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# On Certain Fixed Points for $(\alpha,\varphi,\mathcal{F})$-Contraction on $S_b$- Metric Spaces with Applications

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**Abstract:** This work establishes unique fixed point theorems (UFPT) for self mapping in complete S_b-metric spaces (S_b-MS) with the concept of (α,φ,F)-contraction in the context of S_b-MS Furthermore, we show how the results may be used and present applications to integral equations and homotopy theory.

**Introduction**: In previous work authors were discussed fixed pointon various metric spaces with F -contractions, α-type almost -F- contractions, α-type F -Suzuki contractions, (φ, F)-contraction, F - weak contractions, α −ψ-contractive type, α −ψ-Meir-Keeler contractive mapping, α-ratonal contractive mappings, (α, β) − (φ,ψ)-rational contractive type mappings.

**Objectives**: Finding the unique fixed point for self mapping in S_b-MS, via (α,φ,F)-contraction.

**Methods**: with the help of α- admisible mapping, (φ, F)-contraction, α type F -contraction and (α.φ,F)-contraction we have fixed point findings in complete S_b-metric spaces.

**Results**: we obtained unique fixed point results via (α,φ,F) contractive type for self mapping in complete S_b-MS.

**Conclusions**: In this article, Contractive mappings of the (α,φ,F) type are used to show certain fixedpoint results in the context of S_b-MS, along with appropriate example that illustrate the key findings. Appications to integral equations and homotopy are also offered.

**Keywords**: $S_b$ −metric spaces, α-admissible mappings,$(\alpha,\varphi,\mathcal{F})$ contraction and completeness.

## 1. Introduction

Fixed point theory is a crucial topic of non-linear analysis. Numerous types of equations that exist in natural, biological, social, engineering, and other branches of science and technology are studied in order to understand their underlying relevance. Examining the situations in which single or multi-valued mappings have solutions is a common application of this technique. In 1922, S. Banach.[1] introduced the notion of Banach contraction principal. It is

most fundamental tool in nonlinear analysis and some results related with generalization of various types of metric spaces (See [2]-[6]). Recently Sedghi et. al. [7] defined $S_b$-metric spaces by using the concept of S and b-metric spaces and established common fixed point results in $S_b$-metric spaces. Subsequently to improve many author established so many results on $S_b$-metric spaces (See [8]-[13]). In 2012, Wardowski [15] introduced the concept of $\mathcal{F}$ -contractions which plays a crucial part in the recent trend of research in fixed point theory.
After that, Wardowski and Dung [16] and Dung and Hang [17] extended the concept of $\mathcal{F}$ -contractions to $\mathcal{F}$ -weak contractions and generalized F-contractions respectively.By mixing up the concept of α-admissible mappings with $\mathcal{F}$-contractions [15] and $\mathcal{F}$ -weak contractions [16], Gopal et al. [18] introduced the concept of α-type $\mathcal{F}$ -contractions and α-type F-weak cont ractions and Hafida Massit et al.[19] introducing (φ, $\mathcal{F}$)-contraction to inspired by the work done in ([20], [21]) and proved some fixed point results in $C^*$ -algebra valued metric spaces. Subsequently, this type contractions extended or generalized by various researchers (See.([22])-([25])). The purpose of the current paper is to provide unique fixed point theorems for $(\alpha,\varphi,\mathcal{F})$-type contractive mappings in the context of complete $S_b$-metric spaces. We can also provide examples that are appropriate and relevant applications to integral equations as well as Homotopy.

## 2. Objectives

In this article we have proved the existence of unique fixed point solution to self mapping of $S_b$-metric space using $(\alpha,\varphi,\mathcal{F})$-contraction and example provided which supports our main results and also shown our results for nonlinear integral equations and Homotopy theory.

## 3. Methods

**Definition 3.1** Let G be the set that is nonempty and assume that $k \geq 1$ is a real number. Consider a function $S_b$: $G^3 \to [0\ \infty)$ that satisfies the following conditions:

$(S_{b_1})$ $0 < S_b(\Delta_1,\Delta_2,\Delta_3)$ *for all* $\Delta_1,\Delta_2,\Delta_3 \epsilon$ G with $\Delta_1 \neq \Delta_2 \neq \Delta_3$

$(S_{b_2})$ $S_b(\Delta_1,\Delta_2,\Delta_3) = 0 \Leftrightarrow \Delta_1 = \Delta_2 = \Delta_3$

$(S_{b_3})$ $S_b(\Delta_1,\Delta_2,\Delta_3) \leq k(S_b(\Delta_1,\Delta_1,\Delta_4) + S_b(\Delta_2,\Delta_2,\Delta_4) + S_b(\Delta_3,\Delta_3,\Delta_4))$

*for all* $\Delta_1,\Delta_2,\Delta_3 \epsilon$ G. Then the pair (G, $S_b$) is referred as a $S_b -$ MS and the function $S_b$ referred as a $S_b$ metric on G.

**Example 3.1** Define $S_*$: $G^3 \to [0\ \infty)$ by $S_*(\nabla_1,\nabla_2,\nabla_3) = (S(\nabla_1,\nabla_2,\nabla_3))^t$ $when\ k = 2^{2(t-1)}$ then $S_*$ $is$ $S_b -$ MS.

**Definition 3.2** Assuming a space (G, $S_b$) is a $S_b -$ MS. It is argued that a sequence $\{\zeta_p\}$ is a sequence in G.

(i) If $p_0$ in N exists such that $S_b(\zeta_p,\zeta_p,\zeta_q) <\in$ each p, q $\geq p_0$, then the sequence is known as the $S_b$-Cauchy sequence.
(ii) $S_b$ is convergent to a point ζ in G, if for each $\in > 0, \exists\ p_0 \in N$ such that

$S_b\left(\zeta_p, \zeta_p, \zeta\right) < \in \forall\, p \geq p_0$ and write is as $\lim_{p\to\infty} \zeta_p = \zeta$.

(iii) An $S_b$-metric space (G, $S_b$) is said to be complete, if every $S_b$-cauchy sequence must be $S_b$ –convergent in G.

**Lemma 3.1** In $S_b$-metric space, we have

$$S_b(\zeta,\zeta,\aleph) \leq kS_b(\aleph,\aleph.\zeta) and\, S_b(\aleph,\aleph.\zeta) \leq S_b(\zeta,\zeta,\aleph)$$

**Lemma 3.2** In $S_b$-metric space, we have

$$S_b(\zeta,\zeta,\aleph) \leq 2kS_b(\zeta,\zeta,ð\,) + k^2 S_b(ð,ð,\aleph)$$

**Lemma 3.3** Given that $\{\alpha_p\}$ is $S_b$ convergent to $\alpha$ and that (G, $S_b$) is $S_b$ metric space k ≥ 1, then

i. $\frac{1}{2k} S_b(\beta,\beta,\alpha) \leq \lim_{p\to\infty} inf\, S_b\left(\beta,\beta,\alpha_p\right) \leq \lim_{p\to\infty} sup\, S_b\left(\beta,\beta,\alpha_p\right) \leq 2kS_b(\beta,\beta,\alpha)$

ii. $\frac{1}{k^2} S_b(\alpha,\alpha,\beta) \leq \lim_{p\to\infty} inf\, S_b\left(\alpha_p,\alpha_p,\beta\right) \leq \lim_{p\to\infty} sup\, S_b\left(\alpha_p,\alpha_p,\beta\right) \leq k^2 S_b(\alpha,\alpha,\beta)$

**Definition 3.3**

Let $\Gamma$:G → G be a self mapping and $\alpha$: G × G × G → $R^+$ be a function then $\Gamma$ is called $\alpha$-admisible mapping, if $\alpha(\ell,\ell,\zeta) \geq 1 \Rightarrow \alpha(\Gamma\ell, \Gamma\ell, \Gamma\zeta) \geq 1 \quad \forall \ell,\zeta \in G$.

**Definition 3.4** Let $\Gamma$: G→ $G$ be a self mapping and

$\alpha$: G × G × G → $R^+$ are two mappings defined on a nonempty set G, $\Gamma$ is called triangular $\alpha$-admisible mapping, if

(i) $\Gamma$ is an $\alpha$- admisible mapping.

(ii) $\alpha(\ell,\ell,\zeta) \geq 1, \alpha(\zeta,\zeta,\aleph) \geq 1 \Rightarrow \alpha(\ell,\ell,\aleph) \geq 1 \quad \forall \ell,\zeta,\aleph \in G$.

**Definition 3.5**

Let $\chi = \{\mathcal{F} \,/\, \mathcal{F}{:}R^+ \to R\}$ and $\mathcal{L} = \{\varphi/\varphi{:}\,(0,\infty) \to (0,\infty)\}$ be a family of mappings satisfying

(a) For ${\ell_1}^1, {\ell_2}^1 \in R^+$ such that ${\ell_1}^1 < {\ell_2}^1, \mathcal{F}({\ell_1}^1) < \mathcal{F}({\ell_2}^1)$ (i.e., $\mathcal{F}$ is strictly increasing);

(b) $\lim_{p\to\infty} \chi_p = 0$ if and if $\lim_{p\to\infty} \mathcal{F}\left(\chi_p\right) = -\infty$ ;

(c) $\lim_{\ell\to{\ell_1}^1} inf\ \varphi(\ell) > 0$ for all $\ell > 0$;

(d) there exist $\lambda \in (0,1)$ such that $\lim_{\beta\to 0^+} \beta^\lambda \mathcal{F}(\beta) = 0$;

(e) $\mathcal{F}$ is continuous on (0,∞).

**Definition 3.6**

Assume that $\Gamma$: G→ $G$ is a mapping and (G, d) is a complete metric space then $\Gamma$ is called a $(\varphi,\mathcal{F})$-contraction if there exist $\varphi \in \mathcal{L}$, $\mathcal{F} \in \chi$ with

$$d(\Gamma\ell, \Gamma\zeta) > 0 \Rightarrow \varphi\big(d(\ell,\zeta)\big) + \mathcal{F}\,(d(\Gamma\ell, \Gamma\zeta)) \leq \mathcal{F}\big(d(\ell,\zeta)\big) \ \forall\, \ell,\zeta \in G$$

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**Definition 3.7**

Assuming that (G, d) is a metric space, $\Gamma: G \to G$ and $\alpha: G \times G \to R^+$ be the functions,

if there exists $\theta > 0$ such that for any $\ell, \zeta \in G$ and $\digamma \in \chi$

$d(\Gamma\ell, \Gamma\zeta) > 0 \Rightarrow \theta + \alpha(\ell,\zeta)\ (\mathcal{F}(d(\Gamma\ell, \Gamma\zeta)) \leq \mathcal{F}(d(\ell,\zeta))\ \forall\, \ell, \zeta \in G.$

then it is claimed that the function $\Gamma$ is called an $\alpha$ type $\mathcal{F}$ -contraction.

## 4. Result

In this part, we will demonstrate some basic fixed point theorems for
Self-mapping using novel contractive condition of the $(\alpha.\varphi,\digamma)$ -type in $S_b - MS$.

**Definition 4.1** Let $\Gamma: G \to G$ be a mapping and $(G, S_b)$ be an $S_b - MS$. Assume that $\mathcal{F} \in \chi$, $\varphi \in \mathcal{L}$ and $\alpha: G^3 \to R^+$ are functions such that for $\ell, \zeta \in G$ with j= 3,4,5, then we say that $\Gamma$ is $(\alpha.\varphi,\mathcal{F})$-contraction if

$$S_b(\Gamma\ell, \Gamma\ell, \Gamma\zeta) > 0 \Rightarrow \varphi(S_b(\ell,\ell,\zeta)) + \alpha(\ell,\ell,\zeta)\, \digamma(S_b(\Gamma\ell, \Gamma\ell, \Gamma\zeta)) \leq \mathcal{F}\left(\frac{1}{2k^4} M_s^j(\ell,\zeta)\right) \quad (4.1.1)$$

Where

$$M_s^3((\ell,\zeta)) = \max\left\{S_b(\ell,\ell,\zeta), \frac{S_b(\ell,\ell,\Gamma\ell)+S_b(\zeta,\zeta,\Gamma\zeta)}{2k^4}, \frac{S_b(\ell,\ell,\Gamma\zeta)+S_b(\zeta,\zeta,\Gamma\ell)}{2k^4}\right\}$$

$$M_s^4((\ell,\zeta)) = \max\left\{S_b(\ell,\ell,\zeta), S_b(\ell,\ell,\Gamma\ell), S_b(\zeta,\zeta,\Gamma\zeta)), \frac{S_b(\ell,\ell,\Gamma\zeta)) + S_b(\zeta,\zeta,\Gamma\ell)}{2k^4}\right\}$$

$$M_s^5((\ell,\zeta)) = \max\{S_b(\ell,\ell,\zeta), S_b(\ell,\ell,\Gamma\ell), S_b(\zeta,\zeta,\Gamma\zeta)), S_b(\ell,\ell,\Gamma\zeta)), S_b(\zeta,\zeta,\Gamma\ell)\}$$

**Theorem 4.2**

A $CS_b$MS with coefficient $k > 1$ is defined as $(G, S_b)$.Let $\Gamma: G \to G$ be a mapping of the type $(\alpha.\varphi,\mathcal{F})$-contraction with j = 5, where $\mathcal{F} \in \chi$, $\varphi \in \mathcal{L}$. Assume that each of the following statements is true.

$(i_0)$ $\Gamma$ is an $\alpha$- admisible mapping,

$(i_1)$ there exists $\ell_0 \in G$ such that $\alpha(\ell_0, \ell_0, \Gamma\ell_0) \geq 1$,

$(i_2)$ If $\{l_p\}$ is sequence in G with $\alpha(\ell_p, \ell_p, \ell_{p+1}) \geq 1$ for all positive integers p and

$\ell_p \to v$ as $p \to \infty$, we have $\alpha(\ell_p, \ell_p, v) \geq 1$ for all $p \in N$

Then, in G, there is a unique fixed point of $\Gamma$.

Proof: According to the theory, there are $\zeta_0 \in G$ such that $\alpha(\zeta_0, \zeta_0, \Gamma\zeta_0) \geq 1$.

Create a sequence $\{\zeta_p\}$ by setting $\zeta_{p+1} = \Gamma\zeta_p$ for all $p \in N$. If an integer p exists and $\zeta_p = \Gamma\zeta_p$, then $\zeta_p$ is a Fixed point of $\Gamma$, and the proof is thus complete. We therefore suppose that there is no number p such that $\zeta_{p+1} = \zeta_p$. This means that that $\alpha(\zeta_0, \zeta_0, \Gamma\zeta_0) \geq 1$ also holds for $\alpha(\zeta_0, \zeta_0, \zeta_1) \geq 1$. Since $\Gamma$ is an $\alpha$-admissible mapping, we obtain $\alpha(\zeta_p, \zeta_p, \zeta_{p+1}) \geq 1$ for every $p \in N$.

Assuming the above definition and $S_b(\ulcorner\zeta_{p-1}, \ulcorner\zeta_{p-1}, \ulcorner\zeta_p) \geq 1$ for all p$\in N$.

We have

$$\begin{aligned}\mathcal{F}\left((S_b(\zeta_p,\zeta_p,\zeta_{p+1})\right) &= \mathcal{F}\left((S_b(\ulcorner\zeta_{p-1}, \ulcorner\zeta_{p-1}, \ulcorner\zeta_p)\right) \\ &\leq \alpha(\zeta_{p-1},\zeta_{p-1},\zeta_p)\,\mathcal{F}\left((S_b(\ulcorner\zeta_{p-1}, \ulcorner\zeta_{p-1}, \ulcorner\zeta_p)\right) \\ &\leq \mathcal{F}\left(\tfrac{1}{2k^4}M_S^j(\zeta_{p-1},\zeta_p)\right) - \varphi(S_b(\zeta_{p-1},\zeta_{p-1},\zeta_p)) \\ &\leq \mathcal{F}\left(\tfrac{1}{2k^4}M_S^j(\zeta_{p-1},\zeta_p)\right). \qquad (4.2.1)\end{aligned}$$

Now, by simple computations, we have

$$M_S^j(\zeta_{p-1},\zeta_p) = max\begin{Bmatrix}S_b(\zeta_{p-1},\zeta_{p-1},\zeta_p), S_b(\zeta_{p-1},\zeta_{p-1},\ulcorner\zeta_{p-1}), S_b(\zeta_p,\zeta_p,\ulcorner\zeta_p), \\ S_b(\zeta_{p-1},\zeta_{p-1},\ulcorner\zeta_p), S_b(\zeta_p,\zeta_p,\ulcorner\zeta_{p-1})\end{Bmatrix}$$

$$= max\begin{Bmatrix}S_b(\zeta_{p-1},\zeta_{p-1},\zeta_p), S_b(\zeta_{p-1},\zeta_{p-1},\zeta_p), S_b(\zeta_p,\zeta_p,\zeta_{p+1}), \\ S_b(\zeta_{p-1},\zeta_{p-1},\zeta_{p+1}), S_b(\zeta_p,\zeta_p,\zeta_p)\end{Bmatrix}$$

$$= max\{S_b(\zeta_{p-1},\zeta_{p-1},\zeta_p), S_b(\zeta_p,\zeta_p,\zeta_{p+1}), S_b(\zeta_{p-1},\zeta_{p-1},\zeta_{p+1})\}$$

From the definition, we have

$\mathcal{F}((S_b(\zeta_p,\zeta_p,\zeta_{p+1})) <$

$\mathcal{F}(\max\left\{\frac{1}{2k^4}S_b(\zeta_{p-1},\zeta_{p-1},\zeta_p), \frac{1}{2k^4}S_b\left(\zeta_p,\zeta_p,\zeta_{p+1}), \frac{1}{2k^4}S_b(\zeta_{p-1},\zeta_{p-1},\zeta_{p+1})\right)\right\}$

But

$$\frac{1}{2k^4}S_b(\zeta_{p-1},\zeta_{p-1},\zeta_{p+1}) \leq \frac{1}{2k^4}(2kS_b(\zeta_{p-1},\zeta_{p-1},\zeta_p) + k^2S_b(\zeta_p,\zeta_p,\zeta_{p+1}))$$

$$\leq \max\left\{\frac{1}{k^3}S_b(\zeta_{p-1},\zeta_{p-1},\zeta_p), \frac{1}{2k^2}S_b(\zeta_p,\zeta_p,\zeta_{p+1})\right\}$$

Therefore,

$((S_b(\zeta_p,\zeta_p,\zeta_{p+1})) \leq (\max\{\frac{1}{k^3}S_b(\zeta_{p-1},\zeta_{p-1},\zeta_p), \frac{1}{2k^2}S_b(\zeta_p,\zeta_p,\zeta_{p+1})\})$

We get a contradiction if $\frac{1}{2k^2}S_b(\zeta_p,\zeta_p,\zeta_{p+1})$ is the maximum. Hence

$$\mathcal{F}\left((S_b(\zeta_p,\zeta_p,\zeta_{p+1})\right) < \mathcal{F}\left(\tfrac{1}{k^3}S_b(\zeta_{p-1},\zeta_{p-1},\zeta_p)\right)$$

$$\Rightarrow S_b(\zeta_p,\zeta_p,\zeta_{p+1}) < \frac{1}{k^3}S_b(\zeta_{p-1},\zeta_{p-1},\zeta_p)$$

This demonstrates that the $\{\zeta_p\}$ sequence of nonnegative real numbers is a decreasing sequence. We assert that $\lim_{p\to\infty} S_b(\zeta_p,\zeta_p,\zeta_{p+1}) = 0$.If possible, Assume that $\lim_{p\to\infty} S_b(\zeta_p,\zeta_p,\zeta_{p+1}) = \gamma$ for some $\gamma > 0$. Therefore, for every p $\in N$, we have $S_b(\zeta_p,\zeta_p,\zeta_{p+1}) \geq \gamma$. By using condition $(i_0)$ and the equation 4.1.2, we have

$$\mathcal{F}\,(\gamma) \leq \mathcal{F}\,((S_b(\zeta_p, \zeta_p, \zeta_{p+1})) < \mathcal{F}((\tfrac{1}{k^3} S_b\big(\zeta_{p-1}, \zeta_{p-1}, \zeta_p\big))$$

$$< \mathcal{F}((\tfrac{1}{(k^3)^2} S_b\big(\zeta_{p-2}, \zeta_{p-2}, \zeta_{p-1}\big)$$

$$.$$

$$.$$

$$< \mathcal{F}((\tfrac{1}{(k^3)^p}(S_b(\zeta_0, \zeta_0, \zeta_1))\ .$$

As $\lim_{p\to\infty} \mathcal{F}((\frac{1}{(k^3)^p}(S_b(\zeta_0, \zeta_0, \zeta_1)) = -\infty$, so we can find some q$\in N$ such that $\mathcal{F}((\frac{1}{(k^3)^p}(S_b(\zeta_0, \zeta_0, \zeta_1)) < \mathcal{F}(\gamma)$ for all p$> q$, which contradicts the above equation. Therefore, we must have $\lim_{p\to\infty} S_b\big(\zeta_p, \zeta_p, \zeta_{p+1}\big) = 0$. As of right now, we have established that Cauchy sequence in (G, $S_b$) exists. On the other hand, we assume that $\{\zeta_p\}$ is not Cauchy. The sequence of natural integers $\{q_k\}$ and $\{p_k\}$ exists when $\in > 0$ and monotonically increases such that $p_k > q_k$

$$S_b\big(\zeta_{qk}, \zeta_{qk}, \zeta_{pk}\big) \geq \in \tag{4.2.3}$$

and

$$S_b\big(\zeta_{qk}, \zeta_{qk}, \zeta_{pk-1}\big) < \in \tag{4.2.4}$$

from Lemma (3.1), (3.2) and (3.3), we have

$$\in \ \leq \ S_b\big(\zeta_{qk}, \zeta_{qk}, \zeta_{pk}\big)$$

$$\leq 2\mathrm{k}S_b\big(\zeta_{qk}, \zeta_{qk}, \zeta_{qk+1}\big) + k^2 S_b\big(\zeta_{qk+1}, \zeta_{qk+1}, \zeta_{pk}\big)$$

so that

$$\frac{\in}{k^2} \leq \frac{2}{k} S_b\big(\zeta_{qk}, \zeta_{qk}, \zeta_{qk+1}\big) + S_b\big(\zeta_{qk+1}, \zeta_{qk+1}, \zeta_{pk}\big)$$

We obtain that by setting k$\to\infty$ and applying $\mathcal{F}$ on both sides

$$\mathcal{F}\,(\tfrac{\in}{k^2}) \ \leq \ \lim_{k\to\infty} \mathcal{F}\,(S_b\big(\zeta_{qk+1}, \zeta_{qk+1}, \zeta_{pk}\big))$$

$$\leq \lim_{k\to\infty} \mathcal{F}\ (S_b\big(\ulcorner\zeta_{qk}, \ulcorner\zeta_{qk}, \ulcorner\zeta_{pk-1}\big))$$

$$\leq \lim_{k\to\infty} \alpha\big(\zeta_{qk}, \zeta_{qk}, \zeta_{pk-1}\big)\mathcal{F}\ (S_b\big(\ulcorner\zeta_{qk}, \ulcorner\zeta_{qk}, \ulcorner\zeta_{pk-1}\big)$$

$$\leq \lim_{k\to\infty} \mathcal{F}(\tfrac{1}{2k^4} M_S^j(\zeta_{qk}, \zeta_{pk-1})) - \lim_{k\to\infty} \varphi(S_b(\zeta_{qk}, \zeta_{qk}, \zeta_{pk-1}))$$

$$\leq \lim_{k\to\infty} \mathcal{F}(\tfrac{1}{2k^4} M_S^j(\zeta_{qk}, \zeta_{pk-1})) - \lim_{k\to\infty} \varphi(\in)$$

$$< \lim_{k\to\infty} \mathcal{F}(\tfrac{1}{2k^4} M_S^j(\zeta_{qk}, \zeta_{pk-1})) \tag{4.2.5}$$

Now, by simple computations, we have

$$\lim_{k\to\infty} M_s^j(\zeta_{qk}, \zeta_{pk-1})$$

$$= \lim_{k\to\infty} \max\{S_b(\zeta_{qk}, \zeta_{qk}, \zeta_{pk-1}), S_b(\zeta_{qk}, \zeta_{qk}, \ulcorner\zeta_{qk}), S_b(\zeta_{pk-1}, \zeta_{pk-1}, \ulcorner\zeta_{pk-1}),$$
$$S_b(\zeta_{qk}, \zeta_{qk}, \ulcorner\zeta_{pk-1})), S_b(\zeta_{pk-1}, \zeta_{pk-1}, \ulcorner\zeta_{qk}))\}$$

$$= \lim_{k\to\infty} \max\{S_b(\zeta_{qk}, \zeta_{qk}, \zeta_{pk-1}), S_b(\zeta_{qk}, \zeta_{qk}, \ulcorner\zeta_{qk+1}), S_b(\zeta_{pk-1}, \zeta_{pk-1}, \ulcorner\zeta_{pk}))$$
$$S_b(\zeta_{qk}, \zeta_{qk}, \zeta_{pk})), S_b(\zeta_{pk-1}, \zeta_{pk-1}, \ulcorner\zeta_{pk+1}))\}$$

$$= \lim_{k\to\infty} \max\{\epsilon, 0, S_b(\zeta_{qk}, \zeta_{qk}, \zeta_{pk}), S_b(\zeta_{pk-1}, \zeta_{pk-1}, \zeta_{pk+1}))\}$$

$$\le \lim_{k\to\infty} \max\{\in, 0, 2k\, S_b(\zeta_{qk}, \zeta_{qk}, \zeta_{pk-1}) + k^2 S_b(\zeta_{pk-1}, \zeta_{pk-1}, \zeta_{pk}),$$
$$2kS_b(\zeta_{pk-1}, \zeta_{pk-1}, \zeta_{qk}) + k^2 S_b(\zeta_{qk}, \zeta_{qk}, \zeta_{pk+1}))\}$$

$$\le \max\{\in, 0, 2k \in, 2k^2 \in\} = 2k^2 \in$$

Therefore from Eq. (4.2.5) and condition (e), we deduce

$$\mathcal{F}\left(\frac{\in}{k^2}\right) < \lim_{k\to\infty} \mathcal{F}\left(\frac{1}{2k^4} M_S^j(\zeta_{qk}, \zeta_{pk-1})\right) < \mathcal{F}\left(\frac{\in}{k^2}\right)$$

This is incongruous. Hence a $S_b - CS$ called $\{\zeta_p\}$ exists in C$S_b MS$ (G, $S_b$). The sequence $\{\zeta_p\}$ are convergent to $v \in (\mathrm{G}, S_b)$, according to the completeness of $(\mathrm{G}, S_b)$.

$$\lim_{p\to\infty} \ulcorner\zeta_p = v = \lim_{p\to\infty} \ulcorner\zeta_{p+1} \qquad (4.2.6)$$

Assume that $\ulcorner$ is $\alpha$-admissible mapping, there is a sub sequence $\{\zeta_{pk}\}$ of $\{\zeta_p\}$ such that $\alpha(\zeta_{pk}, \zeta_{pk}, \zeta_{pk+1})) \ge 1$ for all k$\in N$ and we have $\alpha(\mathrm{v}, v, \zeta_{pk+1})) \ge 1$. At this stage, we must demonstrate that $v$ is a fixed point of $\ulcorner$, Lemma (3.3) says that

If $\ulcorner v \neq v$, then we have that

$$\frac{1}{2k} S_b(\ulcorner v, \ulcorner v, v) \le \lim_{p\to\infty} inf\, S_b(\ulcorner v, \ulcorner v, \ulcorner\zeta_{p+1}).$$

Again, by the property of $\mathcal{F}$ and Eq (4.2.1), we obtain

$$\mathcal{F}\left(\frac{1}{2k} S_b(\ulcorner v, \ulcorner v, v) \le \lim_{p\to\infty} \inf \mathcal{F}(S_b(\ulcorner v, \ulcorner v, \ulcorner\zeta_{p+1})\right.$$
$$\le \lim_{p\to\infty} \inf\ \alpha(\mathrm{v}, \mathrm{v}, \zeta_{p+1})\mathcal{F}(S_b(\ulcorner v, \ulcorner v, \ulcorner\zeta_{p+1}))$$
$$\le \liminf_{k\to\infty} \mathcal{F}\left(\frac{1}{2k^4} M_S^j(v, \zeta_{p+1})\right) - \lim_{p\to\infty} \inf \varphi((S_b(v, v, \zeta_{p+1}))$$
$$< \liminf_{k\to\infty} \mathcal{F}\left(\frac{1}{2k^4} M_S^j(v, \zeta_{p+1})\right) \qquad (4.2.7)$$

Where

$\liminf_{k\to\infty} \mathcal{F}(\frac{1}{2k^4} M_S^j(v,\zeta_{p+1})) =$

$\lim_{p\to\infty} \inf \max \left\{ \begin{matrix} S_b(v,v,\zeta_{p+1}), S_b(v,v,\ \Gamma v), S_b(\zeta_{p+1},\zeta_{p+1},\ \Gamma\zeta_{p+1}), \\ S_b(v,v,\ \Gamma\zeta_{p+1}), S_b(\zeta_{p+1},\zeta_{p+1},\ \Gamma v) \end{matrix} \right\}$

$\leq \lim_{p\to\infty} \sup \max \left\{ \begin{matrix} S_b(v,v,\zeta_{p+1}), S_b(v,v,\ \Gamma v), S_b(\zeta_{p+1},\zeta_{p+1},\ \Gamma\zeta_{p+1}), \\ S_b(v,v,\ \Gamma\zeta_{p+1}), S_b(\zeta_{p+1},\zeta_{p+1},\ \Gamma v) \end{matrix} \right\}$

$\leq \max\ \{0, S_b(v,v,\ \Gamma v)\}$

$= kS_b(\ \Gamma v,\ \Gamma v, v)$.

From Eq (4.2.7), we get

$$\mathcal{F}\ (\frac{1}{2k} S_b(\ \Gamma v,\ \Gamma v, v) < \mathcal{F}(\frac{1}{2k^3} S_b(\ \Gamma v,\ \Gamma v, v)$$

This is incongruous. $v$ is hence a fixed point of $\Gamma$. Assume that there is a fixed point in $\Gamma$ called $v^1$ such that $v \neq v^1$

Consider

$\mathcal{F}\ (S_b(v,v,v^1)) = \mathcal{F}\ (S_b(\ \Gamma v,\ \Gamma v,\ \Gamma v^1))$

$$\leq \alpha(\ v,v,v^1))\ \mathcal{F}\ (S_b(\ \Gamma v,\ \Gamma v,\ \Gamma v^1))$$

$$\leq \mathcal{F}\left(\frac{1}{2k^4} M_S^j(v,v^1)\right) - \varphi(S_b(v,v,v^1))$$

$$\leq\ \mathcal{F}\left(\frac{1}{2k^4} M_S^j(v,v^1)\right)$$

$$\leq \mathcal{F}(\frac{1}{2k^4} \max\{0, S_b(v,v,v^1), S_b(v^1,v^1,v)\})$$

$$<\ \mathcal{F}(\frac{1}{2k^3} S_b(v,v,v^1))$$

This is incongruous. Consequently $v = v^1$.

Therefore, the unique fixed point of $\Gamma$ is $v$.

**Corollary 4.1**

A $CS_bMS$ with coefficient $k > 1$ is defined as (G, $S_b$). Let $\Gamma: G \to G$ be a $(\alpha.\varphi,\mathcal{F})$-contraction type mapping with j = 3 or 4, assuming that all the requirements in Theorem (4.2) are true. Then, in G, there is a UFP of $\Gamma$.

**Corollary 4.2**

If (G, $S_b$) is a $CS_bMS$ with coefficient $k > 1$, then suppose that a self map $\Gamma: G \to G$ meets the condition.

$S_b(\ \Gamma\ell,\ \Gamma\ell\ ,\ \Gamma\ \zeta) > 0 \Rightarrow\ \varphi\big(S_b(\ell,\ell,\zeta)\big) +\ \mathcal{F}\ (S_b(\ \Gamma\ell,\ \Gamma\ell\ ,\ \Gamma\ \zeta)) \leq \mathcal{F}\ (\frac{1}{2k^4} M_S^j(\ell,\zeta))$
$where\ \mathcal{F} \in \chi\ and\ \varphi \in \mathcal{L}$, j = 5, 4 or 3. Then, in G there is UFP of $\Gamma$.

**Corollary 4.3**

If (G, $S_b$) is a C$S_b$MS with coefficient k $>1$, then suppose that a self map $\lceil: G \to G$ meets the condition.

$S_b(\lceil\ell, \lceil\ell, \lceil\zeta) > 0 \Rightarrow \tau + \mathcal{F}(S_b(\lceil\ell, \lceil\ell, \lceil\zeta)) \leq \mathcal{F}(\frac{1}{2k^4} M_s^j(\ell,\zeta))$

$where\ \mathcal{F} \in \chi\ and\ \varphi \in \mathcal{L}$, j = 5, 4 or 3. Then, in G there is UFP of $\lceil$.

**Corollary 4.4**

If (G, $S_b$) is a C$S_b$MS with coefficient k $>1$, then suppose that a self map $\lceil: G \to G$ meets the condition. $S_b(\lceil\ell, \lceil\ell, \lceil\zeta)) \leq \boldsymbol{\lambda}\, M_s^j(\ell,\zeta))$, where $\mathcal{F} \in \chi\ and\ \varphi \in \mathcal{L}$, j = 5, 4 or 3. Then, in G there is UFP of $\lceil$ and $\boldsymbol{\lambda} \in [0, \frac{1}{2k^4}]$.

**Example 4.1**

Let $S_b: G^3 \to R^+$ be a mapping defined as

$S_b(\nabla_1, \nabla_2, \nabla_3) = (|\nabla_2 + \nabla_3 - 2\nabla_1| + |\nabla_2 - \nabla_3|)^2$ , where G = [0, ∞).

Clearly (G, $S_b$) is C$S_b$MS with k = 2

Define $\lceil: G \to G$ by $\lceil(a) = \frac{a}{3\sqrt{8}}$. Let $\varphi: (0,\infty) \to (0,\infty)$ and $\mathcal{F}: R^+ \to R$ as

$\varphi(t) = \frac{t}{4k^4}\ and\ \mathcal{F}(t) = t$ and $\alpha: G^3 \to R^+\ as\ \alpha(i,i,j) = \begin{cases} 1 & i,j \in [0,1] \\ \frac{1}{2} & otherwise \end{cases}$

Let i,j ∈ $G$, $\alpha(i,i,j) \geq 1$ for i,j∈ [0,1]. On the other hand, $\lceil i \leq 1$ for every i,j∈ [0,1].

As a result, $\alpha(\lceil i, \lceil i, \lceil j) \geq 1$. As a result, the prediction is correct. In support of the preceeding argument, $\alpha(0,0,0) > 1$. If $\{i_p\}$ and $\{j_p\}$ are sequences in G such that $\alpha(i_p, i_p, j_p) > 1$ and $i_p \to i$,

$j_p \to j \in G$, for all $p \in N \cup \{0\}$. Then $i_p, j_p \subseteq [0,1]$, and hence $i,j \in [0,1]$ which implies $\alpha(i,i,j) \geq 1$.

Let i,j∈ $[0,\infty)$, we have

$$\mathcal{F}(S_b(\lceil(i), \lceil(i), \lceil(j))) = S_b(\lceil(i), \lceil(i), \lceil(j)$$

$$= 4|\lceil i - \lceil j|^2$$

$$= 4|\frac{i}{3\sqrt{8}} - \frac{j}{3\sqrt{8}}|^2$$

$$\leq \frac{1}{64} S_b(i,i,j) \leq \frac{1}{2k^4} S_b(i,i,j) - \frac{1}{4k^4} S_b(i,i,j)$$

$$\leq \frac{1}{2k^4} \max\{S_b(i,i,j), S_b(i,i,\lceil i), S_b(i,i,j) S_b(j,j,\lceil j), S_b(i,i,\lceil j) S_b(j,j,\lceil i)\} - \frac{1}{4k^4} S_b(i,i,j)$$

$$\leq \frac{1}{2k^4} M_S^5(i,j) - \frac{1}{4k^4} S_b(i,i,j)$$

$$\leq \mathcal{F}\left(\frac{1}{2k^4} M_S^5(i,j)\right) - \varphi\big(S_b(i,i,j)\big).$$

As a result the theorem (4.2) presumptions are all met, and 0 is the only fixed point of ⎾.

## 5. Application to Integral Equation:

We take into account the subsequent integral equation as an application:

$$\alpha(t) = \mathcal{A}(t) + 2k^4 \int_0^t K(t,\lambda)\, \mathfrak{h}\big(\lambda, \alpha(\lambda)\big)\, d\lambda \text{ for all } t \in [0\ 1] \quad (5.1)$$

Let ⊕ = $\{\mathcal{H}/\mathcal{H}:(0,\infty) \to (0,\infty)\}$ be a family of non-decreasing functions and

Ɓ = $\{ \mathcal{F} / \boldsymbol{\mathcal{F}}: R^+ \to R\}$ be a family of increasing functions satisfying

$$(\mathcal{H}(t))^\theta \leq (t^\theta) - 2k^4 t^\theta \text{ for all } \theta \geq 1 \text{ and } t \geq 0$$

Eq (5.1) will be examined under the ensuing presumptions

$(i_0)$ $\mathcal{A}: [0\ 1] \to [\frac{\alpha_0}{2k^4}, \infty]$ is a continuous function

$(i_1)$ $\mathfrak{h}$: I × $\left[\frac{\alpha_0}{2k^4}, \infty\right] \to [\frac{\alpha_0}{2k^4}, \infty]$ is continuous function, $\mathfrak{h}(t,\alpha) \geq 0$ and ∃ $\mathcal{H}$ ∈⊕

Such that $\forall\, \alpha, \beta \in \left[\frac{\alpha_0}{2k^4}, \infty\right]$

$|\mathfrak{h}(t,\alpha) - \mathfrak{h}(t,\beta)| \leq \mathcal{H}(|\alpha - \beta|)$ with $\mathcal{H}(t_p) \to \frac{1}{2^{\theta-1}}$ as p→ ∞ implies $\lim_{p\to\infty} t_p = 0$.

$(i_2)$ $K: [0\ 1] \times [0\ 1] \to R$ is continuous function in t∈ [0 1] and is measurable in each $\boldsymbol{\lambda}$∈ [0 1], for every t in such a way that $K(t,\lambda) \geq 0$ and $\int_0^1 K(t,\lambda) d\lambda \leq \frac{1}{\sqrt{2^{3+\frac{3}{\theta}}}}$.

Consider the space of continuous functions G=C (I, $\left[\frac{\alpha_0}{2k^4}, \infty\right]$) and

$S_b: G^3 \to R^+$ be a mapping defined as

$S_b(\nabla_1, \nabla_2, \nabla_3) = (|\nabla_2 + \nabla_3 - 2\nabla_1| + |\nabla_2 - \nabla_3|)^2$ where G=[0,∞).

Clearly (G, $S_b$) is C$S_b$MS with k = 2

**Theorem 5.1**

Equation (5.1) has a unique solution in C (I, $\left[\frac{\alpha_0}{2k^4}, \infty\right]$) under assumptions $(i_0) - (i_2)$

Proof: Define ⎾: $G \to G$ by ⎾$(\alpha)$ (t) = $\frac{\mathcal{A}(t)}{2k^4} + \int_0^t K(t, \lambda)\, \mathfrak{h}\big(\tau, \alpha(\lambda)\big) d\lambda$, t∈ [0 1]

Now, for $\rho, \mu \in G$

$$\mathcal{F}\,(S_b\Big(⎾(\rho)(t), ⎾(\rho)(t), \big(⎾(\mu)(t)\big)\Big) = S_b\big(⎾(\rho)(t), ⎾(\rho)(t), ⎾(\mu)(t)\big)$$

$$= (2|⎾(\rho)(t) - ⎾(\mu)(t)|)^2$$

$$=4\left|\int_0^t K(t, \lambda\right)\mathfrak{h}\big(\tau, \alpha(\lambda)\big)d\lambda - \int_0^t K(t, \lambda)\mathfrak{h}\big(\tau, \mu(\lambda)\big)d\lambda|^2$$

$$\leq (2\int_0^t K(t,\ \lambda)|\mathfrak{h}(\tau,\alpha(\lambda)) - \mathfrak{h}(\tau,\mu(\lambda))|)^2$$

$$\leq \frac{1}{2^{\frac{3\theta+3}{\theta}}}((\mathcal{H}(2|\rho-\mu|))^2$$

$$\leq \frac{1}{2^{\frac{3\theta+3}{\theta}}}(\mathcal{F}(2|\rho-\mu|)^2 - 2k^4(2|\rho-\mu|)^2)$$

$$\leq \frac{1}{2^{\frac{3\theta+3}{\theta}}}(\mathcal{F}(S_b(\rho,\rho,\mu)) - 2k^4 S_b(\rho,\rho,\mu))$$

$$\leq \mathcal{F}\,(\frac{1}{2k^4}S_b(\rho,\rho,\mu)) - \frac{1}{2}S_b(\rho,\rho,\mu)$$

$$\leq \mathcal{F}(\frac{1}{2k^4}M_s^j(\rho,\mu)) - \varphi(S_b(\rho,\rho,\mu))$$

In which $\frac{\theta+1}{\theta} = k, \mathcal{F}(\mathrm{t}) = \mathrm{t}$ and $\varphi(t) = \frac{t}{2}$.

According to theorem 4.2, equation (5.1) has a unique solution in C (I, $\left[\frac{\alpha_0}{2k^4},\infty\right]$).

## 6. Application to Homotopy

This section investigates the existence oef a unique solution to Homotopy

**Theorem 6.1**

If (G, $S_b$) is a C$S_b$MS, then $\Delta$ and $\bar{\Delta}$ are open and closed subsets of G respectively such that $\Delta \subseteq \bar{\Delta}$. Let $\mathcal{k}_b: \bar{\Delta} \times [0\ 1] \to G$ be an operator meeting the requirements listed below.

$(\tau_0)$ $\alpha \neq \mathcal{k}_b(\alpha,s)$, for each $\alpha \in \partial\Delta$ and s$\in$ [0 1] (Here $\partial\Delta$ is boundary of $\Delta$ in G

$(\tau_1)$ $S_b(\mathcal{k}_b(\alpha,s),, \mathcal{k}_b(\alpha,s), \mathcal{k}_b(\beta,s)) \geq 0$ implies

$$\tau + \mathcal{F}(2k^4\mathcal{k}_b(\alpha,s),, \mathcal{k}_b(\alpha,s), \mathcal{k}_b(\beta,s)) \leq \mathcal{F}(S_b(\alpha,\alpha,\beta)$$

for all $\alpha,\beta \in \bar{\Delta}$ with $\tau > 0$ and s $\in$ [0 1]

$(\tau_2)$ $\exists\, M_b \geq 0 \ \ni S_b(\mathcal{k}_b(\alpha,s),, \mathcal{k}_b(\alpha,s), \mathcal{k}_b(\alpha,t)) \leq M_b|s-t|$ for every $\alpha \in \bar{\Delta}$ and s, t$\in$ [0 1].

Then $\mathcal{k}_b(.,0)$ has Fixed Point $\Leftrightarrow$ $\mathcal{k}_b(.,1)$ has Fixed Point.

Proof: Take into account the set A= {s$\in$ [0 1]: $\alpha = \mathcal{k}_b(\alpha,s)$, for some $\alpha \in \Delta$}.

Due to the fact that $\mathcal{k}_b(.,0)$ has a Fixed Point in $\Delta$, we have that 0$\in A$. The set A is not empty as a result. We will prove that A = [0 1] by establishing that A is both open and closed in [0 1]. $\mathcal{k}_b(.,1)$ has a Fixed Point as a result. The first thing we do is show that A$\subseteq$ [0 1] is closed. To observe this, assign $s_p \to s \in$ [0 1] as p$\to \infty$ and let $\{s_p\}_{p=1}^{\infty} \subseteq A$.

We must show that s is in A. Given that $s_p$in A for p=0, 1, 2……., there is $\alpha_p$ in $\Delta$ with

$\alpha_{p+1} = \Bbbk_b(\alpha_p, s_p)$.The proof is successful if n∈ $N$ exists such that $S_b\left(\alpha_p, \alpha_p, \Bbbk_b(\alpha_p, s_p)\right) = 0$. So we assume that , for any p ∈ $N$, we have

$S_b(\alpha_p, \alpha_p, \alpha_{p+1}) = S_b(\Bbbk_b(\alpha_{p-1}, s_{p-1}), \Bbbk_b(\alpha_{p-1}, s_{p-1}), \Bbbk_b(\alpha_p, s_p))$

$\leq 2k(S_b\left(\Bbbk_b(\alpha_{p-1}, s_{p-1}), \Bbbk_b(\alpha_{p-1}, s_{p-1}), \Bbbk_b(\alpha_p, s_{p-1})\right)$

$+k^2 S_b(\Bbbk_b(\alpha_p, s_{p-1}), \Bbbk_b(\alpha_p, s_{p-1}), \Bbbk_b(\alpha_p, s_p))$

$\leq 2k(S_b\left(\Bbbk_b(\alpha_{p-1}, s_{p-1}), \Bbbk_b(\alpha_{p-1}, s_{p-1}), \Bbbk_b(\alpha_p, s_{p-1})\right) + k^2 M_b |s_p - s_{p-1}|$

Letting p → ∞, we obtain

$$\lim_{p\to\infty} S_b(\alpha_p, \alpha_p, \alpha_{p+1}) \leq 2k(S_b\left(\Bbbk_b(\alpha_{p-1}, s_{p-1}), \Bbbk_b(\alpha_{p-1}, s_{p-1}), \Bbbk_b(\alpha_p, s_{p-1})\right).$$

Since $\mathcal{F}$ is continuous and increasing function, we obtain

$\operatorname{Lim}_{p\to\infty} \mathcal{F}(k^3\, S_b(\alpha_p, \alpha_p, \alpha_{p+1}))$

$= \lim_{p\to\infty} \mathcal{F}(2k^4\, S_b\left(\Bbbk_b(\alpha_{p-1}, s_{p-1}), \Bbbk_b(\alpha_{p-1}, s_{p-1}), \Bbbk_b(\alpha_p, s_{p-1})\right)$

$\leq \operatorname{Lim}_{p\to\infty}(\mathcal{F}(\, S_b(\alpha_{p-1}, \alpha_{p-1}, \alpha_p)) - \tau)$

$\leq \operatorname{Lim}_{p\to\infty}(\mathcal{F}(\, S_b(\alpha_{p-2}, \alpha_{p-2}, \alpha_{p-1})) - 2\tau)$

$\leq \operatorname{Lim}_{p\to\infty}(\mathcal{F}(\, S_b(\alpha_{p-3}, \alpha_{p-3}, \alpha_{p-2})) - 3\tau)$

.

.

$\leq \operatorname{Lim}_{p\to\infty}(\mathcal{F}(\, S_b(\alpha_0, \alpha_0, \alpha_1)) - p\tau)$ (6.1.1)

In the above inequality Eq (2.3.1), we have

$$\operatorname{Lim}_{p\to\infty} \mathcal{F}(k^3\, S_b(\alpha_p, \alpha_p, \alpha_{p+1})) = -\infty.$$

Which together the above definition (3.5) (b), gives $\lim_{p\to\infty} S_b(\alpha_p, \alpha_p, \alpha_{p+1}) = 0$. It is now time to demonstrate the $S_b$ −Cauchy Sequence in (G, $S_b$). on the other hand, Assume $\{\alpha_p\}$ is not a $S_b$ −Cauchy Sequence. Natural numbers $\{q_k\}$ $and$ $\{p_k\}$ can be arranged in a monotone increasing sequence with ∈> 0 such that $p_k > q_{k,}$

$$S_b(\alpha_{qk,}\, \alpha_{qk,} \alpha_{pk}) \geq \epsilon \qquad (6.1.2)$$

and

$$S_b(\alpha_{qk,}\, \alpha_{qk,} \alpha_{pk-1}) < \epsilon \qquad (6.1.3)$$

From Eq (4.2.1) and Eq (6.1.2), we have

$$\varepsilon \leq \quad S_b(\alpha_{qk,}\, \alpha_{qk,} \alpha_{pk})$$

$$\leq 2k\, S_b(\alpha_{qk,}\ \alpha_{qk,}\alpha_{qk+1}) + k^2 S_b(\alpha_{qk+1,}\ \alpha_{qk+1,}\alpha_{pk})$$

$$\leq 2kS_b\big(\alpha_{qk,}\ \alpha_{qk,}\alpha_{qk+1}\big) + k^2 S_b(\big(\hbar_b(\alpha_{qk}, s_{qk}), \hbar_b(\alpha_{qk}, s_{qk}), \hbar_b(\alpha_{pk-1}, s_{pk-1})\big)$$

$$\leq 2kS_b\big(\alpha_{qk,}\ \alpha_{qk,}\alpha_{qk+1}\big) + 2k^3 S_b(\big(\hbar_b(\alpha_{qk}, s_{qk}), \hbar_b(\alpha_{qk}, s_{qk}), \hbar_b(\alpha_{pk-1}, s_{qk})\big) +$$

$$k^4 S_b(\big(\hbar_b(\alpha_{pk-1}, s_{qk}), \hbar_b(\alpha_{pk-1}, s_{qk}), \hbar_b(\alpha_{pk-1}, s_{pk-1})\big).$$

So that

$$\mathrm{k}\epsilon \leq\ 2k^2 S_b\big(\alpha_{qk,}\ \alpha_{qk,}\alpha_{qk+1}\big) + 2k^4 S_b(\big(\hbar_b(\alpha_{qk}, s_{qk}), \hbar_b(\alpha_{qk}, s_{qk}), \hbar_b(\alpha_{pk-1}, s_{qk})\big)$$
$$+k^5 \big|s_{qk} - s_{pk-1}\big|.$$

We obtain that by setting k→ ∞ and applying $\mathcal{F}$ on both sides.

$$\mathcal{F}(\mathrm{k}\epsilon) \leq \lim_{p\to\infty} \mathcal{F}(\ 2k^4 S_b(\big(\hbar_b(\alpha_{qk}, s_{qk}), \hbar_b(\alpha_{qk}, s_{qk}), \hbar_b(\alpha_{pk-1}, s_{qk})\big)$$

$$\leq \lim_{p\to\infty} \mathcal{F}(\ S_b(\alpha_{qk,}\ \alpha_{qk,}\alpha_{pk-1})) - \tau$$

$$\leq\ \mathcal{F}(\epsilon) - \tau$$

This leads to the conclusion that $\tau +\ \mathcal{F}(\mathrm{k}\epsilon) \leq\ \mathcal{F}(\epsilon)$. It contradicts itself.

In the $S_b$ −metric space (G, $S_b$), the sequence $\{\alpha_p\}$ is a $S_b$ −Cauchy sequence. The sequence $\{\alpha_p\} \to v$ ∈(G, $S_b$) comes from the completeness of (G, $S_b$)

$$\lim_{p\to\infty} \alpha_{p+1} = v = \lim_{p\to\infty} \alpha_p$$

We can prove $v$=$\hbar_b(v, s)$). Suppose that $S_b(\hbar_b(v, s), \hbar_b(v, s), v) > 0$

From Lemma (3.3) (i), we have

$$\mathcal{F}(k^3 S_b(\hbar_b(v, s), \hbar_b(v, s), v)) \leq \lim_{p\to\infty} \inf \mathcal{F}(2k^4\, S_b\left(\hbar_b(v, s), \hbar_b(v, s), \hbar_b(\alpha_p, s)\right)$$

$$\leq \lim_{p\to\infty} \mathcal{F}(S_b(v, v, \alpha_p) - \tau$$

This is $\mathcal{F}(k^3 S_b(\hbar_b(v, s), \hbar_b(v, s), v) = -\infty\ as\ \tau \to \infty$.

Accordingly, $v$ =$\hbar_b(v, s)$ indicates that $S_b(\hbar_b(v, s), \hbar_b(v, s), v) = 0$. So, $s$ in A.

It is obvious that A is closed in [0 1]. Let $s_0 \in A$, then $\alpha_0$ exists in Δ such that

$\alpha_0 = \hbar_b(\alpha_0, s_0)$. Because Δ is open $\delta > 0$ must exist for$B_{S_b}(\alpha_{0,}\delta) \subseteq \Delta$. Select the value of $s\ \in (s_0 - \epsilon, s_0 + \epsilon)$ such that$|s - s_0| \leq \frac{1}{M^p} < \epsilon$.

Consequently, for $\overline{B_b(\alpha_0, \delta)}$= $\{\alpha \in G: S_b(\alpha, \alpha, \alpha_0) \leq \delta + k^2 S_b(\alpha_0, \alpha_0, \alpha_0)$

Now, $S_b(\hbar_b(\alpha, s), \hbar_b(\alpha, s), \alpha_0)$ =$S_b\big(\hbar_b(\alpha, s), \hbar_b(\alpha, s), \hbar_b(\alpha_0, s_0)\big)$

$\leq$ 2k $(S_b\big(\hbar_b(\alpha, s), \hbar_b(\alpha, s), \hbar_b(\alpha, s_0)\big) + k^2 S_b\big(\hbar_b(\alpha, s_0), \hbar_b(\alpha, s_0), \hbar_b(\alpha_0, s_0)\big)$

$\leq 2kM|s - s_0| + k^2 S_b\big(\hbar_b(\alpha, s_0), \hbar_b(\alpha, s_0), \hbar_b(\alpha_0, s_0)\big)$.

If p is allowed to reach ∞ and $\mathcal{F}$ is applied on both sides, then

$$\begin{aligned}\mathcal{F}(S_b(\mathscr{k}_b(\alpha,s),\mathscr{k}_b(\alpha,s),\alpha_0))) &\leq \mathcal{F}(2k^3 S_b(\mathscr{k}_b(\alpha,s),\mathscr{k}_b(\alpha,s),\alpha_0))) \\ &\leq \mathcal{F}(2k^4 S_b\big(\mathscr{k}_b(\alpha,s_0),\mathscr{k}_b(\alpha,s_0),\mathscr{k}_b(\alpha_0,s_0)\big) \\ &\leq \mathcal{F}\big(S_b(\alpha,\alpha,\alpha_0)\big)-\tau \\ &\leq \mathcal{F}\big(S_b(\alpha,\alpha,\alpha_0)\big).\end{aligned}$$

Therefore,

$$S_b(\mathscr{k}_b(\alpha,s),\mathscr{k}_b(\alpha,s),\alpha_0) \leq S_b(\alpha,\alpha,\alpha_0) \leq \delta + k^2 S_b(\alpha_0,\alpha_0,\alpha_0).$$

Thus for each fixed $s \in (s_0-\epsilon, s_0+\epsilon)$, $\mathscr{k}_b(.;s)$: $\overline{B_b(\alpha_0,\delta)} \to \overline{B_b(\alpha_0,\delta)}$. Then, all of the criteria of Theorem (4.2) are satisfied. As a result, we conclude that $\mathscr{k}_b(.;s)$ has fixed point in $\bar{\Delta}$. However, this must be in $\Delta$. As a result, $s \in A$ for $s \in (s_0-\epsilon, s_0+\epsilon)$. As a result, $(s_0-\epsilon, s_0+\epsilon) \subseteq A$. Clearly A is open in the range [0 1].

A similar method can be used to demonstrate the converse.

## 7. Discussion

We have discussed unique fixed point theorems via $(\alpha,\varphi,\mathcal{F})$-contraction in complete $S_b$-MS and aillustrated an example also applications to nonlinear integral equations and Homotopy theory.

## Refrences

[1] Banach S, Sur les operations dans les ensembles abstraits etleur applications aux equations integrales, Fund. Math. 3. 133-181(1922).

[2] M. Bina Devi, N. Priyobarta, Y. Rohen, Fixed point theorems for (α, β) − (φ, ψ)-rational contractive type mappings, J. Math. Comput. Sci. 11 (2021), 955-969.

[3] B. Khomdram, M. Bina Devi, Y. Rohen, Fixed point of theorems of generalised α-ratonal contractive mappings on rectangular b-metric spaces, J. Math. Comput. Sci. 11 (2021), 991-1010.

[4] Mustafa Z, Jaradata M. M. M., Jaradat H. M., Some common fixed point results of graphs on b - metric space, Journal of . Nonlinear Science and Applications, 9 (2016) Issue: 6, 4838-4851.

[5] E. Karapinar, P. Kumam and P. Salimi, on α −ψ-Meir-Keeler contractive mapping, Fixed Point Theory Appl. 2013:9, 2013.

[6] B. Samet, C. Vetro and P. Vetro, Fixed point theorems for α −ψ-contractive type mappings, Nonlinear Anal. 75, no. 4 (2012), 21542165.

[7] Sedghi S, Gholidahneh, Dosenovic T, Esfahani J and Radenovic S, Common fixed point of four maps in Sb-metric spaces, Journal of Linear and Topological Algebra, Vol.5(2), (2016), 93 - 104.

[8] Souayah Nizar, Mlaiki A fixed point theorems in Sb-metric spaces, Journal of mathematics and computer science, 16(2016), 131-139.

[9] Souayah Nizar, A fixed point in partial Sb-metric spaces, An. St. univ ovidius constante, vol.24 (3),2016, 315-362.

[10] N. Mlaiki, A. Mukheimer, Y. Rohen, N. Souayah, T. Abdeljawad, Fixed point theorems for αcontractive mapping in Sb-metric spaces, J. Math. Anal. 8 (2017), 40-46.

[11] Y. Rohen, T. Dosenovic, S. Radenovic, A Note on the Paper A Fixed point Theorems in Sb-Metric Spaces,Filomat. 31 (2017), 3335-3346.

[12] Rohen y, Dosenovic' T, Randanovic' S, A note on a paper A fixed point theorems in Sb-metric spaces, Filomat 31(11), 3335-3346(2017).

[13] Kishore GNV, Rao KPR, Panthi D, Srinuvsa Rao B, Satyanaraya S, Some applications via fixed point results in partially ordered Sb-metric spaces, Fixed point theory and applications (2017), 2017;10, doi: 10.1186/s13663-017-0603-2.

[14] D. Wardowski, Solving existence problems via $\mathcal{F}$ - contractions, Proc. Amer. Math. Soc. 146 (2018) 1585-1598.

[15] D. Wardowski, Fixed points of new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012:94, 2012.

[16] D. Wardowski and N. V. Dung, Fixed points of $\mathcal{F}$ -weak contractions on complete metric spaces, Demonstr. Math. 47, no. 1 (2014), 146-155.

[17] N. V. Dung and V. L. Hang, A fixed point theorem for generalized $\mathcal{F}$ -contractions on complete metric spaces, Vietnam. J. Math. 43, no. 4 (2015), 743-753.

[18] D. Gopal, M. Abbas, D. K. Patel and C. Vetro, Fixed points of α-type $\mathcal{F}$ -contractive mappings with an application to nonlinear fractional differential equation, Acta Math. Scientia 36, no. 3 (2016), 957-970.

[19] Hafid Massit and Mohamed Rossafi, Fixed point theorem for (φ, $\mathcal{F}$)-contraction on $C^*$ -algebra valued metric spaces, Eur. J. Math. Appl. (2021)1:14

[20] H. Piri, S. Rahrovi, H. Marasi, P. Kumam, $\mathcal{F}$-contraction on asymmetric metric spaces, J. Math. Computer Sci. 17 (2017) 32-40.

[21] A. Taheri, A. Farajzadeh, A new generalization of α-type almost- $\mathcal{F}$ -contractions and α-type $\mathcal{F}$ -Suzuki contractions in metric spaces and their fixed point theorems, carpathian Math. Publ. 11 (2019) 475-492.

[22] Secelean, NA, Iterated function systems consisting of $\mathcal{F}$ -contractions. Fixed Point Theory Appl. 2013, Article ID 277(2013).

[23] L. K. Dey and S. Mondal, Best proximity point of $\mathcal{F}$ -contraction in complete metric space, Bull. Alahabad Math. Soc. 30, no. 2 (2015), 173-189.

[24] H. Piri and P. Kumam, Some fixed point theorem concerning $\mathcal{F}$ -contractions in complete metric spaces, Fixed Point Theory Appl. 2014:210, 2014.

[25] G. Stan, O. Popescu, Two fixed point theorems concerning $\mathcal{F}$ -contraction in complete metric spaces, Symmetry. 12 (2020), 58.