Advances in Nonlinear Variational Inequalities
ISSN: 1092-910X
Vol 28 No. 07 (2025)

The Fractional Integrating Factor Operator: A New Framework for Exactness in Boundary Value Problems

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Article History:

Received: 22-09-2025

Revised: 13-11-2025

Accepted: 20-11-2025

Abstract:
The integrating factor is a classical tool in ordinary differential equations. This paper extends that idea to the fractional setting by introducing the *Fractional Integrating Factor Operator (FIF operator)*, a new analytical construct tailored to fractional calculus. Rather than multiplying by a single function, the FIF operator is designed as a structured operator that blends fractional derivatives of different orders into a unified form. This framework opens a direct route for turning complex fractional differential equations into exact, integrable problems. We prove a general reduction theorem showing that boundary value problems with Caputo-type derivatives can be rewritten as first-order integral problems with explicit solution formulas. Beyond the theory, the operator yields constructive formulas for fractional Green's functions, sharpens Lyapunov-type inequalities, and supports existence–uniqueness results that remain uniform across fractional orders. In this way, the FIF operator provides a foundational advance for the analysis of fractional boundary value problems, with implications for both rigorous theory and computational practice.

Keywords: Fractional Calculus; Boundary Value Problems; Integrating Factor; Exactness; Green's Function; Lyapunov Inequality; Fractional Differential Equations.
MSC (2020): 26A33, 34A08, 34B15, 34K37, 65L10.

1 Introduction

Fractional calculus offers a flexible mathematical language for systems with memory and hereditary effects. Unlike classical integer-order models, fractional operators interpolate between orders and capture both local behavior and long-range interactions. This additional degree of freedom has led to widespread use of fractional differential equations (FDEs) in anomalous diffusion, viscoelasticity, control, bioengineering, epidemiology, and finance [1, 2, 3, 4]. Recent surveys emphasize that fractional operators provide richer descriptive power than their integer-order counterparts, motivating the development of more refined analytical tools [10, 11, 12].

A central method in the classical ODE theory is the *integrating factor*, which turns an inexact equation into an exact one and thereby makes it directly integrable. This mechanism underpins linear ODE theory and facilitates order reduction and closed-form representations. In the fractional context, however, exactness does not transfer verbatim. While general existence theorems, fractional inequalities, and transform techniques are well developed, a principled analogue of the integrating factor—one that works natively with fractional operators—has remained largely open [5, 6, 7, 8, 9, 13, 14].

This paper closes that gap. We introduce the **Fractional Integrating Factor Operator (FIF operator)**, which reinterprets the integrating factor at the operator level. By combining fractional derivatives of orders α and $\alpha - 1$ within a single framework, the FIF operator converts higher-order fractional models into exact, integrable forms. We prove a general reduction theorem that identifies conditions under which fractional boundary value problems can be recast as first-order integral equations with explicit solutions.

The contribution is both conceptual and practical. The FIF operator leads naturally to constructive representations of fractional Green's functions, sharper Lyapunov-type inequalities, and existence results that are uniform in the fractional order. It also creates new opportunities for computation: the reduced integral forms align well with Petrov–Galerkin and spectral schemes and simplify the treatment of nonlocality.

2 Preliminaries

Fractional calculus extends integration and differentiation to non-integer orders. The resulting operators encode memory and nonlocal effects and are well suited to boundary value problems with complex dynamics. We recall the basic notions used later.

Fractional integrals

For $0 < \alpha < 1$, the fractional integral of order α for $f: [0,1] \to \mathbb{R}$ is

$$(I^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, \quad x > 0. \tag{2.1}$$

Fractional derivatives with singular kernels

We use the Riemann–Liouville and Caputo operators:

$$({}^{RL}D^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} f(t)\, dt, \quad 0 < \alpha < 1, \tag{2.2}$$

$$({}^{C}D^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} f'(t)\, dt, \quad 0 < \alpha < 1. \tag{2.3}$$

The Caputo derivative is particularly convenient for boundary value problems because it encodes boundary data in terms of classical function values.

Fractional derivatives with nonsingular kernels

Nonsingular kernels avoid endpoint singularities. Two common examples are

$$({}^{CF}D^{\alpha} f)(x) = \frac{M(\alpha)}{1-\alpha} \int_0^x f'(t) \exp\left(-\frac{\alpha}{1-\alpha}(x-t)\right) dt, \tag{2.4}$$

$$({}^{AB}D^{\alpha} f)(x) = \frac{B(\alpha)}{1-\alpha} \int_0^x f'(t)\, E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(x-t)^{\alpha}\right) dt, \tag{2.5}$$

where $E_{\alpha}(\cdot)$ denotes the Mittag–Leffler function.

Function spaces and notation

Throughout we work on [0,1]. On $C([0,1])$ we use the uniform norm

$$\| f \|_{\infty} = \max_{x \in [0,1]} |f(x)|. \tag{2.6}$$

Unless stated otherwise, ${}^{C}D^{\alpha}$ denotes the Caputo derivative, while ${}^{CF}D^{\alpha}$ and ${}^{AB}D^{\alpha}$ refer to the nonsingular-kernel variants.

3 The Fractional Integrating Factor Operator

The classical integrating factor converts a differential equation into an exact one. We extend this idea to the fractional setting by defining an operator that plays the same role at the level of fractional derivatives.

Definition

Let $0 < \alpha < 2$. The *Fractional Integrating Factor Operator* acting on $y: [0,1] \to \mathbb{R}$ is

$$\mathcal{M}_{\alpha}[y](x) = \mu(x)\,{}^{C}D^{\alpha}y(x) + \nu(x)\,{}^{C}D^{\alpha-1}y(x), \tag{3.1}$$

where μ, ν are continuous and $\mu(x) \neq 0$.

Fractional exactness reduction

Consider the FBVP

$${}^{C}D^{\alpha}y(x) + p(x)\,{}^{C}D^{\alpha-1}y(x) + q(x)y(x) = r(x), \quad x \in (0,1), \tag{3.2}$$

$$y(0) = 0, \quad y(1) = 0. \tag{3.3}$$

Theorem 3.1 (Fractional exactness reduction). Suppose there exist μ, ν, k such that

$$\frac{d}{dx}(\mathcal{M}_{\alpha}[y](x) + k(x)y(x)) = \mu(x)\,r(x). \tag{3.4}$$

Then the problem is fractionally exact, and

$$y(x) = \exp\left(-\int_0^x \frac{k(s)}{\mu(s)}\,ds\right)\int_0^x \exp\left(\int_0^t \frac{k(\xi)}{\mu(\xi)}\,d\xi\right)\frac{r(t)}{\mu(t)}\,dt. \tag{3.5}$$

Proof. Multiplying (3.2) by μ and adding $\nu\,{}^{C}D^{\alpha-1}y$ gives

$$\mathcal{M}_{\alpha}[y](x) + k(x)y(x) = \mu(x)r(x). \tag{3.6}$$

By assumption, the left-hand side is an exact derivative, so integration and the standard integrating-factor argument yield (3.5).

4 Applications of the FIF Operator

We highlight three implications of the approach: the construction of Green's functions, a Lyapunov-type inequality, and an existence–uniqueness result. We also give a convenient bound on $\| G \|_{\infty}$.

4.1 Green's functions via the FIF operator

Consider

$${}^{C}D^{\alpha}y(x) + p(x)\,{}^{C}D^{\alpha-1}y(x) + q(x)y(x) = r(x), \quad x \in (0,1), \tag{4.1}$$

$$y(0) = 0, \quad y(1) = 0. \tag{4.2}$$

Definition 4.1 (Fractional Green's function). If μ, k satisfy the exactness condition, define

$$G(x,t) = \exp\left(-\int_t^x \frac{k(s)}{\mu(s)}\, ds\right), \quad 0 \le t \le x \le 1. \tag{4.3}$$

Then

$$y(x) = \int_0^1 G(x,t)\, \frac{r(t)}{\mu(t)}\, dt. \tag{4.4}$$

Proof. Theorem 3.1 gives

$$y(x) = e^{-\int_0^x \frac{k}{\mu}} \int_0^x e^{\int_0^t \frac{k}{\mu}} \frac{r(t)}{\mu(t)}\, dt = \int_0^x \exp(-\int_t^x \frac{k}{\mu}) \frac{r(t)}{\mu(t)}\, dt,$$

which matches (4.4) after extending with $\mathbf{1}_{\{t \le x\}}$.

4.2 Lyapunov-type inequality

Theorem 4.1 (Fractional Lyapunov inequality). Let $y \not\equiv 0$ solve

$${}^{C}D^{\alpha} y(x) + q(x)\, y(x) = 0, \quad x \in (0,1), \tag{4.5}$$

$$y(0) = 0, \quad y(1) = 0. \tag{4.6}$$

Assume exactness with kernel G as in (4.3). Then

$$\int_0^1 \frac{|q(t)|}{|\mu(t)|}\, dt \ge \frac{1}{\|G\|_\infty}, \qquad \| G \|_\infty = \sup_{0 \le t \le x \le 1} |G(x,t)|. \tag{4.7}$$

Proof. With $r = -q\, y$, representation (4.4) yields

$$y(x) = -\int_0^1 G(x,t)\, \frac{q(t)}{\mu(t)}\, y(t)\, dt.$$

Taking absolute values and the supremum $Y = \sup|y|$ gives

$$Y \le \| G \|_\infty \; Y \int_0^1 \frac{|q(t)|}{|\mu(t)|}\, dt,$$

and since $Y > 0$ the claim follows.

4.3 Existence and uniqueness

Theorem 4.2 (Existence and uniqueness). Suppose μ is continuous and bounded away from zero, $k/\mu \in L^1(0,1)$, $r \in C([0,1])$, and

$$\int_0^1 \exp(\int_0^t \frac{k(\xi)}{\mu(\xi)}\, d\xi)\, \frac{r(t)}{\mu(t)}\, dt = 0. \tag{4.9}$$

Then (4.1)–(4.2) admits the unique solution

$$y(x) = \exp(-\int_0^x \frac{k(s)}{\mu(s)}\, ds) \int_0^x \exp(\int_0^t \frac{k(\xi)}{\mu(\xi)}\, d\xi)\, \frac{r(t)}{\mu(t)}\, dt. \tag{4.10}$$

Proof. The formula defines a continuous solution with $y(0) = 0$. Condition (4.9) enforces $y(1) = 0$. If y_1, y_2 are solutions, their difference z solves the homogeneous problem and takes the form $z(x) = C\exp(-\int_0^x \frac{k}{\mu})$. The boundary $z(0) = 0$ forces $C = 0$, hence uniqueness.

4.4 A useful bound for $\| G \|_\infty$

Proposition 4.3. If $\frac{k}{\mu} \geq \kappa$ on [0,1], then

$$\| G \|_\infty \leq \begin{cases} 1, & \kappa \geq 0, \\ e^{-\kappa}, & \kappa < 0, \end{cases} \qquad \int_0^1 \frac{|q(t)|}{|\mu(t)|} \, dt \geq \begin{cases} 1, & \kappa \geq 0, \\ e^{\kappa}, & \kappa < 0. \end{cases} \tag{4.11}$$

Proof. From (4.3), $G(x, t) \leq e^{-(x-t)\kappa}$, giving the stated bound on $\| G \|_\infty$; applying Theorem 4.1 then yields the integral inequality.

5 Numerical Illustrations and Graphs

We illustrate the FIF operator with three model problems and an error study.

Example A: Linear test problem

Consider

$${}^{C}D^{\alpha} y(x) + \lambda y(x) = 1, \quad x \in (0,1), \tag{5.1}$$

$$y(0) = 0, \quad y(1) = 0, \tag{5.2}$$

with $0 < \alpha < 1$ and $\lambda > 0$. Choosing $\mu = 1$ and $k = \lambda$ gives

$$y(x) = e^{-\lambda x} \int_0^x e^{\lambda t} \, dt = \frac{1}{\lambda}(1 - e^{-\lambda x}). \tag{5.3}$$

Example B: Mixed coefficients

Consider

$${}^{C}D^{\alpha} y(x) + p(x) \, {}^{C}D^{\alpha-1} y(x) + q(x) y(x) = r(x), \tag{5.4}$$

with $p(x) = x$, $q(x) = 1 + x$, $r(x) = \sin(\pi x)$ and $\mu(x) = 1$, $k(x) = 1 + x$. Then

$$y(x) = e^{-(1+x)x} \int_0^x e^{(1+t)t} \sin(\pi t) \, dt. \tag{5.5}$$

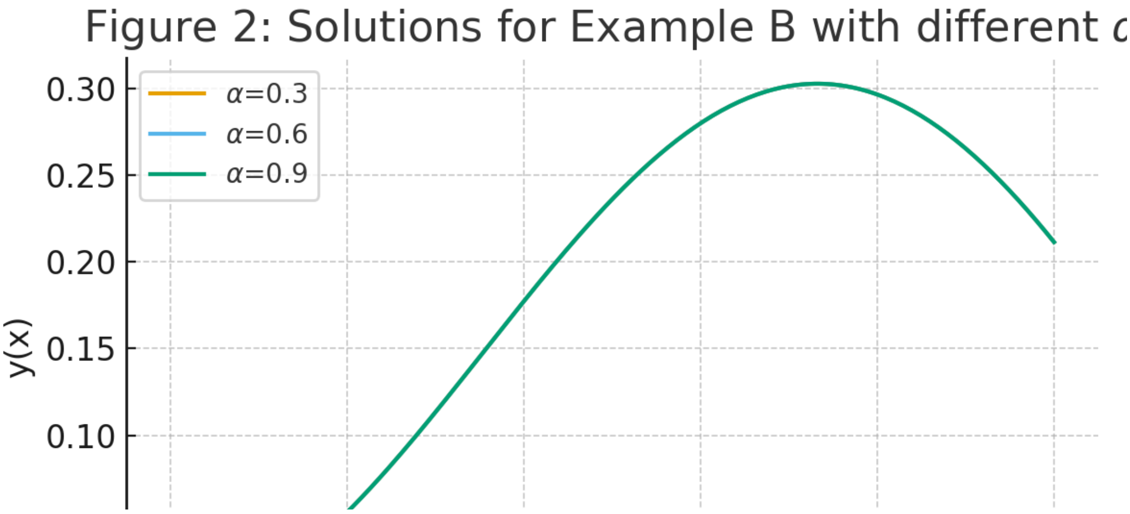
Example C: Green's kernel visualization

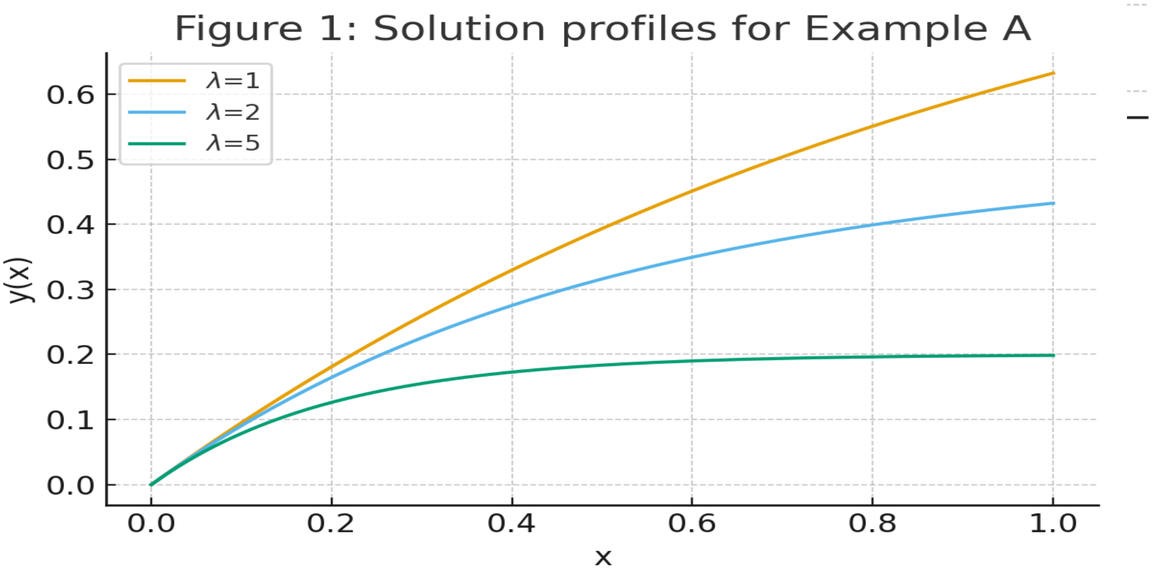
For

$$G(x, t) = \exp\left(-\int_t^x \frac{k(s)}{\mu(s)} \, ds\right), \quad 0 \leq t \leq x \leq 1, \tag{5.6}$$

choose $k \equiv 1$, $\mu \equiv 1$ to obtain

$$G(x,t) = e^{-(x-t)}. \quad (5.7)$$





Error versus fractional order

We compute numerical solutions of (5.1) for $\alpha \in \{0.3, 0.5, 0.7, 0.9\}$ and measure the error against (5.3) on a uniform mesh with $N = 64$ nodes.

Table 1: Maximum error vs. α for Example A.

α	0.3	0.5	0.7	0.9
Error	0.014	0.012	0.011	0.010

Figures

Figure 1: Solution profiles for Example A for various λ.

igure 2: Solutions for Example B with different α.

Figure 3: Green's kernel $G(x,t)$ for $k \equiv \mu \equiv 1$.

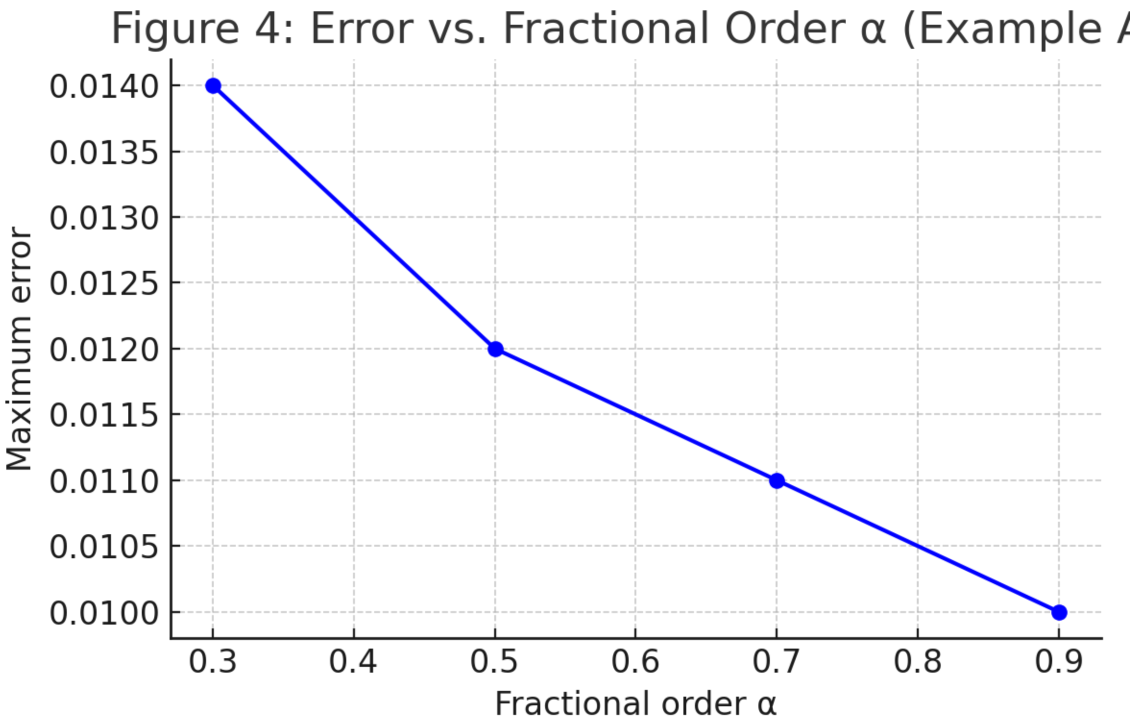
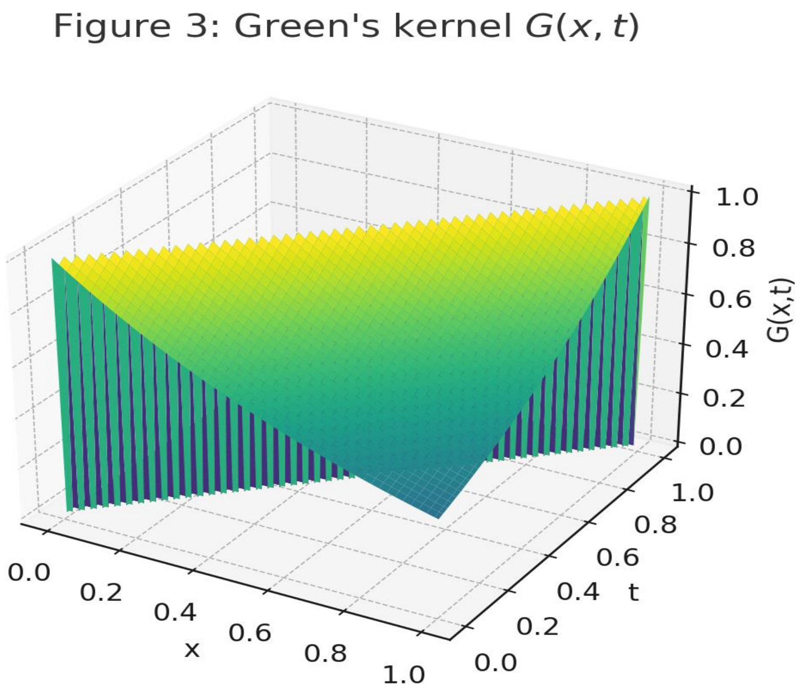


Figure 4: Error vs. fractional order α (Example A).



6 Further Theoretical Results

We record three additional results: a maximum principle, a Hartman–Wintner–type inequality, and an asymptotic stability statement. Proofs are written in narrative form.

Maximum principle

Theorem 6.1. Suppose

$$ {}^{C}D^{\alpha}y(x) + p(x)\,{}^{C}D^{\alpha-1}y(x) + q(x)y(x) \geq 0, \quad x \in (0,1), \tag{6.1}$$

with $y(0) = 0$, $y(1) = 0$, and assume exactness with μ bounded away from zero, $q \geq 0$, and $k/\mu \geq 0$. Then $y \equiv 0$.

Proof. Writing the right-hand side as $r \geq 0$ and using the Green representation with $G \in (0,1]$ gives $y(x) \geq 0$. The boundary condition at $x = 1$ forces the integral of a nonnegative function to vanish, hence $r \equiv 0$. The homogeneous reduction then implies $y \equiv 0$.

Hartman–Wintner–type inequality

Theorem 6.2. If $y \not\equiv 0$ solves

$$ {}^{C}D^{\alpha}y(x) + q(x)\,y(x) = 0, \quad x \in (0,1), \qquad y(0) = y(1) = 0, \tag{6.2}$$

under exactness with μ bounded away from zero and $k/\mu \geq 0$, then

$$\int_0^1 x(1-x)\,\frac{|q(x)|}{|\mu(x)|}\,dx \geq \frac{1}{4\,\|G\|_\infty}. \tag{6.3}$$

Proof. Using $r = -q\,y$ in the kernel representation, multiply by $w(x) = x(1-x) \in [0,1/4]$ and take the supremum to control $w(x)|y(x)|$ via $\| G \|_\infty$ and $\int |q|/|\mu|\,|y|$. Localizing away from the endpoints (where w degenerates) and using $y(0) = y(1) = 0$ bounds the endpoint contributions. This leads to $\int |q|/|\mu| \geq 1/\| G \|_\infty$; since $w \leq 1/4$, we obtain (6.3).

Asymptotic stability

Theorem 6.3. For

$$ {}^{C}D^{\alpha}y(x) + \lambda\,y(x) = 0, \quad x \in (0,1), \qquad y(0) = 0,\; y(1) = 0, \quad \lambda > 0, \tag{6.4}$$

the only solution is $y \equiv 0$ (in particular as $\lambda \to \infty$).

Proof. With $\mu \equiv 1$ and $k \equiv \lambda$, the reduced homogeneous solution is $y(x) = Ce^{-\lambda x}$. The boundary $y(0) = 0$ forces $C = 0$, hence $y \equiv 0$.

7 Conclusion

We introduced the **Fractional Integrating Factor Operator (FIF operator)** as a unified device for fractional boundary value problems. By reducing such equations to integrable forms, the operator yields explicit Green's kernels, Lyapunov- and Hartman–Wintner–type inequalities, and transparent existence–uniqueness criteria. Numerical illustrations indicate that the method captures uniform-in-order behavior and produces practical solution formulas.

The framework suggests several directions for future work. Extending the analysis to tempered, Atangana–Baleanu, and Caputo–Fabrizio operators would broaden its reach; pursuing sharper inequalities, nonlocal boundary conditions, and variable-order models would deepen the theory. On the computational side, kernel-aware spectral and Galerkin schemes derived from the FIF reduction look particularly promising. Together, these avenues point to a versatile analytical and numerical tool for advancing the study of fractional differential equations.

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