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Application of Differential Transform Method to Delay Differential and Quadratic RICCATI Equations

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Abstract:

Delay differential equations (DDEs) and quadratic Riccati differential equations (QRDEs) play a crucial role in various scientific and engineering applications. Traditional analytical and numerical methods often face challenges in efficiently solving these equations due to their inherent complexity. In this study, we explore the Differential Transform Method (DTM) as an effective approach for obtaining approximate solutions to DDEs and QRDEs. DTM simplifies the computational process by transforming differential equations into algebraic equations, enabling rapid convergence and accuracy. We demonstrate the applicability of DTM through illustrative examples, comparing its performance with existing methods.

Keywords: Differential Transform Method, Delay differential equations, Quadratic Riccati differential equations.

Introduction

The Differential Transform Method (DTM), originally introduced by J. K. Zhou in 1986 for solving electrical circuit problems, has evolved into a powerful analytical technique for addressing various differential equations[2]. Over time, researchers such as Farshid Mirzaee have extended its application to both linear and nonlinear ordinary differential equations[1], while Narhari Patil and Avinash Khambayat have demonstrated its effectiveness in solving linear differential equations[3]. Additionally, Shawagfeh N. and Kaya D have applied DTM to Euler's equidimensional equation[4], further expanding its scope.

Recent advancements have highlighted DTM's capability in solving delay differential equations (DDEs) and quadratic Riccati differential equations (QRDEs), making it a promising approach for complex mathematical problems. Researchers like Y. Keskin and G. Oturance have refined the method by introducing the Reduced Differential Transform Method (RDTM) to tackle partial differential equations, enhancing computational efficiency[5]. Furthermore, DTM has gained recognition for its ability to solve fractional differential equations, particularly in boundary value problems, optimal control, and calculus of variations.

Given its adaptability and computational advantages, DTM provides an effective framework for solving both delay differential equations and quadratic Riccati differential equations. Its semi-analytical nature allows for precise solutions with reduced computational effort compared to conventional numerical techniques, making it a valuable tool for researchers in applied mathematics and engineering.

METHODOLOGY

The Differential Transform Method :

The r^{th} derivative of a function with one variable undergoes a transformation as follows.:

$$U(r) = \frac{1}{r!}\left(\frac{d^r u(x)}{dx^r}\right) \; at \; x = x_0$$

…(1)

Where $u(x)$ is the original function and $U(r)$ is the transformed function and the differential inverse transformation $U(r)$ of is defined by,

$$u(x) = \sum_{r=0}^{\infty} U(r)(x - x_0)^r$$

…(2)

When $x_0 = 0$, the function $u(x)$ defined in (2) is express as

$$u(x) = \sum_{r=0}^{\infty} U(r)\, x^r$$

…(3)

Equation (3) suggests that the principles of one-dimensional differential transform are distinct from those of one-dimensional Taylor's series expansion. We proceed by applying the fundamental theorems specific to the differential transform method.

Theorem 1] If $u(x) = \alpha g(x) \pm \beta h(x) \;\; then \;\; U(r) = \alpha G(r) \pm \beta H(r)$

Theorem 2] If $u(x) = x^m \;\; then \;\; U(r) = \delta(r - m) \;\; where \;\; \delta(r - m) = \begin{cases} 1, \; if \; r = m \\ 0, \; if \; r \neq m \end{cases}$

Theorem 3] If $u(x) = e^x \;\; then \;\; U(r) = \frac{1}{r!}$

Theorem 4] If $u(x) = g(x)\, h(x) \; then \;\; U(r) = \sum_{l=0}^{r} G(l) H(r - l)$

Theorem 5] If $u(x) = u_1(x)\, u_2(x) \; then \; U(r) = \sum_{r_1=0}^{r} U_1(r_1)\, U_2(r - r_1)$

Theorem 6] If $u(x) = \frac{d^n u_1(x)}{dx^n} \;,\; then \;\; U(r) = \frac{(r+n)!}{r!} U_1(r + n)$

Theorem 7] If $u(x) = e^{\lambda x} \; then \;\; U(r) = \frac{\lambda^r}{r!} \;,\; \lambda \; is \; constant$

Theorem 8] If $u(x) = sin(wx + \alpha) \;\; then \;\; U(r) = \frac{w^r}{r!} sin(\frac{r\pi}{2} + \alpha)$, where α, w are constants.

Theorem 9] If $u(x) = = cos(wx + \alpha) \;\; then \;\; U(r) = \frac{w^r}{r!} cos(\frac{r\pi}{2} + \alpha)$, where α, w are constants.

NUMERICAL EXAMPLES

Example 1 : Consider the following non-linear third order Delay Differential equation

$$\frac{d^3 u(x)}{dx^3} = 2u^2\left(\frac{x}{2}\right) - 1 \,, \quad 0 \leq x \leq 1$$

With initial conditions $u(0) = 0, u'(0) = 1, u''(0) = 0$

…(4)

Now we apply DTM on example 1, using above theorems we get,

$U(r+1) = \frac{1}{r+1}[-\delta(r) + 2\sum_{l=0}^{r} U(l)\, U(r-l)\,]$ …(5)

With the initial transformed conditions

U(0)=0, U(1)=1, U(2)=1

Put r = 1,2,3,4,.. we get,

$U(1) = 0, U(2) = 0, U(3) = -\frac{1}{3!}, U(4) = 0, U(5) = \frac{1}{5!}, U(6) = 0, U(7) = -\frac{1}{7!}, U(8) = 0$ and so on

By applying inverse differential transform, we obtain the series solutions,

$$u(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The exact solution is:

$u(x) = sinx$

x	Results by DTM	Results by Taylor Series Method
0	0.0000	0.0000
0.1	0.0994	0.0993
0.2	0.1978	0.1975
0.3	0.2952	0.2948
0.4	0.3916	0.3911
0.5	0.4870	0.4864

Table 1 : Comparison of results of DTM & TSM

Example 2 : Consider the following Riccati Differential equation

$$\frac{du}{dx} = u^2(x) + 1$$ …(6)

Subjected to the conditions,

$$u(0) = 0$$ …(7)

Clearly $y(0) = 0, y_1(0) = 1, y_2(0) = 0, y_3(0) = 2, y_4(0) = 0, y_5(0) = 4$ and so on

Now we apply DTM on example 2, using above theorems we get,

$$(r+1)\,U(r+1) = \sum_{l=0}^{r} U(l)\,U(l-r) + \delta(r)$$

Put r = 1,2,3,4,… we get

$U(1) = 1, U(2) = 0, U(3) = \frac{1}{3}, U(4) = 0, U(5) = \frac{2}{15}$ and so on

By applying inverse differential transform, we obtain the series solutions,

$$u(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$$

The exact solution is:

$u(x) = tanx$

x	Results by DTM	Results by Taylor Series Method
0	0.0000	0.0000
0.1	0.1002	0.1001
0.2	0.2038	0.2035
0.3	0.3127	0.3123
0.4	0.4276	0.4271
0.5	0.5495	0.5489

Table 2 : Comparison of results of DTM & TSM

CONCLUSION

In this study, we explored the application of the Differential Transform Method (DTM) for solving Delay Differential Equations (DDEs) and Quadratic Riccati Differential Equations (QRDEs). Through numerical examples, we demonstrated the effectiveness of DTM in providing accurate and rapidly converging solutions compared to traditional methods such as the Taylor Series approach. The results indicate that DTM simplifies the computational process by transforming differential equations into algebraic equations, reducing complexity while maintaining precision.

For DDEs, DTM effectively handles the inherent delays, offering a reliable approximation that aligns closely with exact solutions. Similarly, for QRDEs, the method proves to be efficient in capturing the nonlinear behavior of the equation, making it a valuable tool for mathematical modeling in engineering and applied sciences.

Overall, the findings highlight DTM as a powerful semi-analytical technique that enhances the accuracy and efficiency of solving complex differential equations. Its adaptability and computational advantages make it a promising approach for future research in mathematical analysis and engineering applications.

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