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Comparison of Numerical Solutions for Volterra Integro-Differential Equations Using the Differential Transform Method

Rohit Nagargoje[1], **Avinash Khambayat**[2]

1. Research Scholar, Department of Mathematics, School of Science, Sandip University Nasik, Maharashtra

Email: rnagargoje92@gmail.com

2. Professor, Department of Mathematics, School of Science, Sandip University Nasik, Maharashtra

Email: avinash.khambayat@sandipuniversity.edu.in

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Abstract:

Volterra integro-differential equations are widely used to describe systems with memory effects in science and engineering. This paper presents a comparison of numerical solutions for such equations using the Differential Transform Method (DTM). The method is applied to selected examples and the obtained results are compared with exact solutions. The accuracy and efficiency of DTM are demonstrated through numerical values and graphical representations. The results show that DTM provides reliable approximations with minimal computational effort, making it an effective approach for solving Volterra integro-differential equations.

Keywords: Volterra Integro-Differential Equations, Differential Transform Method,

Introduction:

Volterra integro-differential equations arise naturally in the mathematical modelling of many real-world phenomena where the present state of a system depends not only on its current conditions but also on its past behavior [1]. Such equations are commonly encountered in fields including physics, engineering, biology, and economics, particularly in problems involving memory and hereditary effects. The combined presence of differential and integral terms makes these equations more complex than ordinary differential equations and often prevents the use of closed-form analytical solutions [2]. As a result, numerical methods play a crucial role in obtaining approximate solutions for Volterra integro-differential equations. Over the years, several numerical techniques have been developed, such as decomposition methods, collocation approaches, quadrature-based schemes, and perturbation techniques. Although these methods can produce accurate results, they may require discretization, linearization, or significant computational effort, which can limit their efficiency and practical implementation [3-4].

The Differential Transform Method (DTM) has emerged as an effective alternative for solving functional equations, including integro-differential equations [5-6]. This method is based on the Taylor series expansion and provides solutions in the form of rapidly convergent series without the need for discretization or small perturbation parameters [7]. Its straightforward implementation and reduced computational complexity make it particularly attractive for solving Volterra integro-differential equations [8]. In this paper, a comparative study of numerical solutions for Volterra integro-differential equations is presented using the Differential Transform Method [9]. The method is applied to selected test problems, and the obtained numerical solutions are compared with exact solutions and existing numerical approaches [10]. The accuracy and efficiency of the method are examined through numerical results and graphical representations [11]. The study demonstrates that the Differential Transform Method is a reliable and efficient tool for solving Volterra integro-differential equations and offers significant advantages over traditional numerical techniques [12].

Let $y(t)$be a function that is sufficiently smooth in the neighborhood of $t = t_0$. The differential transform of the function $y(t)$is defined as,

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(t)}{dt^k}\right]_{t=t_0}, k = 0,1,2, \dots$$

where $Y(k)$represents the transformed function and corresponds to the coefficients of the Taylor series expansion of $y(t)$about the point t_0.

The inverse differential transform reconstructs the original function $y(t)$from its transformed components and is given by,

$$y(t) = \sum_{k=0}^{\infty} Y(k)\,(t - t_0)^k$$

In practical computations, this infinite series is truncated to a finite number of terms to obtain an approximate solution [13].

Integro-Differential Equation:

$$\frac{d^n y(t)}{dt^n} = f\,(t, y(t)) + \int_a^b K(t,s)\,G\,(y(s))\,ds,$$

Where $y(t)$is the unknown function, $K(t,s)$ is a known kernel function, and $G$are given functions, a and $b$denote the limits of integration [14].

Volterra Integral Equation (VIE):

A $g(\propto)u(\propto) = f(x) + \lambda \int_a^{\propto} K(\propto \tau)u(\tau)d\tau$. Where a,b are constant $g(\propto), f(\propto)$ and $K(\propto \tau)$ are known while $u(\propto)$ are unknown function. Is a non-zero or complex parameter, is called VIE of third kind. The function $K(\propto \tau)$ known as the kernel of the IE [15].

VIE Types:

VIE of the first kind:

A linear IE of the form $g(\propto)$=0 in equation $f(x) + \lambda \int_a^{\propto} K(\propto \tau)u(\tau)d\tau = 0$

VIE of the second kind:

A linear IE of the form $g(\propto)$=1 in equation $u(\propto) = f(\propto)\lambda \int_a^{\propto} K(\propto \tau)u(\tau)d\tau$

Theorem [1] If $y(x) = e^{\lambda x}$ then $Y(k) = \frac{\lambda^k}{k!}$, where λ is constant [9]

Proof: Using the definition (3.1.1),

$$Y(k) = \frac{1}{k!}\left[\frac{d^k e^{\lambda x}}{dx^k}\right]_{x=0}, \quad Y(k) = \frac{\lambda^k}{k!}$$

Theorem [2]: If $y(x) = \int_{x_0}^{x} g(x)\,dx$ then $Y(k) = \left\{\begin{matrix} 0 & ,k=0 \\ \frac{G(k-l)}{k} & ,k \geq 1 \end{matrix}\right\}$

where $k \geq 0, Y(0) = 0$

If $y(x) = \int_0^x g_1(x)g_2(x)..g_n(x)dx$ then

$$U(k) = \frac{1}{k}\sum_{k_{n-1}=0}^{k-1}\sum_{k-2=0}^{k_n-1}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2-k_1)..G_n(k-k_{n-1}-1)$$

where $k \geq 1, U(0) = 0$ [13]

Theorem [3]: If $x(t), y(t)$ are two uncorrelated functions with time t and $X(k), Y(k)$ are transformed functions corresponding to $x(t), y(t)$ and symbol D denotes the differential transform process, If $X(k) = D[x(t)], Y(k) = D[y(t)]$ and c_1, c_2 are independent of t and k then $D[c_1x(t) + c_2y(t)] = c_1X(k) + c_2Y(k)$ [14].

Theorem [4]: Consider the perturbed Volterra integro-differential equation of the form

$$\frac{dy(t)}{dt} = f(t) + \int_0^t K(t,s)\,y(s)\,ds + \varepsilon\,g(t,y(t)), y(0) = y_0,$$

where $f(t)$, $K(t,s)$, and $g(t,y)$are continuous functions on the interval $t \in [0,T]$, and $\varepsilon$is a small perturbation parameter.

If $y(t)$is analytic in a neighborhood of $t = 0$, then the Differential Transform Method (DTM) yields a unique series solution of the form

$$y(t) = \sum_{n=0}^{\infty} Y(n)\ t^n,$$

where the transformed coefficients $Y(n)$satisfy the recurrence relation

$$(n+1)Y(n+1) = F(n) + \sum_{k=0}^{n} \sum_{m=0}^{k} K(n-k,m)\,Y(m) + \varepsilon\, G(n),$$

with

$$Y(0) = y_0,$$

and where $F(n)$, $K(n,m)$, and $G(n)$denote the differential transforms of $f(t)$, $K(t,s)$, and $g(t,y(t))$, respectively [14-15].

Differential Transform for fundamental function:

Original Function	**Transformed Function**
$z(x) = u(x) \pm v(x)$	$Z(k) = U(k) + V(k)$
$z(x) = \lambda u(x)$	$Z(k) = \lambda U(k)$
$z(x) = \frac{d^n g(x)}{dx^n}$	$Z(k) = \frac{(k+n)!}{\mathrm{k}!}\ G(k+n)$
$z(x) = u(x)v(x)$	$Z(k) = \sum_{l=0}^{k} U(l)V(k-l)$
$z(x) = \lambda x^m$	$Z(k) = \lambda\delta(k-m) = \begin{cases}1, if\ k = m\\0, if\ k \neq m\end{cases}$
$y(t) = e^{\lambda t}$	$\mathrm{Y}(k) = \frac{\lambda^k}{k!}$
$y(t) = c$(constant)	$Y(0) = c$, $Y(k) = 0$, $k \geq 1$
$y(t) = t^n$	$Y(k) = \delta_{k,n}$
$y(t) = t$	$Y(1) = 1$, $Y(k) = 0$, $k \neq 1$
$y(t) = e^{at}$	$Y(k) = \frac{a^k}{k!}$
$y(t) = \sin(at)$	$Y(k) = \frac{a^k}{k!} \sin\left(\frac{k\pi}{2}\right)$
$y(t) = \cos(at)$	$Y(k) = \frac{a^k}{k!} \cos\left(\frac{k\pi}{2}\right)$
$y'(t)$	$(k+1)Y(k+1)$
$\int_0^t y(\tau)\, d\tau$	$\frac{Y(k-1)}{k}$, $k \geq 1$

$y(t)z(t)$	$\sum_{m=0}^{k} Y(m)Z(k-m)$

Example1] Consider a nonlinear Volterra integro-differential equation of the form

$$y'(x)=e^{x}+\frac{1}{2}-\frac{1}{2}e^{2x}+\int_{0}^{x} y^{2}(t)\ dt \text{ with the condition } y(0)=1 \qquad (1.1)$$

Applying the differential transform (1), (2) we obtain

$$Y(k+1)=\frac{1}{(k+1)!}\left[\frac{1}{k!}+\frac{1}{2}\delta(k)-\frac{1}{2}\frac{2^{k}}{k!}+\frac{1}{k}\sum_{l=0}^{k-1} Y(l)\,Y(k-l-1)\right] \qquad (1.2)$$

with $Y(0)=1$

transformation of integrals are considered for $k\geq 1$ according to theorem (1),substituting $y(0)=1$, $k=0,$ in (1.2) to get, $Y(1)=1$

put $k=1,$ we get $Y(2)=\frac{1}{2!}$,put $k=2,$ we get $Y(3)=\frac{1}{3!}$

put $k=3,$ we get $Y(4)=\frac{1}{4!}$, put $k=4,$ we get $Y(5)=\frac{1}{5!}$ …

put $k=k-1,$ we get $Y(k)=\frac{1}{k!}$

Substituting all values of $Y(k)$ in the equation (1.2) we get,

$$y(x)=1+x+\frac{1}{2!}x^{2}+\frac{1}{3!}x^{3}+\frac{1}{4!}x^{4}+\frac{1}{5!}x^{5}+\ldots=e^{x}$$

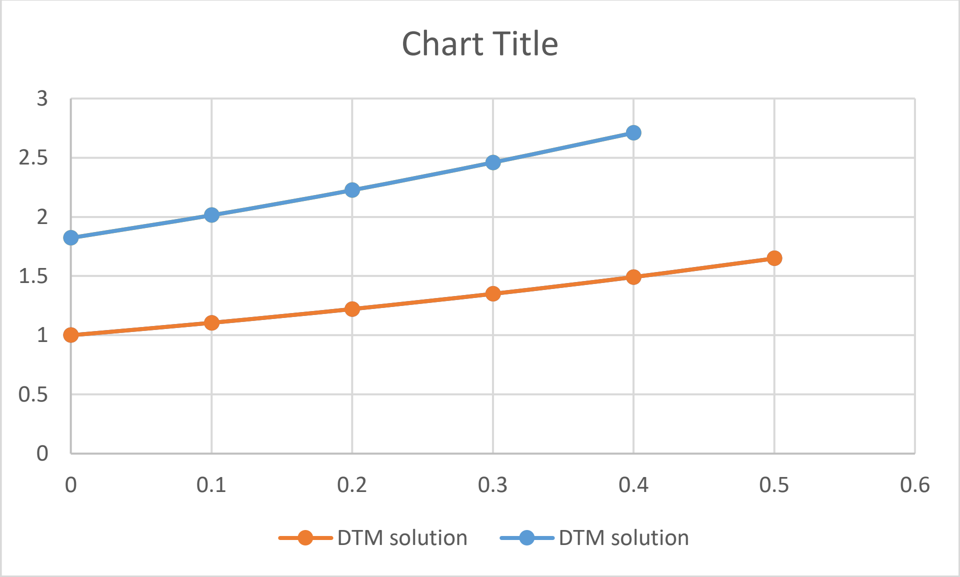
The exact solution for this example is $y=e^{x}$

we compare the exact solution and DTM solution,

x	**Exact solution**	**DTM solution**
0	1	1
0.1	1.105170918	1.105170918
0.2	1.221402758	1.221402758
0.3	1.349858808	1.349858808
0.4	1.491824698	1.491824698

0.5	1.648721271	1.648721271
0.6	1.822118800	1.822118800
0.7	2.0137527707	2.0137527707
0.8	2.225540926	2.225540926
0.9	2.459603103	2.459603103
1	2.71	2.71

Graphical representation of the exact solution and DTM solution



Example2] (Application to perturbed Volterra integral Equation)

Consider the generalized form of perturbed Volterra integral Equation

$$\varepsilon\, y(x) = g(x) + \int_0^x K\big(x, y, y(t)\big)\, dt, \qquad 0 \le t \le X \tag{2.1}$$

where ε is a small parameter satisfying $0 < \varepsilon < 1$ and g, K are smooth functions on $[0, X]$

Consider perturbed Volterra integral Equation

$$\varepsilon\, y(x) = \int_0^x (1 + x - t)\,\big(1 + t - y(t)\big)\, dt \tag{2.2}$$

with the initial condition $y(0) = 0$

by applying differential transform theorem on (4) we get,

$$Y(k)=\frac{1}{\varepsilon}\left[\begin{array}{l}\dfrac{\delta(k-1)-\delta(k-3)}{k}+\sum_{k_1=1}^{k}\delta\left(k-k_1-1\right)\delta\left(k_1-1\right)+\delta\left(k_1-2-Y(k_1-1)+\right.\\ \dfrac{1}{k}\sum_{k_1=1}^{k-1}Y(k_1)\left[\delta(k-k_1-2\right]-\delta\left(k-k_1-1\right)\end{array}\right]$$

$k\geq 1, Y(0)=0$

put $k=1,2,3,4,5,6,7,8,\ldots$ we get,

$$Y(1)=\frac{1}{\varepsilon},$$

$$Y(2)=-\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}$$

$$Y(3)=\frac{1}{6\varepsilon^3}-\frac{1}{2\varepsilon^2}+\frac{1}{6\varepsilon}$$

$$Y(4)=\frac{1}{24\varepsilon^6}+\frac{1}{6\varepsilon^3}-\frac{1}{8\varepsilon^2}$$

$$Y(5)=\frac{1}{120\varepsilon^5}-\frac{1}{24\varepsilon^4}+\frac{1}{20\varepsilon^3}-\frac{1}{120\varepsilon^2}$$

$$Y(6)=-\frac{1}{720\varepsilon^6}+\frac{1}{120\varepsilon^5}-\frac{1}{72\varepsilon^4}+\frac{1}{180\varepsilon^3}$$

$$Y(7)=\frac{1}{5040\varepsilon^7}-\frac{1}{720\varepsilon^6}+\frac{1}{336\varepsilon^5}-\frac{1}{504\varepsilon^4}+\frac{1}{5040\varepsilon^3}$$

$$Y(8)=-\frac{1}{40320\varepsilon^8}+\frac{1}{5040\varepsilon^7}-\frac{1}{1920\varepsilon^6}+\frac{1}{2016\varepsilon^5}-\frac{1}{8064\varepsilon^4}$$

...

Substituting all values of $Y(k)$ in the equation (3) we get,

$$y(x)=\frac{x}{\varepsilon}+\left(-\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}\right)x^2+\left(\frac{1}{6\varepsilon^3}-\frac{1}{2\varepsilon^2}+\frac{1}{6\varepsilon}\right)x^3+\left(-\frac{1}{24\varepsilon^6}+\frac{1}{6\varepsilon^3}-\frac{1}{8\varepsilon^2}\right)x^4+$$
$$\left(\frac{1}{120\varepsilon^5}-\frac{1}{24\varepsilon^4}+\frac{1}{20\varepsilon^3}-\frac{1}{120\varepsilon^2}\right)x^5+\left(-\frac{1}{720\varepsilon^6}+\frac{1}{120\varepsilon^5}-\frac{1}{72\varepsilon^4}+\frac{1}{180\varepsilon^3}\right)$$
$$\left(\frac{1}{5040\varepsilon^7}-\frac{1}{720\varepsilon^6}+\frac{1}{336\varepsilon^5}-\frac{1}{504\varepsilon^4}+\frac{1}{5040\varepsilon^3}\right)x^7+$$
$$\left(\frac{1}{5040\varepsilon^7}-\frac{1}{720\varepsilon^6}+\frac{1}{336\varepsilon^5}-\frac{1}{504\varepsilon^4}+\frac{1}{5040\varepsilon^3}\right)+\ldots$$

The exact solution of this problem is

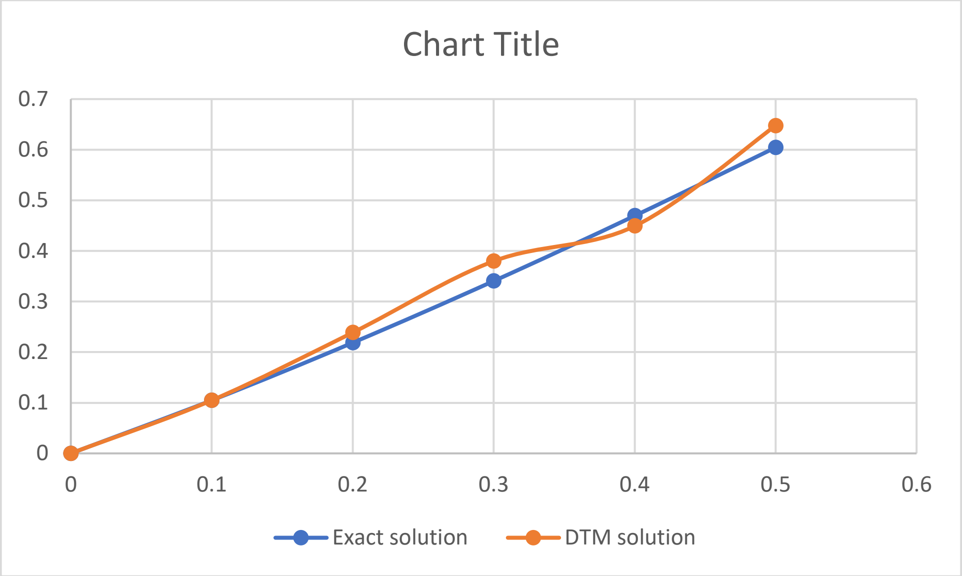
$$y(x)=x+1+\frac{1}{\lambda_1-\lambda_2}\left[\left(\lambda_2-1+\frac{1}{\varepsilon}\right)\exp(\lambda_1(x))-\left(\lambda_1-1+\frac{1}{\varepsilon}\right)\exp(\lambda_2(x))\right]$$

where $\lambda_1=\frac{1}{2\,\varepsilon}\left(-1+\sqrt{1-4\varepsilon}\right)$, $\lambda_2=\frac{1}{2\,\varepsilon}\left(-1-\sqrt{1-4\varepsilon}\right)$

In the following table, we compare the exact solution and DTM solution,

x	**Exact solution**	**DTM solution**
0	0	0
0.1	0.10483	0.10483
0.2	0.21867	0.23871
0.3	0.34052	0.37979
0.4	0.46941	0.44909
0.5	0.60441	0.64755
0.6	0.74458	0.79316
0.7	0.88907	0.92214
0.8	1.03704	1.09123
0.9	1.18769	1.21452
1	1.34038	1.52112

Graphical representation of the exact solution and DTM solution



Conclusion:

In this paper we discussed the comparative study for numerical solutions of Volterra integro-differential equations using the Differential Transform Method. Differential Transform Method

Provides the series solution. Which are very closer to analytical solution. With our study we conclude that Differential Transform Method is very efficient, reliable method with less computational work to get exact solution of Volterra integro-differential equations.

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