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New Generalizations of Nonlinear Retarded Integral Iequalities of Gronwall-Bellman Type and Their Applications

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Abstract: In the current paper we establish some new retarded integral inequalities applicable in the theory of differential equations. These inequalities generalizes some former famous inequalities on the subject. Some applications are also given to convey the importance of our result. .

Keywords: Integral inequality, Retarded integral inequality, Gronwall-Bellman inequality, Boundedness.

1. Introduction

It is well known that integral inequalities have become major role in the development of the theory of linear and nonlinear ordinary differential equations, integral equations, differential equations, and integro-differential equations. see for instance [1-5,9,10,15]. Some integral inequalities for differential and integral equations are introduced by Gronwall [1], Bellman [8] and Pachpatte [9] gives explicit bounds on unknown functions play a fundamental role in the study of qualitative as well as quantitative properties of solutions of nonlinear differential equations. One of the well-known inequalities in the study of nonlinear differential equations is Gronwall-Bellman-inequality and it's generalization can be found in [2, 5-7, 9, 11-14, 17]. Gronwall-Bellman inequality can be stated as

Lemma 1.1 Let f(t) and u(t) be a real-valued nonnegative continuous functions defined on D_1=[0, h]. and let u_0 and h be positive constants for which the inequality

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds, \ \forall t \in D_1.$$

Then

$u(t) \leq u_0 + exp\left(\int_0^t f(s)ds\right), \ \forall t \in D_1.$

2. Preliminaries

Throughout this article, $\mathbb{R}$ denoted the set of real numbers, $\mathbb{I} = 0, \infty) \subset \mathbb{R}$ and $\mathbb{C}(\mathbb{I}, \mathbb{I})$ denotes the set of all continuous functions from $\mathbb{I}$ into $\mathbb{I}$ and $\mathbb{C}'(\mathbb{I}, \mathbb{I})$ denotes the set of all continuously differentiable functions from $\mathbb{I}$into $\mathbb{I}$.

Lemma 2.1 [5] Assume that $a \geq 0, p \geq q \geq 0$ and $p \neq 0$,

$$a^{\frac{q}{p}} \leq \frac{q}{p} k^{\frac{q-p}{p}} a + \frac{p-q}{p} k^{\frac{q}{p}}.$$

for any k > 0.

Lemma 2.2 [5] Let f(t) and g(t) be continuous functions for $t \leq \propto$, Let u(t) be a differentiable function for $t \leq \alpha$, and suppose

$$u'(t) \leq f(t) + g(t)u(t), u(\alpha) \leq u_0. \qquad 2.1$$

Then, for $t \geq \alpha$,

$$u(t) \leq u_0\, exp\left(\int_{\alpha}^{t} g\,(s)ds\right) + \int_0^t f\,(s)\, exp\left(\int_s^t g\,(\lambda)d\lambda\right) ds. \qquad 2.2$$

The main aim of this paper is to establish explicit bounds on retarded Gronwall-Bellman inequalities which play very important role in the study of certain retarded integral and differential equations.

3. Main Results

In this section, we state and prove some new nonlinear retarded integral inequalities of Gronwall-Bellman type, which can be used in estimating the boundedness and global existence of the solution of retarded nonlinear differential and integral equations.

Theorem 3.1 Let $u(t), f(t), g(t), c(t) \in \mathbb{C}(I, I)$and c(t) is nondecreasing. Let $\alpha(t) \in \mathbb{C}'(I, I)$ be a nondecreasing function with $\alpha(t) \leq t,\ \alpha(0) = 0$ and $\phi: I \to I$ be a differentiable increasing function on $(0, \infty)$ with continuous nonincreasing first derivative ϕ'on $(0, \infty)$.

If the inequality

$$u^{w_1(p)}(t) \leq c(t) + \int_0^t f\,(s)u^{w_1(p)}(s)ds + \int_0^{\alpha(t)} g\,(s)\phi\left[u^{w_2(q)}(s)\right]ds, \qquad 3.1$$

holds for $\forall t \in I$, for $w_1(p) > w_2(q) \geq 0$.

Then

$$u(t) \leq \{c(t) + \int_0^t p_1\,(s)\, exp\left(\int_s^t q_1\,(r)dr\right) ds\}^{\frac{1}{w_1(p)}},\ \forall t \in I. \qquad 3.2$$

where,

$$p_1(t) = f(t)c(t) + \alpha'(t)g(\alpha(t))\phi\left(\frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} c(t) + \frac{w_1(p) - w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right)$$

and

$q_1(t) = f(t) + \frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} \alpha'(t) g(\alpha(t)) \times \phi'\left(\frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} c(t) + \frac{w_1(p)-w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right)$.3.3

Proof: Define a function z(t) by

$z(t) = \int_0^t f(s) u^{w_1(p)}(s) ds + \int_0^{\alpha(t)} g(s) \phi\left[u^{w_2(q)}(s)\right] ds.$ 3.4

Then we have

$u(t) \leq [c(t) + z(t)]^{\frac{1}{w_1(p)}} and\ z(0) = 0.$ 3.5

Differentiating z(t) w.r.t. t and using (3.5) we get

$$\begin{aligned} z'(t) &= f(t) u^{w_1(p)}(t) + g(\alpha(t)) \phi\left[u^{w_2(q)}(\alpha(t))\right] \alpha'(t) \\ &\leq f(t)[c(t) + z(t)] + g(\alpha(t)) \phi[c(\alpha(t)) + z(\alpha(t))]^{\frac{w_2(q)}{w_1(p)}} \alpha'(t) \\ &\leq f(t)[c(t) + z(t)] + \alpha'(t) g(\alpha(t)) \phi\left[\left(c(\alpha(t)) + z(\alpha(t))\right)^{\frac{w_2(q)}{w_1(p)}}\right]. \end{aligned}$$

By applying lemma 2.1, we get

$z'(t) \leq f(t)[c(t) + z(t)] + \alpha'(t) g(\alpha(t)) \phi\left(\frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} (c(t) + z(t)) + \frac{w_1(p)-w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right)$.3.6

Now, by mean value theorem for $\phi, x_1 > y_1 > 0$, there exist $c \in (x_1, y_1)$ such that

$$\phi(x_1) - \phi(y_1) = \phi'(c)(x_1 - y_1) \leq \phi'(y_1)(x_1 - y_1).$$

Therefore

$$\begin{aligned} &\phi\left(\frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} (c(t) + z(t)) + \frac{w_1(p) - w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right) \\ &-\phi\left(\frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} c(t) + \frac{w_1(p) - w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right) \\ &\leq \phi'\left(\frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} c(t) + \frac{w_1(p) - w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right) \times \frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} z(t). \end{aligned}$$

Hence inequality (3.6) becomes

$$\begin{aligned} z'(t) \leq f(t)[c(t) + z(t)] + \alpha'(t) g\big(\alpha(t)\big) &\left[\phi\left(\frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} c(t) + \frac{w_1(p) - w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right)\right. \\ &\left.+ \frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} \phi'\left(\frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} c(t) + \frac{w_1(p) - w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right) z(t)\right]. \end{aligned}$$

hence, we get,

$$z'(t) \leq f(t) n(t) + \alpha'(t) g\big(\alpha(t)\big) \phi\left(\frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} n(t) + \frac{w_1(p) - w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right)$$

$+\left[f(t) + \frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} \alpha'(t) g\big(\alpha(t)\big) \phi'\left(\frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} n(t) + \frac{w_1(p)-w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right)\right] z(t)$.3.7

Therefore we get,

$$z'(t) \le p_1(t) + q_1(t)z(t).$$

Applying Lemma 2.2 we get

$$z(t) \le \int_0^t p_1(s)\, exp\left(\int_s^t q_1(r)dr\right) ds,\ \forall\ t \in I. \qquad 3.8$$

Hence inequality (3.5) becomes

$$u(t) \le \{c(t) + \int_0^t p_1(s)\, exp\left(\int_s^t q_1(r)dr\right) ds\}^{\frac{1}{w_1(p)}},\ \forall t \in I. \qquad 3.9$$

Where $p_1(t)$ and $q_1(t)$ are defined as in equation (3.3).

Remark 3.1 If we put $w_1(p) = p, c(t) = u_0,\ \alpha(t) = t, w_2(q) = q, \phi_1(u(s)) = u(s)$then Theorem 3.1 reduces to Theorem 3.1 in [17].

Remark 3.2 If we put $w_1(p) = p, \phi_1(u(s)) = u(s)$ and $w_2(q) = q$.then Theorem 3.1 reduces to Theorem 2.3 in [18].

Remark 3.3 If we put $w_1(p) = p, \phi_1(u(s)) = u(s)$ and $w_2(q) = 1$.then Theorem 3.1 reduces to Theorem 2.1 in [19].

Remark 3.4 If we put $w_1(p) = 1$ in L.H.S. and $w_1(p) = p, \phi_1(u(s)) = u(s), w_2(q) = p$. then Theorem 3.1 reduces to Theorem 2.1 in [21].

Theorem 3.2 Let $u(t), f(t), g(t), c(t) \in \mathbb{C}(I, I)$ and c(t) is nondecreasing. Let $\alpha(t) \in \mathbb{C}'(I, I)$ be a nondecreasing function with $\alpha(t) \le t, \alpha(0) = 0$ and let ϕ_1 and $\phi_2 : I \to I$ be differentiable increasing functions on $(0, \infty)$ with continuous nonincreasing first derivative ϕ'on $(0, \infty)$.

If the inequality

$$u^{w_1(p)}(t) \le c(t) + \int_0^t f(s)\phi_1\left(u^{w_2(q)}(s)\right)ds + \int_0^{\alpha(t)} g(s)\phi_2\left(u^{w_3(r)}(s)\right)ds, \qquad 3.10$$

for all $t \in I, w_1(p) > w_2(q) \ge 0,$ and $w_1(p) > w_3(r) \ge 0$. then

$$u(t) \le \{c(t) + \int_0^t p_2(s)\, exp\left(\int_s^t q_2(r)dr\right) ds\}^{\frac{1}{w_1(p)}},\ \forall t \in I, \qquad 3.11$$

Where,

$$p_2(t) = f(t)\phi_1\left(\frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} c(t) + \frac{w_1(p) - w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right)$$
$$+\alpha'(t)g(\alpha(t))\phi_2\left(\frac{w_3(r)}{w_1(p)} k^{\frac{w_3(r)-w_1(p)}{w_1(p)}} c(t) + \frac{w_1(p) - w_3(r)}{w_1(p)} k^{\frac{w_3(r)}{w_1(p)}}\right),$$

and

$$q_2(t) = \frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} f(t)\phi'_1\left(\frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} c(t) + \frac{w_1(p) - w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right)$$
$$+\alpha'(t)g(\alpha(t))\frac{w_3(r)}{w_1(p)} k^{\frac{w_3(r)-w_1(p)}{w_1(p)}} \phi'_2\left(\frac{w_3(r)}{w_1(p)} k^{\frac{w_3(r)-w_1(p)}{w_1(p)}} c(t) + \frac{w_1(p)-w_3(r)}{w_1(p)} k^{\frac{w_3(r)}{w_1(p)}}\right). \qquad 3.12$$

Proof: Define a function z(t) by

$$z(t) = \int_0^t f(s)\phi_1\left(u^{w_2(q)}(s)\right)ds + \int_0^{\alpha(t)} g(s)\phi_2\left(u^{w_3(r)}(s)\right)ds. \qquad 3.13$$

Then we have z(0)=0, and

$$u(t) \le [c(t)+z(t)]^{\frac{1}{w_1(p)}}. \qquad 3.14$$

Differentiating (3.11) w.r.t. t and using (3.13), we get

$$\begin{aligned} z'(t) &= f(t)\phi_1\left(u^{w_2(q)}(t)\right) + g(\alpha(t))\phi_2\left(u^{w_3(r)}(\alpha(t))\right)\alpha'(t) \\ &\le f(t)\phi_1\left([c(t)+z(t)]^{\frac{w_2(q)}{w_1(p)}}\right) + \alpha'(t)g(\alpha(t))\phi_2\left([c(\alpha(t))+z(\alpha(t))]^{\frac{w_3(r)}{w_1(p)}}\right) \\ &\le f(t)\phi_1\left([c(t)+z(t)]^{\frac{w_2(q)}{w_1(p)}}\right) + \alpha'(t)g(\alpha(t))\phi_2\left([c(t)+z(t)]^{\frac{w_3(r)}{w_1(p)}}\right). \end{aligned}$$

Applying lemma 2.1, we have

$$z'(t) \le f(t)\phi_1\left(\frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}[c(t)+z(t)] + \frac{w_1(p)-w_2(q)}{w_1(p)}k^{\frac{w_2(q)}{w_1(p)}}\right)$$

$$+\alpha'(t)g(\alpha(t))\phi_2\left(\frac{w_3(r)}{w_1(p)}k^{\frac{w_3(r)-w_1(p)}{w_1(p)}}[c(t)+z(t)] + \frac{w_1(p)-w_3(r)}{w_1(p)}k^{\frac{w_3(r)}{w_1(p)}}\right). \qquad 3.15$$

Applying mean value theorem for the functions ϕ_1 and ϕ_2 we obtain

$$\phi(x_1)-\phi(y_1) = \phi'(c)(x_1-y_1) \le \phi'(y_1)(x_1-y_1).$$

Therefore

$$\begin{aligned} &\phi_1\left(\frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}[c(t)+z(t)] + \frac{w_1(p)-w_2(q)}{w_1(p)}k^{\frac{w_2(q)}{w_1(p)}}\right) \\ &\le \phi_1\left(\frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}c(t) + \frac{w_1(p)-w_2(q)}{w_1(p)}k^{\frac{w_2(q)}{w_1(p)}}\right) \\ &+\frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}\phi'_1\left(\frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}c(t) + \frac{w_1(p)-w_2(q)}{w_1(p)}k^{\frac{w_2(q)}{w_1(p)}}\right)z(t), \end{aligned}$$

and

$$\begin{aligned} &\phi_2\left(\frac{w_3(r)}{w_1(p)}k^{\frac{w_3(r)-w_1(p)}{w_1(p)}}[c(t)+z(t)] + \frac{w_1(p)-w_3(r)}{w_1(p)}k^{\frac{w_3(r)}{w_1(p)}}\right) \\ &\le \phi_2\left(\frac{w_3(r)}{w_1(p)}k^{\frac{w_3(r)-w_1(p)}{w_1(p)}}c(t) + \frac{w_1(p)-w_3(r)}{w_1(p)}k^{\frac{w_3(r)}{w_1(p)}}\right) \\ &+\frac{w_3(r)}{w_1(p)}k^{\frac{w_3(r)-w_1(p)}{w_1(p)}}\phi'_2\left(\frac{w_3(r)}{w_1(p)}k^{\frac{w_3(r)-w_1(p)}{w_1(p)}}c(t) + \frac{w_1(p)-w_3(r)}{w_1(p)}k^{\frac{w_3(r)}{w_1(p)}}\right)z(t). \end{aligned}$$

Hence inequality (3.15) becomes

$$\begin{aligned} z'(t) &\le f(t)[\phi_1\left(\frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}c(t) + \frac{w_1(p)-w_2(q)}{w_1(p)}k^{\frac{w_2(q)}{w_1(p)}}\right) \\ &+\frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}\phi'_1\left(\frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}c(t) + \frac{w_1(p)-w_2(q)}{w_1(p)}k^{\frac{w_2(q)}{w_1(p)}}\right)z(t)] \\ &+\alpha'(t)g(\alpha(t))[\phi_2\left(\frac{w_3(r)}{w_1(p)}k^{\frac{w_3(r)-w_1(p)}{w_1(p)}}c(t) + \frac{w_1(p)-w_3(r)}{w_1(p)}k^{\frac{w_3(r)}{w_1(p)}}\right) \end{aligned}$$

$$+\frac{w_3(r)}{w_1(p)}k^{\frac{w_3(r)-w_1(p)}{w_1(p)}}\phi'_2\left(\frac{w_3(r)}{w_1(p)}k^{\frac{w_3(r)-w_1(p)}{w_1(p)}}c(t)+\frac{w_1(p)-w_3(r)}{w_1(p)}k^{\frac{w_3(r)}{w_1(p)}}\right)z(t)].$$

Thus

$$z'(t)\le f(t)\phi_1\left(\frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}c(t)+\frac{w_1(p)-w_2(q)}{w_1(p)}k^{\frac{w_2(q)}{w_1(p)}}\right)$$
$$+\alpha'(t)g(\alpha(t))\phi_2\left(\frac{w_3(r)}{w_1(p)}k^{\frac{w_3(r)-w_1(p)}{w_1(p)}}c(t)+\frac{w_1(p)-w_3(r)}{w_1(p)}k^{\frac{w_3(r)}{w_1(p)}}\right)$$
$$+\left[\frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}f(t)\phi'_1\left(\frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}c(t)+\frac{w_1(p)-w_2(q)}{w_1(p)}k^{\frac{w_2(q)}{w_1(p)}}\right)\right.$$
$$\left.+\alpha'(t)g(\alpha(t))\frac{w_3(r)}{w_1(p)}k^{\frac{w_3(r)-w_1(p)}{w_1(p)}}\phi'_2\left(\frac{w_3(r)}{w_1(p)}k^{\frac{w_3(r)-w_1(p)}{w_1(p)}}c(t)+\frac{w_1(p)-w_3(r)}{w_1(p)}k^{\frac{w_3(r)}{w_1(p)}}\right)\right]z(t).$$

Therefore we get,

$$z'(t)\le p_2(t)+q_2(t)z(t). \qquad 3.16$$

Applying lemma 2.2

$$z(t)\le\int_0^t p_2\,(s)\,exp\left(\int_s^t q_2\,(r)dr\right)ds,\ \forall\ t\in I.$$

Hence inequality (3.14) becomes

$$u(t)\le\{c(t)+\int_0^t p_2\,(s)\,exp\left(\int_s^t q_2\,(r)dr\right)ds\}^{\frac{1}{w_1(p)}}.$$

Where $p_2(t)$ and$q_2(t)$ are defined as in Eq. (3.12).

Remark 3.5 If we put $u^{w_1(p)}(t)=\phi_1(u(t)), w_1(p)=w_2(q)=w_3(r)=1.$ then Theorem 3.2 reduces to Theorem 2.2 in [18].

Remark 3.6 If we put $w_1(p)=p$ in L.H.S. and $\phi_1(u(s))=u(s), w_2(q)=p, g(s)=k(t,s), \phi_2(u(s))=u(s)$, and $w_3(r)=1$ then Theorem 3.2 reduces to Theorem 2.2 in [19].

Remark 3.7 If we put $w_1(p)=p, w_2(q)=p, \phi_1(u(s))=u(s), w_3(r)=1$ and $\phi_2(u(s))=u(s)$ then Theorem 3.2 reduces to Theorem 2.2 in [21].

Theorem 3.3 Let $u(t), f(t), g(t), c(t)\in\mathbb{C}(I,I)$ and let $c(t)$ is nondecreasing. Let $\alpha(t)\in\mathbb{C}'(I,I)$ be a nondecreasing function with $\alpha(t)\le t, \alpha(0)=0$ and let $L, M\in\mathbb{C}(I^2,I)$ satisfy $0\le L(t,x)-L(t,y)\le M(t,y)(x-y), x\ge y\ge 0$.

If the inequality

$$u^{w_1(p)}(t)\le c(t)+\int_0^t f\,(s)u^{w_1(p)}(s)ds+\int_0^{\alpha(x)} f\,g(s)L\big(s,u^{w_2(q)}(s)\big)ds, \qquad 3.17$$

for all $t\in I,\ w_1(p)>w_2(q)\ge 0.$ then

$$u(t)\le\{c(t)+\int_0^t p_3\,(s)\,exp\left(\int_s^t q_3\,(r)dr\right)ds\}^{\frac{1}{w_1(p)}},\ \forall\ t\in I. \qquad 3.18$$

where,

$$p_3(t) = f(t)c(t) + \alpha'(t)\ g(\alpha(t))\ L\left(\alpha(t),\ \frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} c(t) + \frac{w_1(p)-w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right),$$

and

$q_3(t) = f(t) + \frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} \alpha'(t)\ g(\alpha(t))\ M\left(\alpha(t), \frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} c(t) + \frac{w_1(p)-w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right)$.3.19

Proof: Define a function z(t) by

$z(t) = \int_0^t f(s) u^{w_1(p)}(s) ds + \int_0^{\alpha(t)} g(s) L\left(s, u^{w_2(q)}(s)\right) ds.$
3.20

Then we have z(0)=0 and

$u(t) \leq [c(t) + z(t)]^{\frac{1}{w_1(p)}}.$
3.21

Differentiating (3.20) w.r.t. t and using (3.21), we get

$$\begin{aligned} z'(t) &= f(t)\ u^{w_1(p)}(t) + g(\alpha(t)) L\left(\alpha(t), u^{w_2(q)}(\alpha(t))\right) \alpha'(t) \\ &\leq f(t)[c(t) + z(t)] + \alpha'(t) g(\alpha(t)) L\left(\alpha(t), [c(t) + z(t)]^{\frac{w_2(q)}{w_1(p)}}\right). \end{aligned}$$

Applying lemma 2.1

$$z'(t) \leq f(t)[c(t) + z(t)] + \alpha'(t) g(\alpha(t)) L\left(\alpha(t), \frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} [c(t) + z(t)] + \frac{w_1(p)-w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right). \quad 3.22$$

Since

$$0 \leq L(t,x) - L(t,y) \leq M(t,y)(x-y),\ x \geq y \geq 0,$$

we have

$$\begin{aligned} & L\left(\alpha(t), \frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} [c(t) + z(t)] + \frac{w_1(p)-w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right) \\ & \leq L\left(\alpha(t), \frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} c(t) + \frac{w_1(p)-w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right) \\ & +M\left(\alpha(t), \frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} c(t) + \frac{w_1(p)-w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right) \times \frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} z(t). \end{aligned}$$

Hence (3.22) becomes

$$\begin{aligned} z'(t) \leq\ & f(t)[c(t) + z(t)] \\ & + \alpha'(t) g(\alpha(t)) [L\left(\alpha(t), \frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} c(t) + \frac{w_1(p)-w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right) \\ & +M\left(\alpha(t), \frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} c(t) + \frac{w_1(p)-w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right) \times \frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} z(t)]. \end{aligned}$$

Therefore

$$z'(t) \leq f(t)c(t) + \alpha'(t)g(\alpha(t))L\left(\alpha(t), \frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}c(t) + \frac{w_1(p)-w_2(q)}{w_1(p)}k^{\frac{w_2(q)}{w_1(p)}}\right)$$
$$+[f(t) + \frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}\alpha'(t)g(\alpha(t))M\left(\alpha(t), \frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}c(t)\right.$$
$$\left.+\frac{w_1(p)-w_2(q)}{w_1(p)}k^{\frac{w_2(q)}{w_1(p)}}\right)]z(t),$$

$z'(t) \leq p_3(t) + q_3(t)z(t).$ 3.23

Applying lemma 2.2, we get

$z(t) \leq \int_0^t p_3(s)\, exp\left(\int_s^t q_3(r)dr\right) ds \ \forall \ t \in I.$ 3.24

Hence inequality (3.21) becomes

$$u(t) \leq \{c(t) + \int_0^t p_3(s)\, exp\left(\int_s^t q_3(r)dr\right) ds\}^{\frac{1}{w_1(p)}}.$$

where $p_3(t)$ and $q_3(t)$ are defined as in (3.19).

Remark 3.8 If we put $w_1(p) = 2$ in L.H.S. and $w_1(p) = w_2(q) = 1, c(t) = c^2, \alpha(t) = t, L(t,x) = x$ and $f(s) = g(s)$ in R.H.S. of Theorem 3.3 then Theorem 3.3 reduces Theorem 3.4.1 in [9].

Remark 3.9 If we put $w_1(p) = p, L(t,x) = x$, and $w_2(q) = q$ then Theorem 3.3 reduces to Theorem 2.3 in [18]

Remark 3.10 If we put $w_1(p) = p, f(s) = k_2(t,s), g(s) = k_1(t,s), L(t,x) = x$, and $w_2(q) = 1$then Theorem 3.3 reduces to Theorem 2.3 in [19]

Theorem 3.4 Let $u(t), f(t), g(t) \in \mathbb{C}(I,I)$. Let $\alpha(t) \in \mathbb{C}'(I,I)$be a nondecreasing function with $\alpha(t) \leq t, \alpha(0) = 0$ and let $L, M \in \mathbb{C}(I^2, I)$ satisfy

$0 \leq L(t,x) - L(t,y) \leq M(t,y)(x-y), \ x \geq y \geq 0.$

If the inequality

$u(t) \leq u_0 + \int_0^{\alpha(t)} f(s)\left[u^{(2-w_1(p))}(s) + \int_0^s g(r)L\left(r, u^{w_2(q)}(r)\right)dr\right]^{w_1(p)} ds,$ 3.25

holds for $\forall t \in I$, for $u_0 > 0, 0 < w_1(p) \leq 1$, and $0 \leq w_2(q) < 1$ then

$u(t) \leq u_0 + \int_0^{\alpha(t)} \ f(s)\ q_4(\alpha^{-1}(s))\, exp\left(w_1(p)(2-w_1(p))\int_0^s p_4(r)dr\right) ds, \forall \ t \in I.$ 3.26

where, $p_4(t) = f(t) + \frac{w_2(q)}{(2-w_1(p))}k^{(w_2(q)-1)}g(t)M\left(t, (1-w_2(q))k^{w_2(q)}\right),$

and

$q_4(t) = u_0^{(2-w_1(p))} + \int_0^{\alpha(t)} g(t)L\left(t, (1-w_2(q))k^{w_2(q)}\right) \times exp\left(-(2-w_1(p))\int_0^s p_4(r)dr\right) ds.$

Proof: Define a function z(t) by

$z(t) = u_0 + \int_0^{\alpha(t)} f(s)\left[u^{(2-w_1(p))}(s) + \int_0^s g(r)L\left(r, u^{w_2(q)}(r)\right)dr\right]^{w_1(p)} ds,$ 3.27

and $z(0) = u_0$.

Inequality (3.25) becomes

$u(t) \leq z(t).$ 3.28

Differentiating z(t) w.r.t. t and using (3.28), we get

$$z'(t) = f(\alpha(t))\left[z^{(2-w_1(p))}(\alpha(t)) + \int_0^{\alpha(t)} g\,(r)L\big(r, u^{w_2(q)}(r)\big)dr\right]^{w_1(p)} \alpha'(t)$$

$$\leq \alpha'(t)f(\alpha(t))\left[z^{(2-w_1(p))}(t) + \int_0^{\alpha(t)} g\,(s)L\big(s, z^{w_2(q)}(s)\big)ds\right]^{w_1(p)}. \quad 3.29$$

If we take $v(t) = z^{(2-w_1(p))}(t) + \int_0^{\alpha(t)} g\,(s)L\big(s, z^{w_2(q)}(s)\big)ds,$

then we have

$z'(t) \leq \alpha'(t)f(\alpha(t))v^{w_1(p)}(t),$ 3.30

$v(0) = z^{(2-w_1(p))}(0) = u_0^{(2-w_1(p))},$

$w_1(p) \leq 1 \Rightarrow -w_1(p) \geq -1 \Rightarrow 2 - w_1(p) \geq 1,$ and $z^{2-w_1(p)}(t) \leq v(t),$

$z(t) \leq v(t).$ 3.31

Differentiating v(t) w.r.t. t and using (3.30) and (3.31), we obtain

$$\begin{aligned} v^{\cdot}(t) &= (2 - w_1(p))\ z^{(1-w_1(p))}(t)\ z'(t) + g(\alpha(t))L\big(\alpha(t), z^{w_2(q)}(\alpha(t)\big)\alpha'(t) \\ &\leq (2 - w_1(p))v^{(1-w_1(p))}(t)\alpha'(t)f(\alpha(t))v^{w_1(p)}(t) + \alpha'(t)\ g(\alpha(t))\ L\big(\alpha(t, z^{w_2(q)}(t)\big) \\ &\leq (2 - w_1(p))v(t)\alpha'(t)f(\alpha(t)) + \alpha'(t)\ g(\alpha(t))\ L\big(\alpha(t), v^{w_2(q)}(t)\big) \end{aligned}$$

$$\leq (2 - w_1(p))\alpha'(t)f(\alpha(t))v(t) + \alpha'(t)\ g(\alpha(t))\ L\big(\alpha(t), v^{w_2(q)}(t)\big). \quad 3.32$$

taking $w_2(q) = \frac{m}{n}$ with $n > m \geq 0, n \neq 0$ and applying lemma 2.1, we have

$$v^{w_2(q)}(t) \leq w_2(q)k^{(w_2(q)-1)}v(t) + (1 - w_2(q))k^{w_2(q)}.$$

Since $0 \leq L(t,x) - L(t,y) \leq M(t,y)(x - y),\ x \geq y \geq 0,$

$$L(t,x) \leq L(t,y) + M(t,y)(x - y),$$

$$\begin{aligned} &L\big(\alpha(t), w_2(q)k^{(w_2(q)-1)}v(t) + (1 - w_2(q))k^{w_2(q)}\big) \\ &\leq L\big(\alpha(t), (1 - w_2(q))k^{w_2(q)}\big) + w_2(q)k^{(w_2(q)-1)}M\big(\alpha(t), (1 - w_2(q))k^{w_2(q)}\big)v(t). \end{aligned}$$

Inequality (3.32) becomes

$$\begin{aligned} v'(t) &\leq (2 - w_1(p))\alpha'(t)f(\alpha(t))v(t) \\ &+\alpha'(t)g(\alpha(t))[L\big(\alpha(t), (1 - w_2(q))k^{w_2(q)}\big) \\ &+w_2(q)k^{(w_2(q)-1)}M\big(\alpha(t), (1 - w_2(q))k^{w_2(q)}\big)v(t)]. \end{aligned}$$

Thus

$$\begin{aligned} v'(t) &\leq \alpha'(t)\ g(\alpha(t))\ L\big(\alpha(t), (1 - w_2(q))k^{w_2(q)}\big) + [(2 - w_1(p))\alpha'(t)f(\alpha(t)) \\ &+\alpha'(t)\ g(\alpha(t))\ w_2(q)\ k^{(w_2(q)-1)}M\big(\alpha(t), (1 - w_2(q))k^{w_2(q)}\big)]v(t). \end{aligned} \quad 3.33$$

Using lemma 2.2 to the above inequality, we have

$$v(t) \le v(0)\, exp\left(\begin{array}{l}\int_0^t [\,(2-w_1(p))\alpha'(s)f(\alpha(s))n + \alpha'(s)g(\alpha(s)) \\ w_2(q)k^{(w_2(q)-1)}M\big(\alpha(s),(1-w_2(q))k^{w_2(q)}\big)]ds\end{array}\right)$$

$$+\int_0^t \alpha'(s)g(\alpha(s))\, L\big(\alpha(s),(1 - w_2(q))k^{w_2(q)}\big)\, exp\left(\begin{array}{l}\int_s^t [\,(2-w_1(p))\alpha'(\tau)f(\alpha(\tau)) + \alpha'(\tau)g(\alpha(\tau)) \\ w_2(q)k^{(w_2(q)-1)}M\big(\alpha(\tau),(1-w_2(q))k^{w_2(q)}\big)]d\tau\end{array}\right)ds.$$

Therefore

$$v(t) \le u_0^{(2-w_1(p))}\, e^{\left((2-w_1(p))\int_0^t \begin{bmatrix}\alpha'(s)f(\alpha(s)) \\ +\alpha'(s)g(\alpha(s)) \\ \times\frac{w_2(q)}{(2-w_1(p))}k^{(w_2(q)-1)} \\ M\big(\alpha(s),(1-w_2(q))k^{w_2(q)}\big)\end{bmatrix} ds\right)}$$

$$+\int_0^t \left\{\begin{array}{l}\alpha'(s)\ g(\alpha(s))L\big(\alpha(s),(1-w_2(q))k^{w_2(q)}\big) \\ \times\, e^{\left((2-w_1(p))\int_s^t \left\{\begin{bmatrix}\alpha'(\tau)f(\alpha(\tau)) \\ +\frac{w_2(q)}{(2-w_1(p))}k^{(w_2(q)-1)}\times \\ \alpha'(\tau)\ g(\alpha(\tau)) \\ M\big(\alpha(\tau),(1-w_2(q))k^{w_2(q)}\big)\end{bmatrix}\right\} d\tau\right)} d\,s,\end{array}\right\}$$

By using substitution $\alpha(s) = s$ and $\alpha(\tau) = \tau$, we obtain

$$v(t) \le u_0^{(2-w_1(p))}\, exp(\,(2-w_1(p))\int_0^{\alpha(t)} [\,f(s) + \frac{w_2(q)}{(2-w_1(p))}$$
$$k^{(w_2(q)-1)}g(s)M\big(s,(1-w_2(q))k^{w_2(q)}\big)]ds) + \int_0^{\alpha(t)} g\,(s)L\big(s,(1-w_2(q))k^{w_2(q)}\big)$$
$$\times\, exp(\,(2-w_1(p))\int_s^{\alpha(t)} [\,f(\tau) + \frac{w_2(q)}{(2-w_1(p))}k^{(w_2(q)-1)}g(\tau)M\big(\tau,(1-w_2(q))k^{w_2(q)}\big)]d\tau)ds,$$

$$v(t) \le exp(\,(2-w_1(p))\int_0^{\alpha(t)} [\,f(s) + \frac{w_2(q)}{(2-w_1(p))}k^{(w_2(q)-1)}g(s)M\big(s,(1-w_2(q))k^{w_2(q)}\big)]ds)$$
$$[u_0^{(2-w_1(p))} + \int_0^{\alpha(t)} g\,(s)L\big(s,(1-w_2(q))k^{w_2(q)}\big) \times exp(\,-(2-w_1(p))\int_0^s [\,f(\tau)$$
$$+\frac{w_2(q)}{(2-w_1(p))}k^{(w_2(q)-1)}g(\tau)M\big(\tau,(1-w_2(q))k^{w_2(q)}\big)]d\tau)ds],$$

$$v^{w_1(p)}(t) \le exp(\,w_1(p)(2-w_1(p))\int_0^{\alpha(t)} [\,f(s) + \frac{w_2(q)}{(2-w_1(p))}k^{(w_2(q)-1)}g(s)$$
$$M\big(s,(1-w_2(q))k^{w_2(q)}\big)]ds)[u_0^{(2-w_1(p))} + \int_0^{\alpha(t)} g\,(s)L\big(s,(1-w_2(q))k^{w_2(q)}\big)$$
$$\times\, exp(\,-(2-w_1(p))\int_0^s [\,f(\tau)$$
$$+\frac{w_2(q)}{(2-w_1(p))}k^{(w_2(q)-1)}g(\tau)M\big(\tau,(1-w_2(q))k^{w_2(q)}\big)]d\tau)ds]^{w_1(p)},$$

then

$$v^{w_1(p)}(t) \leq exp(w_1(p)(2 - w_1(p)) \int_0^{\alpha(t)} p_4\,(s)ds)q_4(t).$$

where,

$$p_4(t) = f(t) + \frac{w_2(q)}{(2 - w_1(p))} k^{(w_2(q)-1)} g(t)M\big(t, (1 - w_2(q))k^{w_2(q)}\big),$$

$$q_4(t) = \begin{bmatrix} u_0^{(2-w_1(p))} + \int_0^{\alpha(t)} g\,(t)L\big(t, (1 - w_2(q))k^{w_2(q)}\big) \\ \times\, exp(- (2 - w_1(p)) \int_0^s p_4\,(\tau)d\tau)ds \end{bmatrix}^{w_1(p)}. \qquad 3.34$$

Hence inequality (3.30) becomes

$$z'(t) \leq \alpha'(t)f(\alpha(t))\, exp(w_1(p)(2 - w_1(p)) \int_0^{\alpha(t)} p_4\,(s)ds)q_4(t).$$

Integrating w.r.t. t from 0 to t we have

$$z(t) \leq u_0 + \int_0^t \alpha'(t)f(\alpha(t))\, exp(w_1(p)(2 - w_1(p)) \int_0^{\alpha(t)} p_4\,(s)ds)q_4(t)dt, \qquad 3.35$$

By using substitution $\alpha(t) = s$ in (3.35), we obtain

$$z(t) \leq u_0 + \int_0^{\alpha(t)} f\,(s)q_4(\alpha^{-1}(s))\, exp(w_1(p)(2 - w_1(p)) \int_0^s p_4\,(r)dr)ds.$$

Hence inequality (3.28) becomes

$$u(t) \leq u_0 + \int_0^{\alpha(t)} f\,(s)q_4(\alpha^{-1}(s))\, exp(w_1(p)(2 - w_1(p)) \int_0^s p_4\,(r)dr)ds.$$

This is the required inequality.

Remark 3.11 If we put $w_1(p) = p, L(t,x) = x,$ and $w_2(q) = q$ then Theorem 3.4 reduces to Theorem 2.1 in [18].

Remark 3.12 If we put p=1 in L.H.S. and $u_0 = c(t), f(s) = k_1(t,s), g(s) = k_2(t,s),$ $w_1(p) = 1,$ $L(t,x) = x,$ $w_2(q) = 1$ in R.H.S. of Theorem 3.4. then Theorem 3.4 reduces to Theorem 2.4 in [19].

Remark 3.13 If we put $u(t) = \phi_1(u(t)), \phi_2(u(s)) = 1, L(t,x) = x, w_2(q) = 1.$ and $w_1(p) = p$ then Theorem 3.4 reduces to Theorem 2.3 in [20].

Theorem 3.5 Let $u(t), f(t), g(t) \in \mathbb{C}(I, I)$.Let $\alpha(t) \in \mathbb{C}'(I, I)$ be a nondecreasing function with $\alpha(t) \leq t, \alpha(0) = 0$on I.

If the inequality

$$u(t) \leq u_0 + \int_0^{\alpha(t)} f\,(s)u^{w_1(p)}(s)ds$$

$$+ \int_0^{\alpha(t)} f\,(s)L\big(s, u^{w_2(q)}(s)\big) \left[u^{2-w_1(p)}(s) + \int_0^{\alpha(s)} g\,(\lambda)u(\lambda)d\lambda\right]^{w_1(p)} ds, \qquad 3.36$$

holds for ∀ $t \in I,$ for $u_0 > 0, 0 < w_1(p) \leq 1, 0 \leq w_2(q) < 1,$ and $L, M \in \mathbb{C}(I^2, I)$ satisfy

$$0 \leq L(t,x) - L(t,y) \leq M(t,y)(x - y), x \geq y \geq 0.$$

then

$u(t) \le K(t)\, exp\left(\int_0^{\alpha(t)} \psi\,(s)ds\right),\ \forall\ t \in I.$ 3.37

where,

$$K(t) = u_0 + \int_0^{\alpha(t)} f\,(s) \begin{bmatrix} (1-w_1(p))k^{w_1(p)} \\ +L\big(s,(1-w_2(q))k^{w_2(q)}\big) \\ \times\, \theta^{w_1(p)}(\alpha^{-1}(s)) \end{bmatrix} \times exp\left(-\int_0^{s} \psi\,(\lambda)d\lambda\right)ds,$$

and

$\psi(t) = w_1(p)k^{w_1(p)-1}f(t)[1 + \frac{w_2(q)}{w_1(p)}k^{w_2(q)-w_1(p)}M\big(t,(1-w_2(q))k^{w_2(q)}\big)\theta^{w_1(p)}(\alpha^{-1}(t))]$,3.38

Where,

$\theta(t) = \frac{exp\left(\int_0^{\alpha(t)} q_5(s)ds\right)}{c-\int_0^{\alpha(t)} p_5(s)\, exp\left(\int_0^{s} q_5(\tau)d\tau\right)ds},$ 3.39

and

$p_5(t) = \big(2-w_1(p)\big)w_2(q)k^{(w_2(q)-1)}g(t)M\left(t,(1-w_2(q))k^{w_2(q)}\right),$ $q_5(t) = (2-w_1(p))g(t)\big(1 + L(t,(1-w_2(q)))k^{w_2(q)}\big) + g(t).$ 3.40

Proof: Define a function z(t) by

$$z(t) = u_0 + \int_0^{\alpha(t)} f\,(s)u^{w_1(p)}(s)ds$$
$$+\int_0^{\alpha(t)} f\,(s)L\big(s,u^{w_2(q)}(s)\big)\left[u^{2-w_1(p)}(s) + \int_0^{\alpha(s)} g\,(\lambda)u(\lambda)d\lambda\right]^{w_1(p)} ds.$$

Then we have

$z(0) = u_0$ and $u(t) \le z(t).$ 3.41

Differentiating z(t) w.r.t. t and using (3.41), we obtain

$$z'(t) = f(\alpha(t))u^{w_1(p)}(\alpha(t))\alpha' t)$$
$$+f(\alpha(t))L\big(\alpha(t),u^{w_2(q)}(\alpha(t))\big)\left[u^{2-w_1(p)}(\alpha(t)) + \int_0^{\alpha(\alpha(t))} g\,(\lambda)u(\lambda)d\lambda\right]^{w_1(p)} \alpha'(t)$$
$$\le \alpha'(t)f(\alpha(t))u^{w_1(p)}(t)$$
$$+\ \alpha'(t)f(\alpha(t))L\big(\alpha(t),u^{w_2(q)}(t)\big)\left[u^{2-w_1(p)}(t) + \int_0^{\alpha(t)} g\,(\lambda)u(\lambda)d\lambda\right]^{w_1(p)}$$
$$\le \alpha'(t)f(\alpha(t))z^{w_1(p)}(t)$$
$$+\ \alpha'(t)f(\alpha(t))L\big(\alpha(t),z^{w_2(q)}(t)\big)\left[z^{2-w_1(p)}(t) + \int_0^{\alpha(t)} g\,(s)z(s)ds\right]^{w_1(p)}.$$

3.42

Taking $w_2(q)=\frac{m}{n}$ with $n>m\geq 0, n\neq 0$, we have

From lemma 2.1,

$$z^{w_2(q)}(t)\leq w_2(q)k^{(w_2(q)-1)}z(t)+(1-w_2(q))k^{w_2(q)}.$$

Since

$$0\leq L(t,x)-L(t,y)\leq M(t,y)(x-y),\ \ x\geq y\geq 0,$$

$$L(t,x)\leq L(t,y)+M(t,y)(x-y),\ \ x\geq y\geq 0$$

$$L\big(\alpha(t),z^{w_2(q)}(t)\big)=L\big(\alpha(t),w_2(q)k^{(w_2(q)-1)}z(t)+(1-w_2(q))k^{w_2(q)}\big)$$

$$\leq L\big(\alpha(t),(1-w_2(q))k^{w_2(q)}\big)+w_2(q)k^{(w_2(q)-1)}z(t)M\big(\alpha(t),(1-w_2(q))k^{w_2(q)}\big)$$

$L\big(\alpha(t),z^{w_2(q)}(t)\big)\leq L\big(\alpha(t),(1-w_2(q))k^{w_2(q)}\big)+w_2(q)k^{(w_2(q)-1)}M\big(\alpha(t),(1-w_2(q))k^{w_2(q)}\big)z(t)$.Inequality (3.42) becomes

$$z'(t)\leq \alpha'(t)f(\alpha(t))z^{w_1(p)}(t)+\alpha'(t)f(\alpha(t))[L\big(\alpha(t),(1-w_2(q))k^{w_2(q)}\big)$$
$$+w_2(q)k^{(w_2(q)-1)}M\big(\alpha(t),(1-w_2(q))k^{w_2(q)}\big)z(t)]v^{w_1(p)}(t). \qquad 3.43$$

Where,

$$v(t)=z^{2-w_1(p)}(t)+\int_0^{\alpha(t)}g(s)z(s)ds, v(0)=u_0^{2-w_1(p)}, v(t)>0. \qquad 3.44$$

For $w_1(p)<1\Rightarrow -w_1(p)>-1\Rightarrow 2-w_1(p)>1$, and $z^{2-w_1(p)}(t)\leq v(t)$,

$$z(t)\leq z^{2-w_1(p)}(t)\leq v(t). \qquad 3.45$$

Differentiating v(t) w.r.t. t and using (3.43) and (3.45), we obtain

\begin{align*}

\\

$$\begin{aligned}
&v'(t)=(2-w_1(p))z^{(1-w_1(p))}(t)z'(t)+g(\alpha(t))z(\alpha(t))\alpha'(t)\\
&\leq(2-w_1(p))z^{1-w_1(p)}(t)(\alpha'(t)f(\alpha(t))z^{w_1(p)}(t)+\alpha'(t)f(\alpha(t))\\
&[L\big(\alpha(t),(1-w_2(q))k^{w_2(q)}\big)+w_2(q)k^{(w_2(q)-1)}M\big(\alpha(t),(1-w_2(q))k^{w_2(q)}\big)z(t)]v^{w_1(p)}(t))\\
&+g(\alpha(t))z(\alpha(t))\alpha'(t)\\
&\leq(2-w_1(p))z^{(1-w_1(p))}(t)\alpha'(t)f(\alpha(t))z^{w_1(p)}(t)+(2-w_1(p))z^{(1-w_1(p))}(t)\alpha'(t)f(\alpha(t))\\
&[L\big(\alpha(t),(1-w_2(q))k^{w_2(q)}\big)+w_2(q)k^{(w_2(q)-1)}M\big(\alpha(t),(1-w_2(q))k^{w_2(q)}\big)z(t)]v^{w_1(p)}(t)\\
&+g(\alpha(t))z(\alpha(t))\alpha'(t)
\end{aligned}$$

$$\begin{aligned}
&\leq(2-w_1(p))\alpha'(t)f(\alpha(t))v^{(1-w_1(p))}(t)v^{w_1(p)}(t)+(2-w_1(p))\alpha'(t)f(\alpha(t))\\
&[L\big(\alpha(t),(1-w_2(q))k^{w_2(q)}\big)+w_2(q)k^{(w_2(q)-1)}M\big(\alpha(t),(1-w_2(q))k^{w_2(q)}\big)v(t)]\\
&v^{(1-w_1(p))}(t)v^{w_1(p)}(t)+\alpha'(t)g(\alpha(t))v(t)
\end{aligned}$$

$$\begin{aligned}
&\leq(2-w_1(p))\alpha'(t)f(\alpha(t))v(t)+(2-w_1(p))\alpha'(t)f(\alpha(t))v(t)\\
&[L\big(\alpha(t),(1-w_2(q))k^{w_2(q)}\big)+w_2(q)k^{(w_2(q)-1)}M\big(\alpha(t),(1-w_2(q))k^{w_2(q)}\big)v(t)]\\
&+\alpha'(t)g(\alpha(t))v(t)
\end{aligned}$$

$$\begin{aligned}
&\leq[(2-w_1(p))\alpha'(t)f(\alpha(t))\Big(1+L\big(\alpha(t),(1-w_2(q))k^{w_2(q)}\big)\Big)+\alpha'(t)g(\alpha(t))]v(t)\\
&+(2-w_1(p))\alpha'(t)f(\alpha(t))w_2(q)k^{(w_2(q)-1)}M\big(\alpha(t),(1-w_2(q))k^{w_2(q)}\big)v^2(t),
\end{aligned}$$

$$v'(t) \leq A(t)v^2(t) + B(t)v(t).$$

where,

$$A(t) = (2 - w_1(p))\alpha'(t)f(\alpha(t))w_2(q)k^{(w_2(q)-1)}M\big(\alpha(t), (1 - w_2(q))k^{w_2(q)}\big),$$

And

$$B(t) = (2 - w_1(p))\alpha'(t)f(\alpha(t))\left(1 + L\big(\alpha(t), (1 - w_2(q))k^{w_2(q)}\big)\right) + \alpha'(t)g(\alpha(t)).$$

Since

$$v'(t) \leq A(t)v^2(t) + B(t)v(t),$$

Dividing by $v^2(t)$,

$$v^{-2}(t)v'(t) \leq A(t) + B(t)v^{-1}(t),$$

Letting $v^{-1}(t) = y(t)$, we obtain

$$(-1)v^{-2}(t)v'(t) = y'(t) \Rightarrow v^{-2}(t)v'(t) = -y'(t),$$

$$-y'(t) \leq A(t) + B(t)y(t).$$

Then by lemma 2.2 we have

$$-y(t) \leq y(0)\, exp\left(\int_0^t B\,(s)ds\right) + \int_0^t A\,(s)\, exp\left(\int_s^t B\,(\lambda)d\lambda\right)ds,$$

$$\text{i.e. } -y(t) \leq u_0^{-(2-w_1(p))}\, exp\left(\int_0^t B\,(s)ds\right) + \int_0^t A\,(s)\, exp\left(\int_s^t B\,(\lambda)d\lambda\right)ds,$$

$$\Rightarrow y(t) \geq u_0^{-(2-w_1(p))}\, exp\left(-\int_0^t B\,(s)ds\right) - \int_0^t A\,(s)\, exp\left(-\int_s^t B\,(\lambda)d\lambda\right)ds,$$

$$\text{i.e. } y(t) \geq u_0^{-(2-w_1(p))}\, exp(-\int_0^t [\,(2 - w_1(p))\alpha'(s)f(\alpha(s))\left(1 + L\big(\alpha(s), (1 - w_2(q))k^{w_2(q)}\big)\right)$$

$$+\alpha'(s)g(\alpha(s))]ds) - \int_0^t (2 - w_1(p))\alpha'(s)f(\alpha(s))w_2(q)k^{(w_2(q)-1)}M\big(\alpha(s), (1 - w_2(q))k^{w_2(q)}\big)$$

$$exp(-\int_s^t [\,(2 - w_1(p))\alpha'(\lambda)f(\alpha(\lambda))\left(1 + L\big(\alpha(\lambda), (1 - w_2(q))k^{w_2(q)}\big)\right) + \alpha'(\lambda)g(\alpha(\lambda))]d\lambda)ds.$$

By using substitution $\alpha(s) = s$ and $\alpha(\lambda) = \lambda$,we obtain

$$\therefore\ y(t) \geq u_0^{-(2-w_1(p))}\, exp(-\int_0^{\alpha(t)} [\,(2 - w_1(p))f(s)\left(1 + L\big(s, (1 - w_2(q))k^{w_2(q)}\big)\right) + g(s)]ds)$$

$$-(2 - w_1(p))w_2(q)k^{(w_2(q)-1)}\int_0^{\alpha(t)} f\,(s)M\big(s, (1 - w_2(q))k^{w_2(q)}\big)$$

$$\times\, exp(-\int_{\alpha(s)}^{\alpha(t)} [\,(2 - w_1(p))f(\lambda)\left(1 + L\big(\lambda, (1 - w_2(q))k^{w_2(q)}\big)\right) + g(\lambda)]d\lambda)ds,$$

$$\text{i.e. } y(t) \geq u_0^{-(2-w_1(p))}\, exp(-\int_0^{\alpha(t)} [\,(2 - w_1(p))f(s)\left(1 + L\big(s, (1 - w_2(q))k^{w_2(q)}\big)\right) + g(s)]ds)$$

$$-(2-w_1(p))w_2(q)k^{(w_2(q)-1)}\int_0^{\alpha(t)} f(s)M\big(s,(1-w_2(q))k^{w_2(q)}\big)$$

$$\times exp(-\int_s^{\alpha(t)}[(2-w_1(p))f(\lambda)\Big(1+L\big(\lambda,(1-w_2(q))k^{w_2(q)}\big)\Big)+g(\lambda)]d\lambda)ds, y(t)$$

$$\geq exp(-\int_0^{\alpha(t)}[(2-w_1(p))f(s)\Big(1+L\big(s,(1-w_2(q))k^{w_2(q)}\big)\Big)$$

$$+g(s)]ds)\times[u_0^{-(2-w_1(p))}-(2-w_1(p))w_2(q)k^{(w_2(q)-1)}\int_0^{\alpha(t)} f(s)M\big(s,(1-w_2(q))k^{w_2(q)}\big)$$

$$\times exp(\int_0^{s}[(2-w_1(p))f(\lambda)\Big(1+L\big(\lambda,(1-w_2(q))k^{w_2(q)}\big)\Big)+g(\lambda)]d\lambda)ds].$$

Then

$$y(t)\geq exp(-\textstyle\int_0^{\alpha(t)} q_5(s)ds)\times[u_0^{-(2-w_1(p))}-\int_0^{\alpha(t)} p_5(s)\, exp(\int_0^{s} q_5(\lambda)d\lambda)ds]. \qquad 3.46$$

Where,

$p_5(t)=(2-w_1(p))w_2(q)k^{(w_2(q)-1)}f(t)M\big(t,(1-w_2(q))k^{w_2(q)}\big)$,and

$$q_5(t)=(2-w_1(p))f(t)\Big(1+L\big(t,(1-w_2(q))k^{w_2(q)}\big)\Big)+g(t).$$

Inequality (3.46) becomes

$$y(t)\geq\frac{u_0^{-(2-w_1(p))}-\int_0^{\alpha(t)} p_5(s)\, exp(\int_0^{s} q_5(\lambda)d\lambda)ds}{exp(\int_0^{\alpha(t)} q_5(s)ds)},$$

Since

$$y(t)=v^{-1}(t)=\frac{1}{v(t)},$$

$$v(t)\leq\frac{exp(\int_0^{\alpha(t)} q_5(s)ds)}{u_0^{-(2-w_1(p))}-\int_0^{\alpha(t)} p_5(s)\, exp(\int_0^{s} q_5(\lambda)d\lambda)ds}.$$

Let $u_0^{-(2-w_1(p))}=c$, and $\int_0^{\alpha(t)} p_5(s)\, exp(\int_0^{s} q_5(\tau)d\tau)ds<c$,

$$\Rightarrow v(t)\leq\theta(t),$$

$$y(t)=v^{-1}(t)=\frac{1}{v(t)},$$

where,

$$\theta(t)=\frac{exp(\int_0^{\alpha(t)} q_5(s)ds)}{c-\int_0^{\alpha(t)} p_5(s)\, exp(\int_0^{s} q_5(\lambda)d\lambda)ds}.$$

From (3.43) and taking $w_1(p)=\frac{m^*}{n^*}, n^*\geq m^*>0, n^*\neq 0$, we get

$$z'(t)\leq\alpha'(t)f(\alpha(t))z^{w_1(p)}(t)+\alpha'(t)f(\alpha(t))[L\big(\alpha(t),(1-w_2(q))k^{w_2(q)}\big)$$
$$+w_2(q)k^{(w_2(q)-1)}M\big(\alpha(t),(1-w_2(q))k^{w_2(q)}\big)z(t)]v^{w_1(p)}(t).$$

By Lemma 2.1 we have ($w_2(q) = w_1(p)$ and $w_1(p) = 1$)

$$z^{w_1(p)}(t) \le w_1(p)k^{(w_1(p)-1)}z(t) + (1 - w_1(p))k^{w_1(p)}$$

$$z'(t) \le \alpha'(t)f(\alpha(t))\left[w_1(p)k^{(w_1(p)-1)}z(t) + (1 - w_1(p))k^{w_1(p)}\right] + \alpha'(t)f(\alpha(t))$$
$$\left[L\left(\alpha(t),(1-w_2(q))k^{w_2(q)}\right) + w_2(q)k^{(w_2(q)-1)}M\left(\alpha(t),(1-w_2(q))k^{w_2(q)}\right)z(t)\right]v^{w_1(p)}(t),$$ $z'(t) \le$
$\alpha'(t)f(\alpha(t))[(1 - w_1(p))k^{w_1(p)} + L\big(\alpha(t),(1 - w_2(q))k^{w_2(q)}\big)v^{w_1(p)}(t)] + \alpha'(t)f(\alpha(t))$
$[w_1(p)k^{(w_1(p)-1)} + w_2(q)k^{(w_2(q)-1)}M\big(\alpha(t),(1 - w_2(q))k^{w_2(q)}\big)v^{w_1(p)}(t)]z(t).$

Since, $v(t) \le \theta(t) \Rightarrow v^{w_1(p)}(t) \le \theta^{w_1(p)}(t)$,

$z'(t) \le \alpha'(t)f(\alpha(t))[(1 - w_1(p))k^{w_1(p)} + L\big(\alpha(t),(1 - w_2(q))k^{w_2(q)}\big)\theta^{w_1(p)}(t)] + \alpha'(t)f(\alpha(t))$ $[w_1(p)k^{(w_1(p)-1)} + w_2(q)k^{(w_2(q)-1)}M\big(\alpha(t),(1 - w_2(q))k^{w_2(q)}\big)\theta^{w_1(p)}(t)]z(t)$.Applying lemma 2.2 we obtain

$$z(t) \le z(0)\, exp(\int_0^t q\,(s)ds) + \int_0^t p\,(s)\, exp(\int_s^t q\,(\lambda)d\lambda)ds,$$

$$\begin{aligned}
&z(t) \le u_0\, exp(\int_0^t \alpha'(s)f(\alpha(s))\,[w_1(p)k^{(w_1(p)-1)} \\
&+w_2(q)k^{(w_2(q)-1)}M\big(\alpha(s),(1 - w_2(q))k^{w_2(q)}\big)\theta^{w_1(p)}(s)]ds) \\
&+\int_s^t \alpha'(s)f(\alpha(s))\,t[(1 - w_1(p))k^{w_1(p)} + L\big(\alpha(s),(1 - w_2(q))k^{w_2(q)}\big)\theta^{w_1(p)}(s)] \\
&exp(\int_s^t \alpha'(\lambda)f(\alpha(\lambda))\,[w_1(p)k^{(w_1(p)-1)} + w_2(q)k^{(w_2(q)-1)} \\
&M\big(\alpha(\lambda),(1 - w_2(q))k^{w_2(q)}\big)\theta^{w_1(p)}(\lambda)]d\lambda)ds,
\end{aligned}$$

Using the substitution $\alpha(s) = s$ and $\alpha(\lambda) = \lambda$, we obtain

$$\begin{aligned}
\therefore z(t) \le u_0\, exp(&\int_0^{\alpha(t)} f\,(s)[w_1(p)k^{(w_1(p)-1)} \\
&+ w_2(q)k^{(w_2(q)-1)}M\big(s,(1 - w_2(q))k^{w_2(q)}\big)\theta^{w_1(p)}(\alpha^{-1}(s))]ds) \\
+\int_0^{\alpha(t)} &f\,(s)[(1 - w_1(p))k^{w_1(p)} + L\big(s,(1 - w_2(q))k^{w_2(q)}\big)\theta^{w_1(p)}(\alpha^{-1}(s))] \\
exp(\int_{\alpha(s)}^{\alpha(t)} &f\,(\lambda)[w_1(p)k^{(w_1(p)-1)} \\
&+ w_2(q)k^{(w_2(q)-1)}M\big(\lambda,(1 - w_2(q))k^{w_2(q)}\big)\theta^{w_1(p)}(\alpha^{-1}(\lambda))]d\lambda)ds,\text{i.e. } z(t) \\
&\le u_0\, exp(\int_0^{\alpha(t)} f\,(s)[w_1(p)k^{(w_1(p)-1)} + w_2(q)k^{(w_2(q)-1)}M\big(s,(1 - w_2(q))k^{w_2(q)}\big) \\
\theta^{w_1(p)}(\alpha^{-1}(s))]ds) &+ \int_0^{\alpha(t)} f\,(s)[(1 - w_1(p))k^{w_1(p)} + L\big(s,(1 - w_2(q))k^{w_2(q)}\big)\theta^{w_1(p)}(\alpha^{-1}(s))] \\
exp(\int_s^{\alpha(t)} &f\,(\lambda)[w_1(p)k^{(w_1(p)-1)} \\
&+ w_2(q)k^{(w_2(q)-1)}M\big(\lambda,(1 - w_2(q))k^{w_2(q)}\big)\theta^{w_1(p)}(\alpha^{-1}(\lambda))]d\lambda)ds,\text{i.e. } z(t) \\
&\le exp(\int_0^{\alpha(t)} f\,(s)[w_1(p)k^{(w_1(p)-1)} + w_2(q)k^{(w_2(q)-1)}M\big(s,(1 - w_2(q))k^{w_2(q)}\big)
\end{aligned}$$

$\theta^{w_1(p)}(\alpha^{-1}(s))]ds)[u_0 + \int_0^{\alpha(t)} f\,(s)((1-w_1(p))k^{w_1(p)}$

$+ L\big(s,(1-w_2(q))k^{w_2(q)}\big)\theta^{w_1(p)}(\alpha^{-1}(s)))$

$exp(-\int_0^s f\,(\lambda)[w_1(p)k^{(w_1(p)-1)} + w_2(q)k^{(w_2(q)-1)}M\big(\lambda,(1 - w_2(q))k^{w_2(q)}\big)\theta^{w_1(p)}(\alpha^{-1}(\lambda))]d\lambda)ds]$.Thus

$$z(t) \le\ K(t)\,exp(\int_0^{\alpha(t)} \psi\,(s)ds).$$

where,

$$\psi(t) = w_1(p)k^{(w_1(p)-1)}f(t)[1 + \frac{w_2(q)}{w_1(p)}k^{(w_2(q)-w_1(p))}M\big(t,(1-w_2(q))k^{w_2(q)}\big)\theta^{w_1(p)}(\alpha^{-1}(t))],$$

and

$$K(t) = u_0 + \int_0^{\alpha(t)}\left[f(s)\begin{bmatrix}(1-w_1(p))k^{w_1(p)}\\ +L\big(s,(1-w_2(q))k^{w_2(q)}\big)\\ \theta^{w_1(p)}(\alpha^{-1}(s))\end{bmatrix} exp(-\int_0^s \psi\,(\lambda)d\lambda)\right]ds.$$

Hence inequality (3.41) becomes

$u(t) \le\ K(t)\,exp(\int_0^{\alpha(t)} \psi\,(s)ds)$.This completes the proof.

Remark 3.14 If we put $f(s) = 0$ in first integral and $L(t,x) = x, w_2(q) = 0, \alpha(s) = s$,and $w_1(p) = p$ in second integral of R.H.S. of Theorem 3.5 then Theorem 3.5 reduces to Theorem 2.4 in [18].

Remark 3.15 If we put $f(s) = 0$in first integral and $L(t,x) = x, w_1(p) = 1, \alpha(s) = s$in second integral of R.H.S. of Theorem 3.5 then Theorem 3.5 reduces to Theorem 2. 1in [20].

Remark 3.16 If we put $f(s) = 0$ in first integral and $L(t,x) = x, w_2(q) = 1, \alpha(s) = s$, and $w_1(p) = p$in second integral of R.H.S. of Theorem 3.5 then Theorem 3.5 reduces to Theorem 2.3. in [20].

Remark 3.17 If we put $w_1(p) = 1, L(t,x) = x$, and $w_2(q) = 1$.then Theorem 3.5 reduces to Theorem 2.3 in [21].

Remark 3.18 If we put $u_0 = f(t), w_1(p) = 1, u(s) = \phi(u(s))$in first integral and $L(t,x) = \phi(x), w_2(q) = 1$, and $w_1(p) = 0$ in second integral of Theorem 3.5 then Theorem 3.5 reduces to Theorem 3.4 in [21].

Theorem 3.6 Let $u(t), c(t), f(t), g(t) \in \mathbb{C}(I,I)$ and c(t) is nondecreasing. Let $\alpha(t) \in \mathbb{C}'(I,I)$ be a nondrecreasing function with $\alpha(t) \le t, \alpha(0) = 0$ and $L, M \in \mathbb{C}(I^2,I)$and $\phi: I \to I$be a continuous and strictly increasing function with $\phi(0) = 0$ such that

$0 \le L(t,x) - L(t,y) \le m(t,y)\phi^{-1}(x-y),\ x \ge y \ge 0$,for $t \in I$.where ϕ^{-1} is the inverse function of ϕ, and

$\phi^{-1}(xy) \le \phi^{-1}(x)\phi^{-1}(y)$. 3.48

If $u^{w_1(p)}(t) \le c(t) + \phi(\int_0^t f\,(s)L\big(s,u^{w_1(p)}(s)\big)ds + \int_0^{\alpha(t)} g\,(s)L\big(s,u^{w_2(q)}(s)\big)ds)$,holds for $\forall\ t \in I$,for $w_1(p) > w_2(q) \ge 0$, then

$$u(t) \le [c(t) + \phi(\int_0^t p_6\,(s)\,exp\left(\int_s^t q_6\,(\lambda)d\lambda\right)ds)]^{\frac{1}{w_1(p)}}.$$

where,

$$p_6(t) = (2 - w_1(p))w_2(q)k^{(w_2(q)-1)}f(t)M\big(t,(1-w_2(q))k^{w_2(q)}\big),$$
$$q_6(t) = (2 - w_1(p))f(t)\big(1 + L(t,(1-w_2(q)))k^{w_2(q)}\big). \qquad 3.49$$

Proof: Define a function z(t) by

$$z(t) = \int_0^t f\,(s)L\big(s,u^{w_1(p)}(s)\big)ds + \int_0^{\alpha(t)} g\,(s)L\big(s,u^{w_2(q)}(s)\big)ds,$$

$z(0) = 0,$ and $u(t) \le [c(t) + \phi(z(t))]^{\frac{1}{w_1(p)}}.$ 3.50

Differentiating z(t) w.r.t. t and using (3.50), we have

$$z'(t) = f(t)L\big(t,u^{w_1(p)}(t)\big) + g(\alpha(t))L\big(\alpha(t),u^{w_2(q)}(\alpha(t))\big)\alpha'(t)$$
$$\le f(t)L(t,[c(t)+\phi(z(t))]) + \alpha'(t)g(\alpha(t))L\left(\alpha(t),[c(\alpha(t)) + \phi(z(\alpha(t)))]^{\frac{w_2(q)}{w_1(p)}}\right)$$
$$\le f(t)L(t,[c(t)+\phi(z(t))]) + \alpha'(t)g(\alpha(t))L\left(\alpha(t),[c(t) + \phi(z(t))]^{\frac{w_2(q)}{w_1(p)}}\right). \qquad 3.51$$

Since,

$$0 \le L(t,x) - L(t,y) \le m(t,y)\phi^{-1}(x-y),$$

for $x \ge y \ge 0$, and for $t \in I$,

$$\Rightarrow L(t,x) \le L(t,y) + m(t,y)\phi^{-1}(x-y),$$
$$L(t,[c(t)+\phi(z(t))]) \le L(t,c(t)) + m(t,c(t))\phi^{-1}(\phi(z(t))).$$

By lemma 2.1 we have

$$[c(t)+\phi(z(t))]^{\frac{w_2(q)}{w_1(p)}} \le \frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}[c(t)+\phi(z(t))] + \frac{w_1(p)-w_2(q)}{w_1(p)}k^{\frac{w_2(q)}{w_1(p)}}.$$

Inequality (3.51) becomes

$$z'(t) \le f(t)[L(t,c(t)) + m(t,c(t))\phi^{-1}(\phi(z(t)))]$$
$$+\alpha'(t)g(\alpha(t))L\left(\alpha(t),\frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}[c(t)+\phi(z(t))] + \frac{w_1(p)-w_2(q)}{w_1(p)}k^{\frac{w_2(q)}{w_1(p)}}\right)$$
$$\le f(t)[L(t,c(t)) + m(t,c(t))z(t)] + \alpha'(t)g(\alpha(t))[L\left(\alpha(t),\frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}c(t)\right.$$
$$\left.+\frac{w_1(p)-w_2(q)}{w_1(p)}k^{\frac{w_2(q)}{w_1(p)}}\right) + m\left(\alpha(t),\frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}c(t) + \frac{w_1(p)-w_2(q)}{w_1(p)}k^{\frac{w_2(q)}{w_1(p)}}\right)$$
$$\phi^{-1}(\frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}\phi(z(t)))].$$

Using (3.48), we obtain

$$\phi^{-1}(\frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}\phi(z(t))) \le \phi^{-1}(\frac{w_2(q)}{w_1(p)}k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}).z(t)$$

$$z'(t) \le f(t)L(t,c(t)) + \alpha'(t)g(\alpha(t))L\left(\alpha(t), \frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} c(t) + \frac{w_1(p)-w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right)$$
$$+[f(t)m(t,c(t)) + \alpha'(t)g(\alpha(t))m\left(\alpha(t), \frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}} c(t) + \frac{w_1(p)-w_2(q)}{w_1(p)} k^{\frac{w_2(q)}{w_1(p)}}\right)$$
$$\times \phi^{-1}\left(\frac{w_2(q)}{w_1(p)} k^{\frac{w_2(q)-w_1(p)}{w_1(p)}}\right)]z(t).$$

The last inequality can be expressed as

$$z'(t) \le p_6(t) + q_6(t)z(t).$$

Applying lemma 2.2 we have

$$z(t) \le \int_o^t p_6\,(s)\, exp(\int_s^t q_6\,(\lambda)d\lambda)ds.$$

Inequality (3.50) becomes

$u(t) \le [c(t) + \phi(\int_o^t p_6\,(s)\, exp(\int_s^t q_6\,(\lambda)d\lambda)ds)]^{\frac{1}{w_1(p)}}$. This is the required inequality.

Remark 3.19 If we put $u^{w_1(p)}(t) = \phi_1(u(t)), c(t) = u_0, L(t,x) = x,$ and $u^{w_2(p)}(t) = \phi_2(u(t))$ then Theorem 3.6 reduces to Theorem 2.2 in [18].

Remark 3.20 If we put $w_1(p) = p, \phi(u(t)) = u(t), L(t,x) = x,$and $w_2(q) = q$ then Theorem 3.6 reduces to Theorem 2.3 in [18].

Remark 3.21 If we put $w_1(p) = p, \phi(u(t)) = u(t), L(t,x) = x,$ and $w_2(q) = 1$then Theorem 3.6 reduces to Theorem 2.1 in [18].

Remark 3.22 If we put $w_1(p) = p, \phi(u(t)) = u(t), L(t,x) = x,$ $g(s) = k(t,s)$ and $w_2(q) = 1$then Theorem 3.6 reduces to Theorem 2.2 in [19].

Remark 3.23 If we put $w_1(p) = p, \phi(u(t)) = u(t), f(s) = k_2(t,s), L(t,x) = x, g(s) = k_1(t,s)$ and $w_2(q) = 1$then Theorem 3.6 reduces to Theorem 2.3 in [19].

Remark 3.24 If we put $w_1(p) = 1$in L.H. S. and $\phi(u(t)) = u(t), L(t,x) = x,$ and $w_1(p) = w_2(q) = p$ in R.H.S. of Theorem 3.6 then Theorem 3.6 reduces to Theorem 2.1 in [21].

Remark 3.25 If we put $w_1(p) = p, \phi(u(t)) = u(t), L(t,x) = x,$ and $w_2(q) = q$then Theorem 3.6 reduces to Theorem 2.2 in [21].

4. Application

Example 4.1 Consider the following retarded Gronwall-Bellman type integral equation:

$u^p(t) = M\left(t, \int_a^t H_1\,(t,u(s))ds, \int_a^t H_2\,(t,u(\alpha(s)))ds\right),\ \forall\ t \in \mathbb{I}.$

4.1

where $M \in \mathbb{C}(I \times I^2, I)$, and $H_i \in \mathbb{C}(I \times I, I), i = 1,2$.satisfy the following conditions:

$M(t,u,v)| \le c(t) + |u| + |v|\ \forall\ t \in \mathbb{I},$ 4.2

$|H_1(t,u)| \le f(t)\ |u|^p$,and $|H_2(t,u)| \le g(t)\ |u|^q$. 4.3

where $u(t), c(t), f(t), g(t),$ and $\alpha(t)$ are defined in Theorem 3.1 and $w_1(p) = p, w_2(q) = q$. Using conditions (4.2) -(4.3) in (4.1), we get

$$|u(t)|^p \leq c(t) + \int_0^t f(s)|u(s)|^p ds + \int_0^t g(s)|u(\alpha(s))|^q ds.$$

By using substitution $\alpha(s) = s$, we obtain

$$|u(t)|^p \leq c(t) + \int_0^t f(s)|u(s)|^p ds + \int_0^t \alpha(t)\, \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}|u(s)|^q\ ds,$$

$$|u^p(t)| \leq c(t) + \int_0^t f(s)|u^p(s)|ds + \int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\ |u^q(s)|\ ds,$$

for all $t \in I$. Now an application of Theorem 3.1, we have

$$u(t) \leq \{c(t) + \int_0^t (f(s)c(s) + \alpha'(s)g(\alpha(s))\phi(m_1c(t) + m_2)$$
$$exp(\int_s^t [f(r) + m_1\alpha'(r)g(\alpha(r))\phi'(m_1c(r) + m_2)])dr)ds.\}^{\frac{1}{p}},$$

where $m_1 = \frac{q}{p}k^{\frac{q-p}{p}}$ and $m_2 = \frac{p-q}{p}k^{\frac{q}{p}}$.

Thus the estimation of inequality (4.5) implies the existence and boundness of the solution u(t) of retarded Gronwall-Bellman integral equation (4.1).

Example 4.2 Consider the following retarded nonlinear integral inequality:

$$u^{w_1(p)}(t) \leq (t+1) + \int_0^t f(s)\phi_1\big(u^{w_2(q)}(s)\big)ds + \int_0^{\sqrt{t}} g(s)\phi_2\big(u^{w_3(r)}(s)\big)ds. \quad 4.6$$

where u(t) is defined as in Theorem 3.2 and we assume that every solution u(t) of (4.6) exists on I. For this let, $\phi_1(u(t)) = u^2(t), \phi_2(u^r(t)) = u(t), p = 3, q = 2, r = 1, c(t) = t+1, f(s) = s,$ and $g(s) = 2s$,then inequality (4.6) turns to

$$u^3(t) \leq (t+1) + \int_0^t s\, u^2(s)ds + \int_0^{\sqrt{t}} 2\, su(s)ds. \quad 4.7$$

We find that

$m_1 = \frac{2}{3}, m_2 = \frac{1}{3}, m_3 = \frac{1}{3}, m_4 = \frac{2}{3}$,.for any $k > 0$,and $w_1(p) = p, w_2(q) = q$

Assuming k= 1, and $\alpha(t) = \sqrt{t}, \alpha'(t) = \frac{1}{2\sqrt{t}}$. Then

$$p_2(t) = f(t)\phi_1(m_1c(t) + m_2) + \alpha'(t)g(\alpha(t))\phi_2(m_3c(t) + m_4) = \frac{4t^3}{9} + \frac{4t^2}{3} + \frac{4t}{3} + 1, \quad 4.8$$

and

$$q_2(t) = m_2f(t)\phi_1'(m_1c(t) + m_2) + \alpha'(t)g(\alpha(t))m_3\phi_2'(m_3c(t) + m_4) = \frac{4t^2}{9} + \frac{2t}{3} + \frac{1}{3}. \quad 4.9$$

Hence by Theorem 3.2 and inequality (4.6) we have

$$u(t) \leq \{t + 1 + \int_0^t (\frac{4s^3}{9} + \frac{4s^2}{3} + \frac{4s}{3} + 1) \times exp(\int_s^t (\frac{4r^2}{9} + \frac{2r}{3} + \frac{1}{3}) dr) ds\}^{\frac{1}{3}}, \text{for all } t \in I. \quad 4.10$$

The plot of this solution can be plotted using Mathematica as shown in Fig. 1.

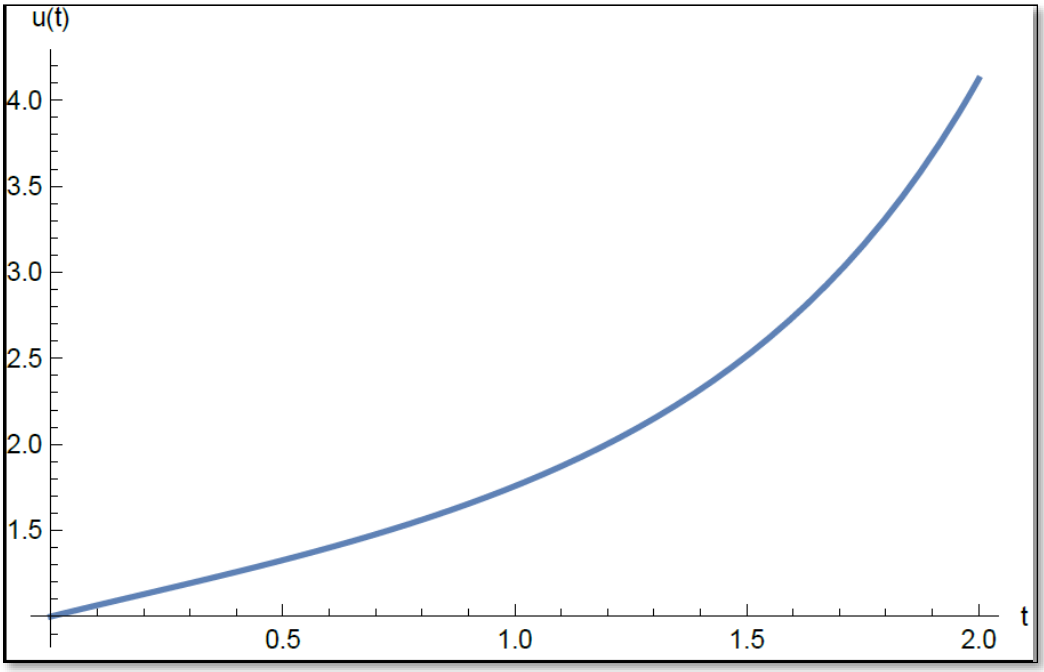

Figure:1

The solution u(t) of retarded integral inequality (4.6) exist and bounded within the interval $[0,4.12]$for $0 \leq t \leq 2$ as shown in Fig.1.

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References

[1] Gronwall, H. T. “Note on the Derivatives with Respect to a Parameter of the Solutions of a System of Differential Equations.” *Annals of Mathematics*, vol. 2, 1918–1919, pp. 292–296.

[2] Kim, Y. H. “On Some New Integral Inequalities for Functions in One and Two Variables.” *Acta Mathematica Sinica*, vol. 21, 2005, pp. 423–434.

[3] Pachpatte, B. G. “A Note on Chebyshev–Grüss Type Inequalities for Differential Functions.” *Tamsui Oxford Journal of Mathematical Sciences*, vol. 22, no. 1, 2006, pp. 29–36.

[4] Hardy, G. H., J. E. Littlewood, and G. Pólya. *Inequalities*. 2nd ed., Cambridge University Press, 1952.

[5] Bainov, D., and P. Simeonov. *Integral Inequalities and Applications*. Kluwer Academic Publishers, 1992.

[6] Bihari, I. A. “A Generalization of a Lemma of Bellman and Its Application to Uniqueness Problem of Differential Equation.” *Acta Mathematica Academiae Scientiarum Hungaricae*, vol. 7, 1956, pp. 81–94.

[7] El-Owaidy, H., A. A. Ragab, and A. Abdeldaim. “On Some Integral Inequalities of Gronwall–Bellman Type.” *Applied Mathematics and Computation*, vol. 106, 1999, pp. 289–303.

[8] Bellman, R. “The Stability of Solutions of Linear Differential Equations.” *Duke Mathematical Journal*, vol. 10, 1943, pp. 643–647.

[9] Pachpatte, B. G. *Inequalities for Differential and Integral Equations*. Academic Press, 1998.

[10] Pachpatte, B. G. *Inequalities for Differential and Integral Equations*. Marcel Dekker, 2002.

[11] Cheung, W. S. “Some New Nonlinear Inequalities and Applications to Boundary Value Problems.” *Nonlinear Analysis*, vol. 64, 2006, pp. 2112–2128.

[12] El-Owaidy, H., A. A. Ragab, and A. Abdeldaim. “On Some Integral Inequalities of Gronwall–Bellman Type.” *Applied Mathematics and Computation*, vol. 106, 1999, pp. 289–303.

[13] El-Owaidy, H., A. A. Ragab, W. Abueleela, and A. A. El-Deeb. “On Some New Nonlinear Integral Inequalities of Gronwall–Bellman Type.” *Kyungpook Mathematical Journal*, vol. 54, 2014, pp. 555–575.

[14] Khan Zareen, A. “Integral Inequalities of Gronwall–Bellman Type.” *Applied Mathematics*, vol. 5, 2015, pp. 3484–3488.

[15] Latpate, S. G., and S. V. Babar. “On Generalizations of Integral Inequalities and Its Applications.” *Mathematics and Computing*, Springer Proceedings in Mathematics and Statistics, vol. 415, 2022, pp. 425–437.

[16] Boukerrioua, Khaled, Brahim Kilani, and Imen Meziri. “Refinement of Some Retarded Integral Inequalities of Gronwall–Bellman–Bihari Type and Their Applications.” *Surveys in Mathematics and Its Applications*, vol. 15, 2020, pp. 233–255.

[17] Abdeldaim, A., and M. Yakout. “On Some New Integral Inequalities of Gronwall–Bellman–Pachpatte Type.” *Applied Mathematics and Computation*, vol. 217, 2011, pp. 7887–7899.

[18] Abdeldaim, A. “Nonlinear Retarded Integral Inequalities of Gronwall–Bellman Type and Applications.” *Journal of Mathematical Inequalities*, vol. 10, no. 1, 2016, pp. 285–299.

[19] El-Deeb, A. A., and Reda Gamal Ahmed. “On Some Generalizations of Certain Nonlinear Retarded Integral Inequalities for Volterra–Fredholm Integral Equations and Their Applications in Delay Differential Equations.” *Journal of the Egyptian Mathematical Society*, vol. 25, 2017, pp. 279–285.

[20] Abdeldaim, A., and A. A. El-Deeb. “On Some Generalizations of Certain Retarded Nonlinear Integral Inequalities with Iterated Integrals and an Application in Retarded Differential Equations.” *Journal of the Egyptian Mathematical Society*, vol. 23, 2015, pp. 470–475.

[21] El-Owaidy, H., A. Abdeldaim, and A. A. El-Deeb. “On Some New Retarded Nonlinear Integral Inequalities and Their Applications.” *Mathematical Sciences Letters*, vol. 3, no. 3, 2014, pp. 157–164.

[22] Abdeldaim, A., and A. A. El-Deeb. “Some New Retarded Nonlinear Integral Inequalities with Iterated Integrals and Their Applications in Retarded Differential Equations and Integral Equations.” *Journal of Functional Calculus and Applications*, vol. 5, no. 9, pp. 1–9.

[23] Abdeldaim, A., A. A. El-Deeb, and Reda Gamal Ahmed. “On Retarded Nonlinear Integral Inequalities of Gronwall and Applications.” *Journal of Mathematical Inequalities*, vol. 13, no. 4, 2019, pp. 1023–1038.

[24] Agarwal, R. P., S. Deng, and W. Zhang. “Generalization of a Retarded Gronwall-Like Inequality and Its Applications.” *Applied Mathematics and Computation*, vol. 165, 2005, pp. 599–612.

[25] Lipovan, O. “A Retarded Gronwall-Like Inequality and Its Applications.” *Journal of Mathematical Analysis and Applications*, vol. 252, 2000, pp. 389–401.

[26] Wang, W. S. “A Generalized Retarded Gronwall-Like Inequality in Two Variables and Applications to BVP.” *Applied Mathematics and Computation*, vol. 191, 2007, pp. 144–154.