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A Glimpse on the Integer Solutions to the Hyperbola $8x^2 = 5y^2 + 27$

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Abstract: The hyperbolas given by $8x^2 = 5y^2 + 27$ is considered to obtain its solutions in integers.Various relations amoung the solutions are exhibited.Using the known solutions of the equations in title hyperbolas,parabolas,special Pythagorean triangle and second order Ramanujan numbers are determind.Facinating properties involving figurate numbers are given.

Keywords: Binary quadratic, Hyperbola, Parabola, Integer solutions, Pell equation, Pythagorean triangle, polygonal numbers.

1. Introduction

The second degree equation with two unknowns given by $y^2 = Dx^2 + N(\neq 0)$ where D> 0 and squarefree, has been analyzed by different mathematicians to obtain non-zero solutions in integers for varies values [1-4].In this context, refer[5-11]. This paper focuses on finding solutions in integers to the hyperbola $8x^2 = 5y^2 + 27$.Various relations among the solutions are exhibited. Using the known solutions of the equation in title hyperbolas, parabolas, special Pythagorean triangle and second order Ramanujan numbers are determined. Fascinating properties involving figurate numbers are given.

2. Method of Analysis

The Pell like equation representing hyperbola under consideration is

$$8x^2 = 5y^2 + 27 \quad (1)$$

whose smallest positive integer solutions are

$$x_0 = 12,\ y_0 = 15$$

Introduction of the linear transformations

x = X+5t, y = X+8t (2)

In (1) leads to the positive pell equation

$$X^2 = 40T^2 + 9 \tag{3}$$

Whose smallest positive integer solution is

$$T_0 = 1, X_0 = 7$$

To obtain the other solutions of (1), consider the pell equation

$$X^2 = 40T^2 + 1$$

whose general solution ($\tilde{T}_n, \tilde{X}_n$) is given by

$$\tilde{T}_n = \frac{1}{2\sqrt{40}} g_n$$

$$\tilde{X}_n = \frac{1}{2} f_n$$

where

$$f_n = \left(19 + 3\sqrt{40}\right)^{n+1} + \left(19 - 3\sqrt{40}\right)^{n+1}$$

$$g_n = \left(19 + 3\sqrt{40}\right)^{n+1} - \left(19 - 3\sqrt{40}\right)^{n+1}$$

Using Brahmagupta lemma between (T_0, X_0) and $\left(\tilde{T}_n, \tilde{X}_n\right)$, the other integer solutions of (3) are obtainedas

$$T_{n+1} = \frac{1}{2} f_n + \frac{7}{2\sqrt{40}} g_n$$

$$X_{n+1} = \frac{7}{2} f_n + \frac{40}{2\sqrt{40}} g_n$$

Using (2), the corresponding integer solution to (1) are given by

$$x_{n+1} = 6f_n + \frac{75}{2\sqrt{40}} g_n$$

$$y_{n+1} = \frac{15}{2} f_n + \frac{48}{\sqrt{40}} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 38x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 38y_{n+2} + y_{n+1} = 0$$

Some numerical examples of the x and y satisfying (1) are given in the following Table 1

n	x_{n+1}	y_{n+1}
-1	12	15
0	453	573
1	17202	21759
2	653223	826269
3	24805272	31376463
4	941947113	1191479325

Table 1: Numerical examples

From the above Table 1, we observe some interesting relations among the solutions which are presented below:

I) The values of x_{n+1} are alternatively even and odd whereas,the value of y_{n+1} are always odd.

II) $x_{n+1}, y_{n+1} \equiv 0 \pmod 3$

III) From the values of x_{n+1} , $y_{n+1} (n > -1)$,one may generate second order Ramanujan numbers with base numbers as real integers.

Illustration :1

Consider

$$x_2 = 17202$$

Write 17202 as the product of two integers.

17202 = 1*17202= 2*8601= 6*2867

The above relation is considered as follows:

$$17202 = 1*17202 = 2*8601 \Rightarrow$$

$$(1+17202)^2 + (8601-2)^2 = (17202-1)^2 + (2+8601)^2$$
$$\Rightarrow (17203)^2 + (8599)^2 = (17201)^2 + (8603)^2 = 369886010$$

$$1*17202 = 6*2867 \Rightarrow$$

$$(1+17202)^2 + (2867-6)^2 = (17202-1)^2 + (6+2867)^2$$
$$\Rightarrow (17203)^2 + (2861)^2 = (17201)^2 + (2873)^2 = 304128530$$

$$2*8601 = 6*2867 \Rightarrow$$

$$(2+8601)^2+(2867-6)^2=(8601-2)^2+(6+2867)^2$$

$$\Rightarrow (8603)^2+(2861)^2=(8599)^2+(2873)^2=82196930$$

Thus,369886010,304128530,82196930are the second order Ramanujan numbers

Illustration:2

Consider

$$y_3=826269$$

Write 826269 as the product of two integers.

$$826269=1*826269=3*275423$$

The above relation is considered as follows:

Now

$$1*826269=3*275423\Rightarrow$$

$$(1+826269)^2+(275423-3)^2=(826269-1)^2+(3+275423)^2$$

$$\Rightarrow (826270)^2+(275420)^2=(826268)^2+(275426)^2$$

$$=758578289300$$

Thus, 758578289300 is the second order Ramanujan number.

IV) Expressions as square integers

- $\frac{1}{405}(5730x_{2n+2}-150x_{2n+3})+2$
- $\frac{1}{15390}(217590x_{2n+2}-150x_{2n+4})+2$
- $\frac{1}{9}(64x_{2n+2}-50y_{2n+2})+2$
- $\frac{1}{513}(7248x_{2n+2}-150y_{2n+3})+2$
- $\frac{1}{6489}(91744X_{2n+2}-50y_{2n+4})+2$
- $\frac{1}{405}(217590x_{2n+3}-5730x_{2n+4})+2$
- $\frac{1}{513}(192\,x_{2n+3}-5730\,y_{2n+2})+2$

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□ $\frac{1}{27}(7248x_{2n+3}-5730y_{2n+3})+2$

□ $\frac{1}{513}(275232x_{2n+3}-5730y_{2n+4})+2$

V) Expressions as cubical integers

□ $\frac{1}{405}[(5730x_{3n+3}-150x_{3n+4})+3(5730x_{n+1}-150x_{n+2})]$

□ $\frac{1}{15390}[(217590x_{3n+3}-150x_{3n+5})+3(217590x_{n+1}-150x_{n+3})]$

□ $\frac{1}{9}(64x_{3n+3}-50y_{3n+3})+3(64x_{n+1}-50y_{n+1})$

□ $\frac{1}{513}[(7248x_{2n+3}-150y_{3n+4})+3(7248x_{n+1}-150y_{n+2})]$

□ $\frac{1}{6489}[(91744x_{3n+3}-50y_{3n+5})+3(91744x_{n+1}-50y_{n+3})]$

□ $\frac{1}{405}[(217590x_{3n+4}-5730x_{3n+5})+3(217590x_{n+2}-5730x_{n+3})]$

□ $\frac{1}{513}[(192\ x_{3n+4}-5730\ y_{3n+3})+3(192\ x_{n+2}-5730\ y_{n+1})]$

□ $\frac{1}{27}[(7248x_{3n+4}-5730y_{3n+4})+3(7248x_{n+2}-5730y_{n+2})]$

□ $\frac{1}{513}[(275232x_{3n+4}-5730y_{3n+5})+3(275232x_{n+2}-5730y_{n+3})]$

VI) Expressions as biquadratic integers

□ $\frac{1}{405}[(5730x_{4n+4}-150x_{4n+5})+\frac{4}{405}(5730x_{n+1}-150x_{n+2})^2]-2$

□ $\frac{1}{15390}[(217590x_{4n+4}-150x_{4n+6})+\frac{4}{15390}(217590x_{n+1}-150x_{n+3})^2]-2$

□ $\frac{1}{9}[(64x_{4n+4}50y_{4n+4})+\frac{4}{9}(64x_{n+1}-50y_{n+1})^2]-2$

□ $\frac{1}{513}[(7248x_{4n+4}-150y_{4n+5})+\frac{4}{513}(7248x_{n+1}-150y_{n+2})^2]-2$

- $\frac{1}{6489}[(91744x_{4n+4}-50y_{4n+6})+\frac{4}{6489}(91744x_{n+1}-50y_{n+3})^2]-2$
- $\frac{1}{405}[(217590x_{4n+5}-5730x_{4n+6})+\frac{4}{405}(217590x_{n+2}-5730x_{n+3})^2]-2$
- $\frac{1}{513}[(192\ x_{4n+5}-5730\ y_{4n+4})+\frac{4}{513}(192\ x_{n+2}-5730\ y_{n+1})^2]-2$
- $\frac{1}{27}[(7248\ x_{4n+5}-5730\ y_{4n+5})+\frac{4}{27}(7248\ x_{n+2}-5730\ y_{n+2})^2]-2$
- $\frac{1}{513}[(275232\ x_{4n+5}-5730\ y_{4n+6})+\frac{4}{513}(275232\ x_{n+2}-5730\ y_{n+3})^2]-2$

VII) Some relations among the solutions are as follows

- $810x_{n+3}+810x_{n+1}-30780x_{n+2}=0$
- $2631690y_{n+1}+3328857x_{n+1}-4617x_{n+3}=0$
- $18y_{n+2}-432x_{n+1}-342y_{n+1}=0$
- $1026y_{n+3}-1296x_{n+1}-38934y_{n+2}=0$
- $12978x_{n+2}-342x_{n+1}-270y_{n+3}=0$
- $270y_{n+1}+12978x_{n+2}-342x_{n+3}=0$
- $1026y_{n+2}-1296x_{n+2}-54y_{n+1}=0$
- $18y_{n+3}-432x_{n+2}-342y_{n+2}=0$
- $1026x_{n+3}-54x_{n+2}-810y_{n+3}=0$

VIII) Remarkable Observations:

(i) Table 2 represents choice of hyperbolas obtained through giver solutions

Table 2: Hyperbolas

S.NO	Hyperbola	X_n, Y_n
1.	$Y_n^2-40X_n^2=656100$	$X_n=24x_{n+2}-906x_{n+1}$ $Y_n=5730x_{n+1}-150x_{n+2}$
2.	$Y_n^2-160X_n^2=947408400$	$X_n=12x_{n+3}-17202x_{n+1}$ $Y_n=217590x_{n+1}-150x_{n+3}$

3.	$Y_n^2 - 40X_n^2 = 324$	$X_n = 8y_{n+1} - 10x_{n+1}$ $Y_n = 64x_{n+1} - 50y_{n+1}$
4.	$Y_n^2 - 40X_n^2 = 1052676$	$X_n = 24y_{n+2} - 1146x_{n+1}$ $Y_n = 7248x_{n+1} - 150y_{n+2}$
5.	$Y_n^2 - 40X_n^2 = 168428484$	$X_n = 8y_{n+3} - 14506x_{n+1}$ $Y_n = 91744x_{n+1} - 50y_{n+3}$
6.	$Y_n^2 - 360X_n^2 = 656100$	$X_n = 302x_{n+3} - 11468x_{n+2}$ $Y_n = 217590x_{n+2} - 5730x_{n+3}$
7.	$Y_n^2 - 40X_n^2 = 1052676$	$X_n = 906y_{n+1} - 30x_{n+2}$ $Y_n = 192x_{n+2} - 5730y_{n+1}$
8.	$Y_n^2 - 360X_n^2 = 2916$	$X_n = 302y_{n+2} - 382x_{n+2}$ $Y_n = 7248x_{n+2} - 5730y_{n+2}$
9.	$Y_n^2 - 40X_n^2 = 1052676$	$X_n = 906y_{n+3} - 43518x_{n+2}$ $Y_n = 275232x_{n+2} - 5730y_{n+3}$

(ii) Table 3 represents choice of pararbolas obtained through giver solutions

Table 3: Parabolas

S.NO	**Parabola**	X_n, α_n
1.	$40X_n^2 = 405\alpha_n - 328050$	$X_n = 24x_{n+2} - 906x_{n+1}$ $\alpha_n = 5730x_{2n+2} - 150x_{2n+3}$
2.	$80X_n^2 = 7695\alpha_n - 236852100$	$X_n = 12x_{n+3} - 17202x_{n+1}$ $\alpha_n = 217590x_{2n+2} - 150x_{2n+4}$

3.	$40X_n^2 = 9\alpha_n - 162$	$X_n = 8y_{n+1} - 10x_{n+1}$ $\alpha_n = 64x_{2n+2} - 50y_{2n+2}$
4.	$40X_n^2 = 513\alpha_n - 526338$	$X_n = 24y_{n+2} - 1146x_{n+1}$ $\alpha_n = 7248x_{2n+2} - 150y_{2n+3}$
5.	$40X_n^2 = 6489\alpha_n - 84214242$	$X_n = 8y_{n+3} - 14506x_{n+1}$ $\alpha_n = 91744X_{2n+2} - 50y_{2n+4}$
6.	$12X_n^2 = 135\alpha_n - 109350$	$X_n = 302x_{n+3} - 11468x_{n+2}$ $\alpha_n = 217590x_{2n+3} - 5730x_{2n+4}$
7.	$40X_n^2 = 513\alpha_n - 526338$	$X_n = 906y_{n+1} - 30x_{n+2}$ $Y_n = 192\ x_{2n+3} - 5730\ y_{2n+2}$
8.	$120X_n^2 = 9\alpha_n - 486$	$X_n = 302y_{n+2} - 382x_{n+2}$ $\alpha_n = 7248x_{2n+3} - 5730y_{2n+3}$
9.	$40X_n^2 = 513\alpha_n - 526338$	$X_n = 906y_{n+3} - 43518x_{n+2}$ $\alpha_n = 275232x_{2n+3} - 5730y_{2n+4}$

(iii). Consider $p = x_{n+1} + y_{n+1}, q = x_{n+1}$. Note that $p>q>0$. Treat p, q as the generators of the Pythagorean triangle $T(X,Y,Z)$ where $X = 2pq$, $Y = p^2 - q^2$, $Z = p^2 + q^2$. Denote the Area and Perimeter by A, P respectively.

Then the following results are obtained:

a) $16X - 5y - 11Z + 54 = 0$

b) $10x - 8Y - 2z - 54 = 0$

c) $\frac{2A}{p} = x_{n+1}y_{n=1}$

d) $3(Z - Y)$ is a nasty number

e) $3(X - 4A\ /\ )$ is a nasty number

f) $X - (4A/P) + Y$is written as the sum of two squares

IX) Let $\alpha_n = \frac{x_{2n}}{2}, \beta_n = \frac{y_{2n} - 1}{2}$

It is observed that

(i) $5t_{3,\beta_n} = 4[(t_{4,\alpha_n})^2 - 1]$

(ii) $2(t_{6,\alpha_n} + \alpha_n) = 5t_{3,\beta_n} + 4$

(iii) $t_{17,\alpha} + t_{19,\alpha} + 14\alpha = 20t_{3,\beta} + 16$

Conclusion:

An attempt has been made to determine solutions in integers to the hyperbola $8x^2 = 5y^2 + 27$. As second-degree equation with two unknowns areplenty, the readers mayanalyse different form of hyperbolas for solutions in integers with interesting properties.

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