpha \neq 1$ is a real constant, then with the above acknowledged conventions and for $n \geq 2$, the succeeding inequalities are determinedly truthful:

$$\sum_{i=1}^n x_i y_i^{\alpha-1} < \sum_{i=1}^n y_i^{\alpha} \quad \text{if } \alpha > 1 \text{ and} \quad (2.22)$$

$$\sum_{i=1}^n y_i x_i^{\alpha-1} > \sum_{i=1}^n x_i^{\alpha} \quad \text{if } 0 < \alpha < 1. \quad (2.23)$$

The inequality (2.22) follows from (2.16) (i) and (2.21) in the case when $\alpha > 1$. If $0 < \alpha < 1$, then by Result 2.9, we acquire the subsequent inequation:

$$\sum_{i=1}^n y_i x_i^{\alpha-1} > \left(\sum_{i=1}^n y_i^{\alpha} \right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^n x_i^{\alpha} \right)^{\frac{\alpha-1}{\alpha}}. \quad (2.24)$$

The inequality (2.23) follows from (2.21) and (2.24).

Consider $n = 2, \alpha = 2$ and $x_1 = 2, x_2 = 3; y_1 = 3, y_2 = 2$. Then $x_1^2 + x_2^2 = 13 = y_1^2 + y_2^2$ and $y_1 x_1 + y_2 x_2 = 12$. Accordingly, we comprehend that, in common, (2.23) does not hold when $\alpha > 1$. Now consider the subsequent equation

$$\sum_{i=1}^n y_i x_i^{\alpha-1} = \sum_{i=1}^n x_i^{\alpha}. \quad (2.25)$$

Obviously, (2.25) holds if $x_i = y_i$. Here, too, let us take $n = 2, \alpha = 2; x_1 = 4, x_2 = 6; y_1 = 1, y_2 = 8$. Then $y_1 x_1 + y_2 x_2 = 52 = x_1^2 + x_2^2$. Consequently, (2.25) holds but $x_1 \neq y_1, x_2 \neq y_2$. This example provides demonstrations that when $\alpha > 1$, (2.25) may be true without being $x_i = y_i$.

Let $p_i \in \Gamma_n^*, q_i \in \Gamma_n^*, n \geq 1$ an integer. Nath [19, 20, 21] defined the subsequent inaccuracy of order α , $\alpha > 0, \alpha \neq 1$ as

$$I_{\alpha}(P; Q) = (1 - \alpha)^{-1} \log_2 \left(\sum_{i=1}^n p_i q_i^{\alpha-1} \right) \quad (2.26)$$

Obviously, $I_{\alpha}(P; P) = H_{\alpha}(P)$, the Renyi's [30] entropy of order α , $\alpha > 0, \alpha \neq 1$. In this common sense, the inaccuracy of order $\alpha, \alpha > 0, \alpha \neq 1$, demarcated by (2.26), is a generalization of the Renyi's [30] entropy of order $\alpha, \alpha > 0, \alpha \neq 1$.

$$\text{Also } \lim_{\alpha \rightarrow 1} I_{\alpha}(P; q) = - \sum_{i=1}^n p_i \log_2 q_i = I(P; Q) \quad (2.27)$$

Thus, the inaccuracy $I(P; Q)$, demarcated by Kerridge [14], may be regarded as the inaccuracy of order 1 and henceforward, $I(P; Q)$ may be written as $I_1(P; Q)$ depending upon the condition.

Let $n \geq 1$ be an integer, then from equations (1.1), (1.9) and Shannon's [2] Lemma, the subsequent inequality holds good:

$$I_1(P; Q) \geq H_1(P) \quad (2.28)$$

The sign of equality, in (2.28), holds good if $n=1$. If $n \geq 2$, then the sign of equivalence in (2.28) holds iff $p_i = q_i \forall i$. Also, $H_1(p) > 0$ for all $(p_1, p_2, \dots, p_n) \in \Gamma_n^*$ whenever $n \geq 2$. Hence (2.28) provides the subsequent manifestation:

$$I_1(P; Q) > 0 \quad (2.29)$$

for all distributions $p_i \in \Gamma_n^*$, $q_i \in \Gamma_n^*$ whenever $n \geq 2$. Also, with $n \geq 2$

$$\sum_{i=1}^n p_i q_i^{\alpha-1} = \sum_{i=1}^n \left(\frac{p_i}{q_i} \right) q_i^\alpha < 1 \text{ or } > 1 \text{ according as } \alpha > 1 \text{ or } 0 < \alpha < 1.$$

Henceforth, we acquire

$$I_\alpha(P; Q) > 0 \quad (2.30)$$

for all $p_i \in \Gamma_n^*$, $q_i \in \Gamma_n^*$ whenever $n \geq 2$. Notice that $I_\alpha(1; 1) = 0$.

Corresponding to (2.28), let us examine the succeeding inequality

$$I_\alpha(P; Q) \geq H_\alpha(P) \text{ when } \alpha > 0, \alpha \neq 1. \quad (2.31)$$

The insignia of egalitarianism holds good in equation (2.31) when $n=1$. If $n \geq 2$, then the insignia of egalitarianism in (2.31) holds iff $p_i = q_i$.

Now, consider the subsequent example:

Example 3.2. Take $n=2$, $p_1 = \frac{1}{2}, p_2 = \frac{1}{2}$; $q_1 = \frac{1}{4}, q_2 = \frac{3}{4}$ and $\alpha=3$. Then $H_3\left(\frac{1}{2}, \frac{1}{2}\right) = 1$ bit

whereas $I_3\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{4}, \frac{3}{4}\right) = \frac{1}{2} \log_2 \left(\frac{16}{5}\right) < H_3\left(\frac{1}{2}, \frac{1}{2}\right)$.

Accordingly, equation (2.31) does not hold for $n=2$ and $\alpha=3$. On the other hand,

$$H_{\frac{3}{2}}\left(\frac{1}{2}, \frac{1}{2}\right) = -2 \log_2(0.7071) \text{ and } I_{\frac{3}{2}}\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{4}, \frac{3}{4}\right) = -2 \log_2 0.6830.$$

Consequently, equation (2.31) holds for $n=2$ and $\alpha = \frac{3}{2}$ as $I_{\frac{3}{2}}\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{4}, \frac{3}{4}\right) > H_{\frac{3}{2}}\left(\frac{1}{2}, \frac{1}{2}\right)$.

However, $H_2\left(\frac{1}{2}, \frac{1}{2}\right) = I_2\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{4}, \frac{3}{4}\right) = 1$ bit.

Now, let us choose $k = \alpha$, $\alpha > 0$, $\alpha \neq 1$ and $a_i = p_i$, $b_i = q_i^{\alpha-1}$, such that $p_i \in \Gamma_n^*$, $q_i \in \Gamma_n^*$, $n \geq 2$ an integer. Then (2.16) (i) and (2.17) (i) condense respectively to the successive inequalities:

$$\sum_{i=1}^n p_i q_i^{\alpha-1} < \left(\sum_{i=1}^n p_i^\alpha \right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^n q_i^\alpha \right)^{\frac{\alpha-1}{\alpha}} \quad (\alpha > 1) \quad (2.32)$$

$$\sum_{i=1}^n p_i q_i^{\alpha-1} > \left(\sum_{i=1}^n p_i^\alpha \right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^n q_i^\alpha \right)^{\frac{\alpha-1}{\alpha}} \quad (0 < \alpha < 1) \quad (2.33)$$

which hold unless $p_i = q_i$. From equations (2.32), (2.33), (2.26) and (1.2), it follows that

$$H_\alpha(P) < \alpha I_\alpha(P; Q) + (1 - \alpha) H_\alpha(Q), \alpha > 0, \alpha \neq 1 \quad (2.34)$$

unless $p_i = q_i$.

Similarly,

$$H_\alpha(Q) \leq \alpha I_\alpha(Q; P) + (1 - \alpha) H_\alpha(P), \alpha > 0, \alpha \neq 1 \quad (2.35)$$

unless $p_i = q_i$.

From equations (2.34) and (2.35), we obtain the succeeding inequality [22]:

$$H_\alpha(P) + H_\alpha(Q) < I_\alpha(P; Q) + I_\alpha(Q; P) \quad (2.36)$$

valid for all $\alpha > 0, \alpha \neq 1$. Accordingly, we have evidenced the obligatory consequence.

Theorem 3.3. Let $\alpha > 0, \alpha \neq 1$ be a prearranged real constant; and $p_i \in \Gamma_n^*, q_i \in \Gamma_n^*, n \geq 2$ an integer. Then (2.36) holds for all $\alpha > 0, \alpha \neq 1$ unless $p_i = q_i$.

Now suppose that for $\alpha > 0, \alpha \neq 1$,

$$I_\alpha(P; Q) = H_\alpha(P) \quad (2.37)$$

Holds for $n \geq 2$ an integer. Then

$$\sum_{i=1}^n p_i q_i^{\alpha-1} = \sum_{i=1}^n p_i^\alpha. \quad (2.38)$$

Obviously, (2.38) holds good if $p_i = q_i$. On the additional hand, if we take $n = 2, \alpha = 2$,

$p_1 = \frac{1}{2}, p_2 = \frac{1}{2}, q_1 = \frac{1}{4}, q_2 = \frac{3}{4}$, then we acquire the succeeding manifestation:

$$\sum_{i=1}^2 p_i q_i^{2-1} = p_1^2 + p_2^2.$$

Accordingly (2.38) holds, when $n = 2$, but $p_1 \neq q_1$ and $p_2 \neq q_2$. Here

$$q_1^2 + q_2^2 = \frac{10}{16} > \frac{1}{2} = p_1^2 + p_2^2.$$

Let us pick out $x_i = p_i, y_i = q_i, i = 1, \dots, n$ and $n \geq 2$ an integer such that $p_i \in \Gamma_n^*, q_i \in \Gamma_n^*$. Let $\alpha > 0, \alpha \neq 1$, be a prearranged real constant. If $\sum_{i=1}^n q_i^\alpha < \sum_{i=1}^n p_i^\alpha$,

then making practice of Lemma 3.1, it monitors that

$$I_\alpha(P; Q) > H_\alpha(P).$$

If the probability distributions $p_i \in \Gamma_n^*, q_i \in \Gamma_n^*$ are such that $\sum_{i=1}^n p_i^\alpha < \sum_{i=1}^n q_i^\alpha; \alpha > 0, \alpha \neq 1$ being a specified real constant, then making practice of (2.21), (2.22), (2.23), (1.2) and (2.26), it follows that

$$I_\alpha(P; Q) > H_\alpha(Q) \quad \text{if } \alpha > 1$$

and

$$I_\alpha(Q; P) > H_\alpha(P) \quad \text{if } 0 < \alpha < 1.$$

Proposition 3.4. Let $\alpha > 0, \alpha \neq 1$ be a specified real constant and with the above acknowledged conventions and also for $n \geq 1$, the subsequent inequality always hold good:

$$\sum_{i=1}^n x_i y_i^{\alpha-1} \log_2 y_i^{1-\alpha} \leq \left(\sum_{i=1}^n x_i y_i^{\alpha-1} \right) \log_2 \left(\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i y_i^{\alpha-1}} \right) \quad (2.39)$$

If $n=1$, then (2.39) holds as an equality. If $n \geq 2$, then the sign of equality in (2.39) holds only for the equivalence in y_i .

Proof. If $n=1$, then the insignia of egalitarianism clutches in (2.39) as both the sides of it reduce to $(1-\alpha)x_1 y_1^{\alpha-1} \log_2 y_1$. Now consider $n \geq 2$. In this situation, by (2.5), we acquire the succeeding manifestation in the form of an inequality:

$$\left(\sum_{i=1}^n x_i \right) \log_2 \left(\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i y_i^{\alpha-1}} \right) \leq \sum_{i=1}^n x_i \log_2 \frac{x_i}{x_i y_i^{\alpha-1}} \text{ which, upon simplification, gives (2.39) with the}$$

insignia of equivalence only for the uniformity in y_i .

Next, we deliberate the significance of Proposition 3.4 in the field of information theory.

We demonstrate that the inaccuracy model $I_\alpha(P; Q)$ is a nonincreasing function of $\alpha, \alpha > 0, \alpha \neq 1$. Indeed, we have

$$\frac{d}{d\alpha} I_{\alpha}(P; Q) = \frac{1}{(1-\alpha)^2} \left[\log_2 \frac{\sum_{i=1}^n p_i q_i^{\alpha-1}}{\sum_{i=1}^n p_i} + \frac{\sum_{i=1}^n p_i q_i^{\alpha-1} \log_2 q_i^{\alpha-1}}{\sum_{i=1}^n p_i q_i^{\alpha-1}} \right] \quad (2.40)$$

If, in equation (2.39) we take $x_i = p_i$, $y_i = q_i$, such that $p_i \in \Gamma_n^*$, $q_i \in \Gamma_n^*$, then the term within brackets on the right hand side of (2.40) is a nonpositive real number.

Consequently, $\frac{d}{d\alpha} I_{\alpha}(P; Q) \leq 0$.

Hence, $I_{\alpha}(P; Q)$ is a nonincreasing function of α .

Now suppose that $q_i \in \Gamma_n^*$ has at least two unequal elements. Then, the term within brackets on the right hand side of (2.40) is a negative real number. Hence, $\frac{d}{d\alpha} I_{\alpha}(P; Q) < 0$.

Accordingly, $I_{\alpha}(P; Q)$ is a strictly decreasing function of α , $\alpha > 0$, $\alpha \neq 1$. In this case, we demonstrate that the succeeding inequalities grip:

$$I_1(P; Q) < I_{\alpha}(P; Q) \text{ if } 0 < \alpha < 1 \quad (2.41)$$

and

$$I_{\alpha}(P; Q) < I_1(P; Q) \text{ if } \alpha > 1. \quad (2.42)$$

Let $\alpha > 0$, $\beta > 0$, $\alpha \neq 1$, $\beta \neq 1$. Without any forfeiture of simplification, we may undertake that $\alpha < \beta$. Then

$$I_{\beta}(P; Q) < I_{\alpha}(P; Q) \quad (2.43)$$

Next, we distribute the above conversation into three circumstances:

Case 1. $0 < \alpha < \beta < 1$.

Letting $\beta \rightarrow 1^-$ in (2.43), equation (2.41) follows.

Case 2. $0 < \alpha < 1 < \beta$.

Letting $\beta \rightarrow 1^+$ in (2.43), equation (2.41) follows.

Case 3. $1 < \alpha < \beta$.

Letting $\alpha \rightarrow 1^+$ in (2.43) and writing α in place of β , equation (2.42) follows.

Both the measures $I_1(P; Q)$ and $I_{\alpha}(P; Q)$, $\alpha > 0$, $\alpha \neq 1$, are additive. Nath [21] furthermore recommended the the not-additive inaccuracy model specified by the consequent manifestation:

$$I^\alpha(P; Q) = (1 - 2^{1-\alpha})^{-1} \left(1 - \sum_{i=1}^n p_i q_i^{\alpha-1} \right) \quad (2.44)$$

where $\alpha > 0$, $\alpha \neq 1$ and $n \geq 1$ an integer. It is informal to authenticate that

$$\lim_{\alpha \rightarrow 1} I^\alpha(P; Q) = \sum_{i=1}^n p_i \log_2 \left(\frac{1}{q_i} \right) = I_1(P; Q).$$

Consequently, the additive inaccuracy $I_1(P; Q)$ is a limiting case of the inaccuracy $I^\alpha(P; Q)$ of order $\alpha > 0$, $\alpha \neq 1$ which is not additive.

For a prearranged real constant $\alpha > 0$, describe the function $g : R \rightarrow R$ as

$$g_\alpha(x) = \begin{cases} (1 - 2^{1-\alpha})^{-1} (1 - 2^{(1-\alpha)x}) & \text{if } \alpha > 0, \alpha \neq 1 \\ x & \text{if } \alpha = 1. \end{cases} \quad (2.45)$$

Then

$$g_\alpha[I_\alpha(P; Q)] = I^\alpha(P; Q), \quad \alpha > 0, \alpha \neq 1 \quad (2.46)$$

Thus, maximum number of properties of $I^\alpha(P; Q)$ may be consequential from those of $I_\alpha(P; Q)$ by exhausting the function g_α .

The inaccuracy model $I^\alpha(P; Q)$ is, certainly a generalization of the entropy model $H^\alpha(P)$, $\alpha > 0$, $\alpha \neq 1$, demarcated by (1.3).

3. Concluding remarks: Inequalities in information theory participate with an important accountability for the management of plentiful looked-for outcomes. These mathematical expressions provide assistance to researchers and specialists to quantify the boundaries of communication systems, coding structures, and information dispensation protocols. By instituting these constraints, inequalities monitor the project and optimization of communication arrangements, confirming effective and dependable information transfer. The investigation of inequalities facilitates the documentation of optimal coding approaches that maximize the rate of information relocation while minimizing the likelihood of errors. In turn, this has insightful consequences for the project of robust and protected communication arrangements in countless presentations, stretching from telecommunications to data packing. All these inequalities, the propositions and the definitions mentioned and demonstrated in the paper are worthwhile in the arena of information theory. All those inequalities which are effective for positive real numbers should be reflected as advantageous from theoretical point of understanding. Such inequalities can be demonstrated by employing additional discrete entropic and inaccuracy models.

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