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Numerous Determinants Identities Involving Jacobsthal and Jacobsthal Lucas Numbers

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Abstract:

Determinants have played an important role in many areas of mathematics. As an example, they are extremely useful in the research and resolution of linear equation and system problems. The study of determinants can be approached from several distinct angles. Throughout the course of this inquiry, we discover a large number of determinant identities involving Jacobsthal and Lucas numbers.

Keywords: Jacobsthal, Jacobsthal Lucas Numbers, Determinants.

1. Introduction

The Jacobsthal numbers possess significant features and are applied in number theory, graph theory, combinatorics, and geometry. While analyzing the relationship between determinants and permanents of two n-square upper Hessenberg matrices, one representing the adjacency matrix of a directed pseudograph, and derived sum formulas for the Jacobsthal sequence. Two higher Hessenberg matrices were analyzed, demonstrating that their permanents correspond to Jacobsthal numbers.

Some families of Toeplitz-Hessenberg matrices are determined by using different translations of the Jacobsthal numbers as the nonzero entries. The determinant identities can be expressed using the Trudi formula as equations that consist of combinations of Jacobsthal numbers and multinomial coefficients. Here, we present analogous results using the generalised Trudi formula. The initial column entries are adjusted and combinatorial proofs are offered in various instances.

As a consequence of this, the Jacobsthal polynomial, $J_n(1) = F_n$, and the Jacobsthal function, $j_n(x)$, are closely related (Jhala *et al*., 2013). It is possible to define the polynomial known as the Jacobsthal-Lucas polynomial for the function $j_n(x)$ with n equal to zero, provided that j_1 equals Ln. A. F. Horadam presented the paper "Horadam, Jacobsthal, Hoggatt, and Bicknell-Johnson Representation Numbers" in May 1994 at the University of New England in Armidale, which is in Australia. Specifically, the [j_n] and [Jn] sequences are of importance to us. The recurrence relations $j_{n+2} = j_{n+1} + 2j_n$, $j_0 = 2$, $j_1 = 1$, $n > 0$, and $J_{n+2} = J_{n+1} + 2 J_n$, $J_0 = 2$, $J_1 = 1$ respectively, define them so that they may be differentiated from one another. Each of the sequences is composed of a number of sets (Assaf E and Gueron S, 2001). These are the Jacobsthal numbers in the order that they are presented: 0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341... The series of JL numbers is comprised of eight numbers: 2, 1, 5, 7, 17, 31, 65, 127, 257, 511, and 1025. These numbers are all integers (Feng, 2011).

2. Fibonacci Numbers

A sequence in mathematics known as the Fibonacci numbers, which are represented by the symbol F_n, is characterized by the fact that each number is equal to the sum of its two preceding integers, beginning with 0 and 1. It is known that this sequence is unique (Yogesh Kumar Gupta and Omprakash Sikhwal Mamta Singh, 2016).

as the Fibonacci sequence. In other words, $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$ and

Binet formula for Fibonacci sequence

$$F_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n + (\frac{1-\sqrt{5}}{2})^n$$

The Fibonacci series and Cassini's formula

$F_nF_{n+1} - F_n^2 = (-1)^n$

Where Fn n^{th} Fibonacci number.

3. Lucas Numbers

In a manner analogous to that of the Fibonacci numbers, the Lucas number is defined as the product of its two terms directly preceding it. Upon completion of this process, an integer sequence in the style of Fibonacci will be produced. The first two numbers in the Fibonacci sequence are $F_0 = 0$ and $F_1 = 1$, whereas the first two numbers in the Lucas sequence are $L_0 = 2$ and $L_1 = 1$. In spite of the fact that they are based on the same idea, the Lucas numbers and the Fibonacci numbers are very different from one another (Horadam, 1971).

Consequently, the following is a description of the Lucas numbers:

$$L_n = \begin{cases} 2 & if \quad n = 0 \\ 1 & if \quad n = 1 \\ L_{n-1} + L_{n-2} & if \quad n > 1 \end{cases}$$

(In which n is a member of the natural numbers)

Numbers 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, and 199 make up the first twelve Lucas numbers.
Binet's formula for the Lucas number

$$L_n = (\frac{1+\sqrt{5}}{2})^n + (\frac{1-\sqrt{5}}{2})^n$$

4. Jacobsthal and Jacobsthal Lucas

It is necessary to make use of the following equations in order to establish the second-order recurrence relations and initial conditions for the Jacobsthal numbers, J_n, n > 0, as well as the JL numbers, j_n, n > 0. For Jacobsthal numbers, $J_{n+2} = J_{n+1} + 2Jn$, $J_0 = 0$, and $J_1 = 1$ one. For JL numbers, $J_{n+2} = J_{n+1} + 2J_n$, $J_0 = 2$, and $J_1 = 1$.

The following is a list of the formulas that govern these sequences, which are referred to as the Binet formulas: $j_n= 2^n - (-1)^n / 3$, and $J_n = 2^n + (-1)^n$. The following is a list of the roots of a characteristic equation that is associated with the values 2 and -1:

The recursive formulas that are presented here are as follows: If $j_0 = 0$ and $j_1= 1$, then $j_{n+2} = j_{n+1} + 2j_n$. On the other hand, if $J_0 = 2$ and $J_1 = 1$, then $J_{n+2} = J_{n+1} +2J_n$.

A representation of these sequences can be found in the Binet formulas, which are as follows: A second way to write $J_n= 2^n - (-1)^n / 3$, and the first way is as $2n + (-1)^n / 3$. Formulas like this are derived from the characteristic equation, which is a mathematical expression that describes the relationship between the numbers 2 and -1 (Campos *et al.*, 2014).

Identity 1.1:

Show that, for any number n≥0,

$$\begin{vmatrix} ɉ_{n+1} & ɉ_{n+2} & ɉ_{n+3} \\ ɉ_{n+4} & ɉ_{n+5} & ɉ_{n+6} \\ ɉ_{n+7} & ɉ_{n+8} & ɉ_{n+9} \end{vmatrix} = 0$$

Identity 1.2:

Show that, for any number n≥0,

$$\begin{vmatrix} ɉ_n - ɉ_{n+1} & ɉ_{n+1} - ɉ_{n+2} & ɉ_{n+2} - ɉ_n \\ .\ ɉ_{n+1} - ɉ_{n+2} & ɉ_{n+2} - ɉ_n & ɉ_n - ɉ_{n+1} \\ ɉ_{n+2} - ɉ_n & ɉ_n - ɉ_{n+1} & ɉ_{n+1} - ɉ_{n+2} \end{vmatrix} = 0$$

Identity1.3:

Show that, for any number n≥0,

$$\begin{vmatrix} 1 & 1 & 1 \\ ɉ_n & ɉ_{n+1} & ɉ_{n+2} \\ ɉ_{n+1} + ɉ_{n+2} & ɉ_n + ɉ_{n+2} & ɉ_n + ɉ_{n+1} \end{vmatrix} = 0$$

Identity1.4:

Show that, for any number n≥0,

$$\begin{vmatrix} ɉ_n & ɉ_n + ɉ_{n+1} & ɉ_n + ɉ_{n+1} + ɉ_{n+2} \\ 2ɉ_n & 2ɉ_n + 3ɉ_{n+1} & 2ɉ_n + 3\,ɉ_{n+1} + 4ɉ_{n+2} \\ 3ɉ_n & 3ɉ_n + 6ɉ_{n+1} & 3ɉ_n + 6ɉ_{n+1} + 12ɉ_{n+2} \end{vmatrix} = 3ɉ_nɉ_{n+1}ɉ_{n+2}$$

Identity1.5:

Show that, for any number n≥0,

$$\begin{vmatrix} 0 & ɉ_nɉ_{n+1}^2 & ɉ_nɉ_{n+2}^2 \\ ɉ_n^2ɉ_{n+1} & 0 & ɉ_{n+1}ɉ_{n+2}^2 \\ ɉ_n^2ɉ_{n+2} & ɉ_{n+2}ɉ_{n+1}^2 & 0 \end{vmatrix} = 2ɉ_n^3ɉ_{n+1}^3ɉ_{n+2}^3$$

Identity 1.6:

$$\begin{vmatrix} 2\jmath_n & 2\jmath_n & 1 \\ \jmath_{n+1} & \jmath_{n+1} & 1 \\ \jmath_{n+2} & \jmath_{n+2} & 1 \end{vmatrix} = 2[\jmath_n\jmath_{n+1} - \jmath_n\jmath_{n+1}]$$

Identity 1.7:

$$\begin{vmatrix} \jmath_n & \jmath_n & \jmath_n + \jmath_n \\ \jmath_{n+1} & \jmath_{n+1} & \jmath_{n+1} + \jmath_{n+1} \\ \jmath_{n+2} & \jmath_{n+2} & \jmath_{n+2} + \jmath_{n+2} \end{vmatrix} = 0$$

Identity 1.8:

$$\begin{vmatrix} \jmath_{n+1} + \jmath_{n+2} & \jmath_n & \jmath_n \\ \jmath_{n+1} & \jmath_{n+2} + \jmath_n & \jmath_{n+1} + \jmath_{n+1} \\ \jmath_{n+2} & \jmath_{n+2} & \jmath_n + \jmath_{n+1} \end{vmatrix} = 4\jmath_n\jmath_{n+1}\jmath_{n+2}$$

Identity1.9:

Show that, for any number n≥0,

$$\begin{vmatrix} 1 + \jmath_n & \jmath_{n+1} & \jmath_{n+2} \\ \jmath_n & 1 + \jmath_{n+1} & \jmath_{n+2} \\ \jmath_n & \jmath_{n+1} & 1 + \jmath_{n+2} \end{vmatrix} = 1 + \jmath_n + \jmath_{n+1} + \jmath_{n+2}$$

Identity1.10:

Show that, for any number n≥0,

$$\begin{vmatrix} \jmath_n & \jmath_n + \jmath_{n+1} & \jmath_n + \jmath_{n+1} + \jmath_{n+2} \\ 2\jmath_n & 3\jmath_n + 2\jmath_{n+1} & 4\jmath_n + 3\jmath_{n+1} + 2\jmath_{n+2} \\ 3\jmath_n & 6\jmath_n + 3\jmath_{n+1} & 10\jmath_n + 6\jmath_{n+1} + 3\jmath_{n+2} \end{vmatrix} = \jmath_n^3$$

5. Jacobsthal Lucas Numbers

$j_{n+2} = j_{n+1} + 2j_n$ is the recurrence relation that defines the JL Numbers. In this equation, n is a number that is higher than or equal to zero, and the beginning values are $j_0 = 2$ and $j_1 = 1$. For example, 2, 1, 5, 7, 17, 31, 65, 127, 257...

JL Numbers were the subject of discussion in this chapter, where we talked about certain determinants and results.

Identity1.11:

For each integer$n \geq 0$ prove that

$$\begin{vmatrix} j_n & j_{n+1} & j_{n+2} \\ j_{n+3} & j_{n+4} & j_{n+5} \\ j_{n+6} & j_{n+7} & j_{n+8} \end{vmatrix} = 0$$

Identity1.12:

Show that, for any number n≥0,

For each integer $n \geq 0$

$$\begin{vmatrix} j_n - j_{n+1} & j_{n+1} - j_{n+2} & j_{n+2} - j_n \\ j_{n+1} - j_{n+2} & j_{n+2} - j_n & j_n - j_{n+1} \\ j_{n+2} - j_n & j_n - j_{n+1} & j_{n+1} - j_{n+2} \end{vmatrix} = 0$$

Identity 1.13:

Show that, for any number n≥0,

For each integer $n \geq 0$ prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ j_n & j_{n+1} & j_{n+2} \\ j_{n+1} + j_{n+2} & j_n + j_{n+2} & j_n + j_{n+1} \end{vmatrix} = 0$$

Identity1.14:

Show that, for any number n≥0,

$$\begin{vmatrix} j_n & j_n + j_{n+1} & j_n + j_{n+1} + j_{n+2} \\ 2j_n & 2j_n + 3j_{n+1} & 2j_n + 3j_{n+1} + 4j_{n+2} \\ 3j_n & 3j_n + 6j_{n+1} & 3j_n + 6j_{n+1} + 12j_{n+2} \end{vmatrix} = 3j_n j_{n+1} j_{n+2}$$

Identity 1.15:

Show that, for any number n≥0,

$$\begin{vmatrix} 0 & j_n j_{n+1}^2 & j_n j_{n+2}^2 \\ j_n^2 j_{n+1} & 0 & j_{n+1} j_{n+2}^2 \\ j_n^2 j_{n+2} & j_{n+2} j_{n+1}^2 & 0 \end{vmatrix} = 2j_n^3 j_{n+1}^3 j_{n+2}^3$$

Identity1.16:

Show that, for any number n≥0,

$$\begin{vmatrix} j_{n+1} + j_{n+2} & j_n & j_n \\ j_{n+1} & j_{n+2} + j_n & j_{n+1} \\ j_{n+2} & j_{n+2} & j_n + j_{n+1} \end{vmatrix} = 4j_n j_{n+1} j_{n+2}$$

Identity 1.17:

Show that, for any number n≥0,

$$\begin{vmatrix} 1 + j_n & j_n & j_{n+2} \\ j_n & 1 + j_{n+1} & j_{n+2} \\ j_n & j_{n+1} & 1 + j_{n+2} \end{vmatrix} = 1 + j_n + j_{n+1} + j_{n+2}$$

Identity 1.18:

Show that, for any number n≥0,

$$\begin{vmatrix} j_n & j_n + j_{n+1} & j_n + j_{n+1} + j_{n+2} \\ 2j_n & 3j_n + 2j_{n+1} & 4j_n + 3j_{n+1} + 2j_{n+2} \\ 3j_n & 6j_n + 3j_{n+1} & 10j_n + 6j_{n+1} + 3j_{n+2} \end{vmatrix} = j_n^3$$

Identity1.19:

Show that, for any number n≥0,

$$\begin{vmatrix} j_n - j_{n+1} - j_{n+2} & 2j_n & 2j_n \\ 2j_{n+1} & j_{n+1} - j_{n+2} - j_n & 2j_{n+1} \\ 2j_{n+2} & 2j_{n+2} & j_{n+2} - j_n - j_{n+1} \end{vmatrix} = (j_n + j_{n+1} + j_{n+2})^3$$

Identity 1.20:

Show that, for any number n≥0,

$$\begin{vmatrix} 1 & j_n & j_n^2 \\ j_n^2 & 1 & j_n \\ j_n & j_n^2 & 1 \end{vmatrix} = (1 - j_n^3)^2$$

6. J& JL numbers using of the Matrix Method,

The Jacobsthal sequence j_n is comprised of fifteen different elements, which are as follows: 0, 1, 3, 5, 11, 21, 43, 171, 341, 683, 1365, 2731, 5461... $j_n = 2^n - (-1)^n / 3$ is one mathematical representation of the sequence that can be used to represent it.

The first ten terms in the JL numbers, denoted by the symbol "J_n," are as follows: 2, 1, 5, 7, 17, 31, 65, 127, 257, and 511.

For the purpose of defining the sequence, the formula $J_n = 2^n + (-1)^n$ is used.

The following is an expression of the formulas with regard to J and JL sequences:

$$J_{n+1} J_{n-1} - J_n^2 = (-1)^n 2^{n-1}$$

$$j_{n+1} j_{n-1} - j_n^2 = 9(-1)^{n+1} 2^{n-1}$$

We define the Jacobsthal K-Matrix as

$K=\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ It is evident.

$\begin{bmatrix} J_{n+1} \\ J_n \end{bmatrix} = K \begin{bmatrix} J_n \\ J_{n-1} \end{bmatrix}$ and

$\begin{bmatrix} j_{n+1} \\ j_n \end{bmatrix} = K \begin{bmatrix} j_n \\ j_{n-1} \end{bmatrix}$ where J_{n+1} denotes the n+1 th JL number and j_(n+1) stands for the JL number.

Identity2.1:

Let K be a 2 X 2 matrix in (1) then $K^n = \begin{bmatrix} Ɉ_{n+1} & 2Ɉ_n \\ Ɉ_n & 2Ɉ_{n-1} \end{bmatrix}$ ------------------- (a)

Proof:

We prove the Identity by mathematical induction

Let n =1 then

K = $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} Ɉ_2 & 2Ɉ_1 \\ Ɉ_1 & 2Ɉ_0 \end{bmatrix}$ so the above result is true

Assuming the outcome is valid for n, ie$K^n = \begin{bmatrix} Ɉ_{n+1} & 2Ɉ_n \\ Ɉ_n & 2Ɉ_{n-1} \end{bmatrix}$

We will now demonstrate that the outcome is valid for n+1 $K^{n+1} = K^n.K$

$$= \begin{bmatrix} Ɉ_{n+1} & 2Ɉ_n \\ Ɉ_n & 2Ɉ_{n-1} \end{bmatrix} . \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} Ɉ_{n+1} + 2Ɉ_n & 2Ɉ_{n+1} \\ Ɉ_n + 2Ɉ_{n-1} & 2Ɉ_n \end{bmatrix} = \begin{bmatrix} Ɉ_{n+2} & 2Ɉ_{n+1} \\ Ɉ_{n+1} & 2Ɉ_n \end{bmatrix}$$

and the result follows.

Corollary:

For all positive integers n,

(i) $det(K^n) = (-2)^n$

(ii) $Ɉ_{n+1}Ɉ_{n-1} - {Ɉ_n}^2$ = $(-1)^n 2^{n-1}$ cassini's like formula

References

[1] Assaf E and Gueron S. Characterization of regular Diophantine quadruples, Elem. Math. 2001; 56:71-81.

[2] Campos. H, Catarino. P, A.P. Aires, P. Vasco and A. Borges, On some identities of k-Jacobsthal-Lucas numbers, Int. Journal of Math. Analysis, 2014: 8(10); 489-494.

[3] Feng A. Fibonacci identities via determinant of tridiagonal matrix, Applied Mathematics and Computation, 2011: 217, pp. 5978—5981.

[4] Jhala. D, Sisodiya .K and G.P.S. Rathore, On some identities for k-Jacobsthal numbers, Int. Journal of Math. Analysis, 2013: 7(12); 551-556

[5] A. F. Horadam, *Pell identities*. Fibonacci Quart. 1971: 9 (3);245-263.

[6] Yogesh Kumar Gupta and Omprakash Sikhwal Mamta Singh. Determinantal Identities of Fibonacci Lucas and Generalized Fibonacci Lucas Sequence, MAYFEB Journal of Mathematics, 2016: Vol 2;pg no.17-23