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Fixed Point Results in General Topology Continuous Mappings and Separation Axioms

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Abstract

The principles underlying the fixed point theory are being methodically explored in this research paper, in the abstract framework of general topology, and no longer in the more traditional distance-based restrictions of the metric spaces. The paper is dedicated to the severe interaction between continuous mappings and separation axioms in the determination of the Fixed Point Property (FPP) in different topological spaces. We consider the way continuous functions and topological retracts conserve key structural properties, e.g. compactness and connectedness, to ensure the existence of fixed points that meet the condition:

$f(x) = x$

Besides, the article offers a mathematically rigorous appraisal of axioms of separation, showing that the Hausdorff (T_2) axiom is a strict requirement in the way of making sure that the fixed-point of a continuous map constitutes a topologically closed set. Through a generalization of the use of classical theorems, such as those of Brouwer, Schauder, and Kakutani, we demonstrate how compactness, convexity and topological resolution are used to give convergence in a single valued and a multi-valued mapping. Lastly, we provide certain counterexamples (e.g. spaces with the cofinite topology and non-compact continuous domains) to illuminate the mathematical pathologies and lack of convergence which occur when these topological axiomatic foundations are weakened or eliminated.

Keywords

General Topology; Fixed Point Theory; Continuous Mappings; Separation Axioms; Hausdorff Space; Topological Retracts; Fixed Point Property (FPP); Brouwer's Theorem.

1. Introduction

The theory of fixed points is a major branch of contemporary mathematics, which connects abstract topology to applied mathematical analysis in a fundamental way. In essence, a fixed point of a mapping is a point which is not moved by that transformation. Consider a space X and a mapping f of X into itself, is a fixed point of f is a point in X such that the following is true:

$f(x) = x$

Traditionally the evolution of the fixed-point theory was strongly rooted in the metric spaces. Instances of such as the Banach Contraction Principle would also be based on the idea of distance in that the concept of distance is viewed as the sole absolute to ensure that the fixed point exists and is unique. Nevertheless, with the change in mathematical concept to a more generalized structure, there was a need to comprehend how these fixed-point outcomes apply to pure topological space, where nothing is given in terms of distance, and space is determined by neighborhoods, open sets, and continuous mappings (Mitrovic et al., 2020).

The shift to topological fixed point theory is traditionally best known by the example of the fixed point theory by Brouwer. This theoretical basis is that any continuous mapping of a compact, convex set to itself will have at least one fixed point (Ben-El-Mechaieh and Mechaiekh, 2022). Their occurrence in a general topological context is inseparably related to the character of the space - in particular its separation properties and the character of the continuous mappings which are defined on it.

Separation axioms: These are the T 0 (Kolmogorov) through T 4 (Normal) spaces that characterize the rigorousness with which distinct points and closed sets can be separated by open neighborhoods. These axioms, together with continuous mappings, constitute the theoretical basis of the topological fixed-point results. As a case in point, to guarantee that the collection of fixed points of a continuous mapping is a closed subset, the space will generally need to fulfill the Hausdorff (T 2) separation axiom (Al-shami and Abo-Tabl, 2021). Moreover, the Fixed Point Property (FPP) according to which a space is claimed to have is strongly affected by topological properties like compactness and connectedness, and the retractability of the space (Kang, Han and Lee, 2019).

Current studies are still building on these classical basics. As of lately, fixed-point theorems have been studied in more generalized settings, such as fuzzy metric spaces (Adhya and Ray, 2021; de Andrade and Wasques, 2024), topological cylinders (Feltrin, 2015), and spaces with interrelated feedback topologies (Radde, 2012). Their abstract topological generalizations do not just exist on paper but have extensive applications in the solution of problems of complex equations, including the use of fractional differential equations to model economic growth and other dynamic systems (Abdou, 2024; Gnanaprakasam et al., 2022).

The purpose of this paper is to systematically investigate the fixed-point of general topology. We shall develop the preliminaries needed on the subject of continuous mappings and separation axioms, explore the way in which these axioms determine the behaviour of fixed-point sets and examine certain topological extensions of classical theorems into more general mathematical fields.

2. Preliminaries and Notation

In order to strictly explore the topic of fixed-point theorems in general topology, a common system of definitions, notation and underlying axioms must first be developed. In contrast to metric spaces, in which distance functions explicitly determine how the points are close to each other, the general topology is based on the open sets construction as the way of defining the convergence, continuity, and separation (Morayne and Ralowski, 2023).

2.1 Topological Spaces and Continuous Mappings

The simplest structure in this field is the topological space. Let X be a non-empty set. One topology on X is denoted by the Greek letter, which is called the topology of X, denoted using the sign τ, and is a set of subsets of X (called open sets) which satisfies three property axioms: the empty set and X belong to the topology; the arbitrary union of any set of sets in the topology τ belong to the topology τ, and the finite intersection of any set of sets in the topology τ belong to the τ topology. Topological space is indicated as:

$$(X, \tau)$$

The points are called the elements of X and a complement of open set is called a closed set. Given a point, x, in X, a neighborhood of x is any set N which contains an open set U, with x being an element of U.

The Fixed Point Property highly relies on the concept of a continuous mapping. Mathematically, the mapping of a topological space, X, to a topological space, Y, of any type, is said to be continuous, when, given any open set V in Y, the preimage of V under f is also an open set in X.

$$f^{-1}(V) \in \tau_X$$

It is the fact that the preservation of structural properties through continuous mappings--as well as their more generalized analogs, e.g., fuzzy mappings and multi-valued contractions (de Andrade and Wasques, 2024; Jleli, Petrov and Samet, 2025) that ultimately enables the abstractability of fixed-point theorems to have no dependency on strict metrics.

Table 1: Hierarchy of Separation Axioms and Fixed Point Implications

Separation Axiom	**Common Name**	**Topological Condition**	**Impact on Fixed-Point Sets**
T_0	Kolmogorov Space	Distinct points are topologically distinguishable by at least one open set.	Minimal impact; insufficient for ensuring uniqueness of limits.
T_1	Fréchet Space	Distinct points are separated by individual open sets; singletons are closed.	Insufficient; the fixed-point set of a continuous mapping is not guaranteed to be closed.
T_2	Hausdorff Space	Distinct points are separated by strictly disjoint open neighborhoods.	**Critical:** Guarantees that the fixed-point set of any continuous mapping is a closed subset.
T_3	Regular Space	Points and closed sets are separated by disjoint open neighborhoods.	Strengthens convergence guarantees in highly abstract contractive mappings.
T_4	Normal Space	Disjoint closed sets are separated by disjoint open neighborhoods.	Allows for powerful extension theorems (e.g., Tietze), useful in proving retracts.

2.2 The Separation Axioms

In general topology spaces may be too cluttered when there are not enough open sets to separate different points or closed sets between each other. Separation axioms are conditions that put a limit on the topology to ensure that point and set may be sufficiently separated by neighbourhoods (Al-shami and Abo-Tabl, 2021). These axioms have a direct effect on the behavior of fixed-point sets based on their hierarchy.

- T_0 **Space (Kolmogorov Space):** For any two distinct points in X, there exists at least one open set containing one of the points but not the other.
- T_1 **Space (Fréchet Space):** Given two different points in X, there is an open set which includes the first one but not the second one, and there is an open set which includes the second one but not the first. The implication of T 1 spaces is that every singleton set is a closed set.
- T_2 **Space (Hausdorff Space):** Given two different points in X, s and w, there are disjointed open neighborhoods U and V where s and w are members of U and V respectively. The Hausdorff condition is probably the most severe separation axiom of the fixed-point theory, since this ensures not only uniqueness of limits of sequences but even the closedness of the fixed-point set of a continuous mapping.

Still more axioms, including Regular (T_3, which additionally partakes points and closed sets) and Normal (T_4, which additionally partakes disjointed closed sets), give even more structural assurances. Generalizations and variations of these axioms, including S_3 separation of geodesic convexity are still actively studied to find fixed points of highly specific network topologies or graph structures (Chepoi, 2024; Radde, 2012).

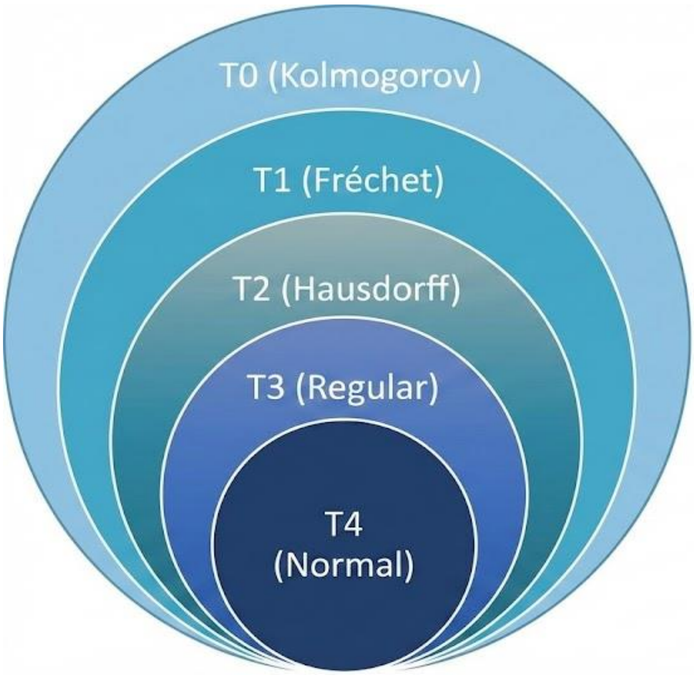

Figure 1: The Hierarchy of Separation Axioms

2.3 The Fixed Point Property (FPP)

Topological space X is considered to have the Fixed Point Property (FPP) when all the continuous mappings f of X to X have at least one point of fixed point:

$f(x) = x$

The Fixed Point Property is a topological invariant, which is one of the most significant background theorems in the field (Kang, Han and Lee, 2019). This is to say that in case a space X has the FPP, any space Y that is homeomorphic to X should also have the FPP.

To prove this strictly, consider X to be, have the FPP and suppose that there is a homeomorphism, h, between X and Y, which is bijective and continuous with a continuous inverse:

$$h^{-1}$$

Let g be any continuous mapping from Y to Y. We can construct a continuous mapping f from X to X defined by the composition (using circ for composition):

$$f = h^{-1} \circ g \circ h$$

Since X possesses the FPP, the mapping f must have a fixed point x in X, meaning:

$$f(x) = x$$

Substituting the definition of f, we obtain:

$$h^{-1}\left(g\big(h(x)\big)\right) = x$$

Applying the function h to both sides yields:

$$g\big(h(x)\big) = h(x)$$

If we let y = h(x), where y is an element of Y, the equation simplifies to:

$$g(y) = y$$

Thus, y is a fixed point of g in Y. It demonstrates that the Fixed Point Property is directly connected to the topology of the space and not to its geometry or particular metric, which is the foundation of studying the role of compactness, retracts, and continuous mappings in imposing fixed points on advanced spaces (Mitrovic et al., 2020).

3. The Role of Continuous Mappings

Continuous mappings are used in general topology as the main tool in the process of moving structural properties between spaces. Whereas contractive inequalities are widely used in metric spaces to compel a sequence to be convergent to a fixed point, the topological spaces use the action of continuous functions to maintain necessary invariants. The study of the interactions of continuous mappings with subsets and topological space is important to determine the existence of a fixed point.

3.1 Preserving Topological Structure

Theoretical strength of a continuous mapping of fixed-point theory Theoretical strength The theoretical strength of a continuous mapping is in its capacity to maintain such properties as compactness and connectedness. As an example, is the classical, fixed point, theorem of Brouwer, which essentially relies on the observation that a continuous mapping of a compact, convex space cannot be used to cut or otherwise spread the space in a manner that leaves a fixed point (Ben-El-Mechaieh and Mechaiekh, 2022).

When a space, X is compact, and f is a continuous mapping of X to itself, then the image of X under f, also, is compact. On the same note, the image of X is also connected when it is connected. These structural assurances restrict the conduct of the function. In even more abstract topological constructions, such as non-retractable spaces or soft parametric metric spaces, more generalizations of continuous mappings, such as single or multi-valued contractions, are used, though they still make use of this most basic principle of structural preservation to guarantee the existence of fixed points (Gunduz, Bayramov and Coskun, 2024; Jleli, Petrov and Samet, 2025).

Table 2: Preservation of Topological Invariants

Topological Property	**Preserved by Continuous Mappings?**	**Relevance to the Fixed Point Property (FPP)**
Compactness	Yes	Forces sequences or nets within the space to possess convergent subnets, preventing escape to infinity.
Connectedness	Yes	Prevents continuous mappings from tearing the space or "jumping" across disjoint open sets to avoid a fixed point.
T_2 **(Hausdorff) Axiom**	No (not inherently)	The target space must possess this axiom independently to ensure the limit of any convergent sequence is unique.
Retractability	Yes (onto subspaces)	If a parent space possesses the FPP, any continuous topological retract of that space inherently inherits the FPP.

3.2 Topological Retracts and the Fixed Point Property

The concept of a topological retract is one of the most potent to be used to extend the Fixed Point Property (FPP) to new spaces. Basically a retract will enable us to project a larger space onto a smaller subspace and still be continuous.

Assume that X is a topological space and that A is a subsection of X. One of the maps X onto A, r, is a retraction provided r is an identity mapping on A.

$$r(a) = a$$

for all points a in A. In case such a mapping allows, the subspace A is said to be a retract of X.

The breakthrough formula of retracts with our research is as follows: In case a topological space X has the Fixed Point Property, then all retracts A of X have the Fixed Point Property as well (Kang, Han and Lee, 2019).

In order to do this rigorously, suppose that X has the FPP, and that A is a retract of X with the continuous retraction r. Suppose f is any mapping of A to A which is continuous. We would like to show that f has a fixed point in A.

By taking a composition of f and r, we can map the whole space X into X that is known as a new mapping g:

$$g = f \circ r$$

Since f and r are continuous mappings, their composition g is a continuous mapping between X and X (Mitrovic et al., 2020). As we have seen that the bigger space will be X with the FPP, the continuous mapping g will have at least one fixed point within X. Let us call this fixed point x_0.

Therefore:

$$g(x_0) = x_0$$

By the definition of our composition g, we can substitute $f\big(r(x_0)\big)$ into the equation:

$$f\big(r(x_0)\big) = x_0$$

Since the image of the retraction r is completely a subset of the subspace A, and the domain of f is A, the resulting value x 0 must, necessarily, be an element of A.

Since x_0 is in A, and r is a retraction, applying r to x_0 simply yields x_0 itself:

$$r(x_0) = x_0$$

And replacing this in our former equation we have:

$$f(x_0) = x_0$$

This proves that x 0 will be a fixed point of the continuous mapping f. As f was an arbitrary continuous transformation between A and A we have shown that the subspace A has the Fixed Point Property.

This evidence underlines the significance of the use of continuous mappings in general topology. They enable mathematicians to determine fixed points in very complicated spaces, as this is done by proving the property of a simpler, parent space and locating an appropriate retraction. The method is commonly used in the case of extending fixed-point theorems to larger parametric metric spaces and topological modules (Mani et al., 2023; Sharma and Kour, 2023).

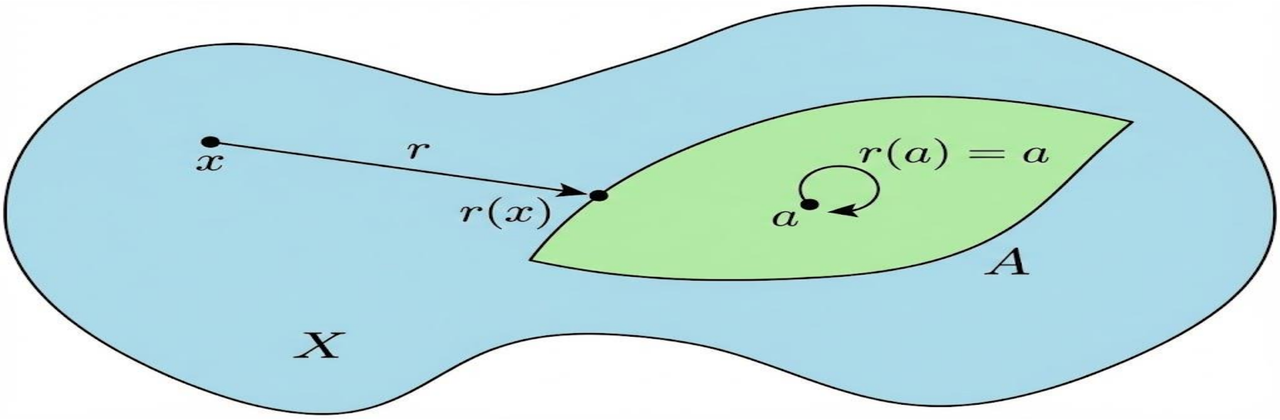

Figure 2: Topological Retraction and Subspaces

4. Interplay Between Fixed Points and Separation Axioms

Seclusion between general topology and fixed-point theory The separation axioms form the basics of bridge. Whereas continuous mappings are a structural preservation of a space, separation axioms give the required resolution in order to identify distinct points and to ensure that sequences do not have multiple limits at the same time. These axioms are then strictly necessary to impose the existence and topological characteristic of fixed points (Al-shami and Abo-Tabl, 2021).

4.1 The Hausdorff Axiom (T_2) and Closed Fixed-Point Sets

The Hausdorff separation axiom (T_2) is the most commonly used condition in topological fixed-point theorems. A space is T_2 when two distinct points can be separated by disjoint open neighborhoods. This property is important in that sense that it ensures that the set of fixed points of any continuous mapping is topologically closed.

We shall write $F = f(X)$ to mean the collection of all fixed points of a continuous mapping f of a topological space X into X:

$F = \{x \in Xmidf(x) = x\}$

Theorem: *If X is a Hausdorff space, and f is a continuous function of X to X, and then fixed-point set F is a sub-set of X that is closed.*

Proof: To establish that F is closed we will have to show that its complement, or X_F, is open. Take any single point in the complement $X_F(x)$. This by definition implies x is not a fixed point and therefore:

$f(x) \neq x$

Since the space X is T_2 separated, there is an open neighborhood U containing x and an open neighborhood V containing f (x) which is entirely discontinuous:

$U \cap V = \emptyset$

Moreover, since the mapping f is continuous the preimage of the open set V should be open in X. Thus, there is an open neighborhood W of x that is represented totally in V:

$f(W) \subseteq V$

It is now possible to build a new open set O taking the intersection of our two open neighborhoods of x:

O = U intersect W

Since O is the finite intersection of two open sets, O is an open neighborhood of x. And now assume any point y in this open set $O. y \in O$, is in W, and this means:

$f(y) \in V$

At the same time, y is in O and therefore, is in U as well. Since U and V are strictly disjointed open sets, f(y) is by no means able to be a member of U, and y is never able to be a member of V. Therefore:

$f(y) \neq y$

This is to show that at no point, y, in the open neighborhood O, is a fixed point. As a result, the open set O is a subset of the complement X\F. As we can locate to each point x of X\F such an open neighborhood, the complement is an open set. The set F, which is fixed, should therefore be a closed subset of X.

This is one of the theses of the field. In T_2 absence, F may not be closed, which in turn can cause pathology of fixed points (Mitrovic et al., 2020).

4.2 Compactness, Separation, and Convergence

The combination of compactness and separation guarantees creates very strong fixed-point guarantees. In a Hausdorff space, which is compact, infinite sequences (or, more generally, nets and filters) have to have convergent subnets, and the T_2 axiom is the property that these limits are absolutely unique.

Generalized forms of continuous mappings, called contractive mappings, are used in generalized mathematical spaces, including complete and weak G-complete fuzzy metric spaces (Adhya and Ray, 2021) or semimetrics spaces (Petrov, Salimov and Bisht, 2025). Even here in these highly abstract settings, the inherent separation of the topology, be it standard T_2 or modified S_3 separation to the geodesic convexity, is what eventually drives the contraction to collapse to a unique, singular fixed point (Chepoi, 2024; Gunduz, Bayramov and Coskun, 2024).

4.3 Common Fixed Points in Commuting Mappings

The naturally occurring extension of this theory is the analysis of a series of continuous mappings. In case of two continuous mappings, f and g, of a topological space X to X, under what conditions do they have a common fixed point?

$f(x) = g(x) = x$

A standard requirement is that the mappings must commute, meaning:

$f\big(g(x)\big) = g\big(f(x)\big)$

In those spaces where the separation axioms are very strong e.g. generalized topological modules or an orthogonal Branciari metric spaces, when f and g commute and have certain containment properties about their images, then the compactness of the space together with Hausdorff separation implies that there is a common fixed point (Gnanaprakasam et al., 2022; Sharma and Kour, 2023). The idea is especially critical in laying the foundations of the existence of shared fuzzy fixation points through fuzzy F-contractions in which several probabilistic mappings need to converge at a single topological coordinate (Kanwal et al., 2024).

5. Classic Fixed Point Theorems in Topology

It is based on the work of continuous mappings and separation axioms that a number of classical fixed-point theorems help demonstrate the strength of general topology. These theorems no longer depend on strict metric dependences but rather depend on topological invariants such as convexity, continuity and compactness.

5.1 Brouwer's Fixed Point Theorem

The most famous and most fundamental finding of topological fixed-point theory is perhaps the Fixed Point Theorem by Brouwer. In its topological formulation, it states that any continuous map of a space which is homeomorphic to the closed unit ball into itself should have at least one fixed point (Ben-El-Mecaieh and Mechaiekh, 2022).

Let D^n denote the closed unit ball in an n-dimensional space. If we define a continuous mapping f such that:

$f: D^n \rightarrow D^n$

then there guarantees to exist at least one point x within D^n that satisfies the condition:

$f(x) = x$

The genius of the Brouwer heroin lies in the fact that no contraction map is necessary. Mapping f may stretch, twist or compress the internal space as long as it is continuous and the space is compact and convex. The lack of holes (property of the underlying group of the space in algebraic topology) such that the void is not filled in by the space), makes the function incapable of mapping the space around a central hole to evade a fixed point.

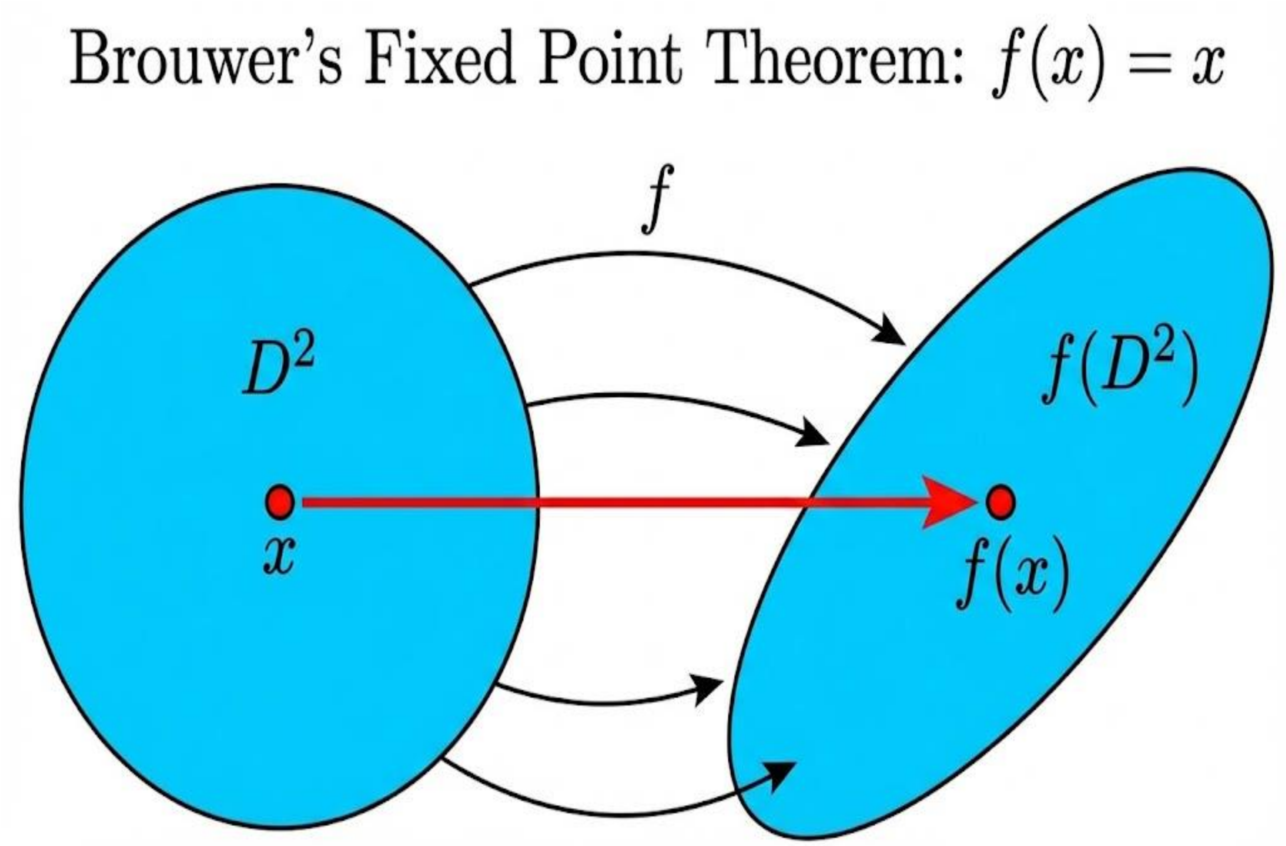

Figure 3: Brouwer's Fixed Point Theorem Dynamics

5.2 Schauder's Fixed Point Theorem

Although the version of the theory of Brouwer is only applicable to finite-dimensional spaces, the Fixed Point Theorem by Schauder generalizes the ideas to infinite-dimensional spaces, namely topological spaces of vectors.

According to the Schauder theory, a non-empty, compact and convex K of a Hausdorff topological vector space, and a continuous mapping f, of K into K, f has a fixed point. The Hausdorff separation axiom (T_2) is entirely essential in this case. As has been determined above, the T_2 axiom is what guarantees the limits of convergent nets of the compact subset K are unique, which imposes the condition of the fixed-point set being well-behaved and closed (Al-shami and Abo-Tabl, 2021).

Variations of the Schauder logic are used in more generalized metric spaces such as semimetric spaces with triangle functions to prove the fixed points of generalized Kannan-type mappings where the classical distance metrics cannot achieve their fixed points (Petrov and Bisht, 2023; Petrov, Salimov and Bisht, 2025).

5.3 Kakutani's Fixed Point Theorem

The Topological abstraction is taken to an even greater extreme by Kakutani in his Fixed Point Theorem which extends it to multi-valued functions, or set-valued mappings. Instead of having one point of input and one point of output, a multi-valued function gives an entire set of points to an input.

In the definition of a fixed point of a multi-valued mapping of a set S into the power set of S, a fixed point is defined in a some what different manner. It is a point x which is an element of the set of f(x):

$x \in f(x)$

Kakutani provides a theorem which states that S is a non-empty, compact, and convex sub-set of a Euclidean space, and f an upper semi-continuous multi-valued fiber mapping of S to the closed and convex subsets of S, f has a fixed point. The topological condition of the upper semi-continuity takes the place of the standard continuity in the case of multi-valued mappings that requires the sets of the mapped sets of f to be non-violent in growth or position.

This has far-reaching applications in applied mathematics. It is also a tool in game theory and economic growth theory to establish the existence of Nash equilibria where inter-depending feedback topologies must have multi-valued probabilistic mappings that must have a common topological coordinate (Abdou, 2024; Jleli, Petrov and Samet, 2025; Radde, 2012).

Table 3: Summary of Classic Fixed Point Theorems in Topology

Theorem Name	**Domain Requirements**	**Type of Mapping**	**Key Topological Guarantee**
Brouwer's Theorem	Compact, convex subset homeomorphic to a closed unit ball.	Continuous, single-valued mapping.	Any such mapping must possess at least one fixed point without requiring a metric contraction.

Schauder's Theorem	Compact, convex subset of a topological vector space.	Continuous, single-valued mapping.	Extends Brouwer into infinite dimensions; strictly relies on the T_2 (Hausdorff) axiom for closed sets.
Kakutani's Theorem	Compact, convex subset of a Euclidean space.	Upper semi-continuous, multi-valued mapping.	Guarantees a fixed point where the point is an element of its mapped set (x in f(x)); used in game theory.

6. Counterexamples and Pathologies

In mathematics The need of an axiom or topological condition is most easily understood by looking at the pathologies that occur when the condition is dropped. General topology is the only kind of topology that can be used to come up with counterexamples to show precisely why strict separation axioms and invariants such as compactness is needed to prove that there are indeed fixed points and that they behave as one would expect.

6.1 Failure of the Closed Fixed-Point Set in T_1 Spaces

Since the Hausdorff separation axiom (T_2) was defined in Section 4, this is the weakest condition that can be given to make the set of fixed points of a continuous mapping be topologically closed. In case we make the topological space any weaker, to meet the T_1 (Frechet) axiom only, then this important structural condition fails absolutely (Al-shami and Abo-Tabl, 2021).

In order to illustrate this pathology, one may take an infinity set, X, with the cofinite topology. The cofinite topology only has an empty set and the subsets of which the complement is finite as open sets. It follows that there are only finite subsets and the space X itself as closed subsets of this space. This space is T_1, since any singleton set is finite and thus closed, but not T_2, since any two non-empty open sets can not be disjointed (their complements are only finite).

Assume that f is some well-designed continuous function of X into itself and that F is the set of fixed-points of f:

$$F = x \in X \vee f(x) = x$$

This can be defined to mean that f fixes an infinitely many number of points in X, but not all. Thus, the set F is an infinite, proper subset of X of the fixed-point set. But the very definition of the cofinite topology implies that the closed sets are finite (or the entire space). Since F is a set which is infinity but not equal to X, F can not possibly be a closed set.

This kind of counterexample identifies that the T_2 axiom cannot be violated given conditions to the fixed-point set being mathematically well-behaved and closed. Without Hausdorff separation, it is possible to have fixed points that are abstract and non-closed and very unstable to topological deformations (Mitrovic et al., 2020).

6.2 The Absence of the Fixed Point Property in Non-Compact Spaces

Compactness implies that a space has to be topologically bound with sequences being not stretched to infinite by continuous mappings. Even the spaces with the most separation axioms (including T_4 Normal spaces) can also lack the Fixed Point Property (FPP) very easily when the notion of compactness is eliminated (Kang, Han and Lee, 2019).

The most fundamental and the most basic example is the real number line, R, the topology being the standard topology. It is real, Hausdorff and complete although not compact. Suppose the continuous translation mapping f:

$f: RrigntarrowR$

defined by:

$f(x) = x + 1$

To find a fixed point, we must satisfy the condition:

$x + 1 = x$

Subtracting x from both sides yields the mathematically absurd statement:

$1 = 0$

Thus, this continuous mapping of any kind has no fixed points whatsoever, which demonstrates that the real line lacks the FPP. This example is what makes Brouwer and Schauder theorems strictly dependent on the domain being compact, as otherwise a continuous mapping is allowed solely to move the entire space further along itself indefinitely without a definite point of contact with a fixed topological coordinate (Ben-El-Mechaieh and Mechaiekh, 2022).

6.3 Pathologies in Generalized and Multi-Valued Mappings

In the case that the classical fixed-point theorems are generalized to generalized spaces, e.g. soft parametric metric spaces, topological modules, or fuzzy metric spaces, there is more of a possibility that pathological behaviour will occur. Generalized Kannan-type mappings or fuzzy F-contractions are commonly used as substitutes of continuous mappings in such spaces (Kanwal et al., 2024; Petrov and Bisht, 2023).

When the generalized mapping does not conform to rigid contractive constraints or multi-valued mapping in the Kakutani theory relaxes the upper semi-continuity condition, then the recursive sequence of the mapping will not converge. Rather than settling on a point, the series of points:

x_n

will vibrate perpetually or will burst out of the local neighbourhood. In practice, where one wants to solve a model of an intracellular regulation network or a fractional differential equation modeling an economic system, the breakdown of these topological properties implies that the mathematical model will not have an equilibrium or a steady state, and make the system unsolvable (Abdou, 2024; Radde, 2012).

7. Conclusion

The analysis of fixed points in topology is an extreme abstraction of classical mathematical analysis. Topological fixed-point theory eliminates the rigorous reliance on distance functions of the metric spaces which show the basic structural requirements needed by mappings to achieve equilibrium. We have shown in this paper that it is precisely due to the presence and action of a fixed point:

$$f(x) = x$$

exist purely as functions of the interaction of continuous mappings with the axioms of separation which are inherent in the topological space. By definition, continuous mappings maintain such important invariants as compactness and connectedness. Moreover, the topological retracts can be used to extend the Fixed Point Property (FPP) of complex spaces to sub spaces, a pivotal point of advanced theorems (Kang, Han and Lee, 2019; Mitrovic et al., 2020).

Structural preservation is not however enough in absence of appropriate topological resolution. The axioms of separation allow the framework to make the fixed points proper. Hausdorff separation axiom (T_2) is a set of requirements that are strictly needed to ensure that the set of fixed points of any continuous mapping is a topologically closed set (Al-shami and Abo-Tabl, 2021). With no such axiom, or the compactness property, spaces are prone to pathological phenomena, in which sequences do not converge, mappings are endlessly translated without their end.

The shift of classical principles of metric contraction to the purely topological ones is best illustrated by theorems of Brouwer, Schauder, and Kakutani (Ben-El-Mechaieh and Mechaiekh, 2022). The theorems determine that when a space has enough compactness, convexity and separation, the existence of continuous (and even multi-valued upper semi-continuous) mappings is obliged to have a fixed point.

Recent studies keep extending these limits and use these topological generalized principles to fuzzy metric spaces, non-retractable spaces and extended parametric modules (Adhya and Ray, 2021; Mani et al., 2023). Finally, the interpretation of fixed-point solutions in terms of general topology does not only enhance pure mathematics, but also offers the rigorous foundations machine that one would need to address complicated, real-life dynamic systems in the fields of economics, biology and applied sciences.

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