

# Existence of Random Solutions for Weakly Contractive Random Integral Equations in Separable Banach Spaces

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## Abstract:

**Introduction:** This paper focuses on studying random integral equations in separable Banach spaces. Such equations often appear in mathematical models that involve uncertainty, random effects, or stochastic processes. We describe simple and general conditions that guarantee the existence of at least one measurable random solution. The analysis is based on basic properties of random operators and fixed point ideas that help to connect randomness with deterministic techniques. The presented method is flexible and can be applied to several types of random integral equations that occur in science, engineering, and applied mathematics. In addition, the paper provides a verified example that illustrates how the theoretical result can be applied to a specific case. This study aims to offer a clear and straightforward approach for researchers working with random equations and related mathematical models.

**Keywords:** Random integral equation, Random fixed point, Banach space, Random operator, Existence of solution.

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## 1. Introduction

Random integral equations (RIEs) have become an essential tool in the analysis of stochastic systems, random dynamical models, and probabilistic differential equations. They appear naturally in models where uncertainty or randomness is involved, such as population dynamics, engineering control systems, economics, and environmental processes. The study of these equations provides a mathematical framework to establish the existence and behaviour of random solutions, which correspond to measurable mappings that satisfy the integral relation for almost every  $\omega \in \Omega$ . The concept of a random fixed point emerged from the pioneering work of Bharucha-Reid [1], who extended Banach's contraction principle to random operators in separable Banach spaces. He demonstrated that if a random operator satisfies a uniform contraction condition, then it admits a measurable random fixed point. This result laid the foundation for the development of random fixed point theory, which now serves as a crucial analytical method in random functional analysis.

Subsequently, Sehgal and Singh [2] generalized the contraction condition by introducing the notion of  $\alpha$  – condensing random operators. Their theorem established the existence of random fixed points for random operators that are continuous and condensing with respect to the Kuratowski measure of noncompactness. Itoh [3] further extended this idea to random compact operators by proving a random analogue of the classical Schauder fixed point theorem. A significant advance in this area was made by Xu and Xia [4], who proved a random fixed point theorem for weakly contractive random operators. Unlike the strict contraction case, their result allows the contractive coefficient to depend measurably on the random parameter  $\omega$ , thus offering more general and realistic applicability to stochastic models. Their approach unified and generalized previous works by Bharucha-Reid and Sehgal–Singh, bridging the gap between deterministic and random fixed point frameworks.

## 2. Statement of the Problem:

Consider the random integral equation of the form

$$x(\omega, t) = f(\omega, t) + \int_a^b K(\omega, t, s, x(\omega, s)) ds, \quad t \in [a, b], \omega \in \Omega, \quad (1)$$

where  $(\Omega, \Sigma, \mu)$  is a complete probability space, and  $(Y, \|\cdot\|)$  is a separable Banach space.

Here,

- ❖  $f : \Omega \times [a, b] \rightarrow Y$  is a measurable and bounded function;
- ❖  $K : \Omega \times [a, b]^2 \times Y \rightarrow Y$  is a measurable kernel, continuous in the last variable  $x$ ;
- ❖ For each  $\omega \in \Omega$ , the mapping  $x(\cdot) \mapsto K(\omega, t, s, x(\cdot))$  defines a weakly contractive operator.

We seek a measurable mapping  $x^*(\omega, \cdot) \in C([a, b]; Y)$  such that Eq.(1) holds for all  $t \in [a, b]$  and almost every  $\omega \in \Omega$ . The measurable function  $x^*$  satisfying Eq. (2) is called a random solution of the random integral equation. Such random integral equations naturally arise as mild solutions of random or stochastic differential equations [1, 4]. The existence of random solutions can be established using the theory of random fixed points.

## 3. Preliminaries:

Let  $(Y, \|\cdot\|)$  be a separable Banach space and  $(\Omega, \Sigma, \mu)$  a complete probability space. Denote by  $C([a, b]; Y)$  the Banach space of continuous functions from  $[a, b]$  to  $Y$  endowed with the supremum norm

$$\|x\| = \sup_{t \in [a, b]} \|x(t)\|.$$

**Definition 3.1.** A mapping  $T : \Omega \times Y \rightarrow Y$  is called a random operator if, for each fixed  $x \in Y$ , the map  $\omega \mapsto T(\omega, x)$  is measurable (see [6, 5]).

**Definition 3.2.** A random operator  $T$  is said to be a random weak contraction if there exists a measurable function  $k : \Omega \rightarrow [0, 1)$  such that for all  $x, y \in Y$  and every  $\omega \in \Omega$ ,

$$\|T(\omega, x) - T(\omega, y)\| \leq k(\omega) \|x - y\|. \quad (2)$$

**Remark 3.3.** If  $k(\omega) = k < 1$  is constant, then Eq. (2) reduces to the classical Banach contraction condition [1]. Thus, the random weak contraction property provides a measurable generalization suitable for random operators.

**Definition 3.4.** A measurable function  $x^* : \Omega \rightarrow Y$  is called a random fixed point of  $T$  if

$$T(\omega, x^*(\omega)) = x^*(\omega), \quad \text{for all } \omega \in \Omega.$$

The concept of measurable selection (see [6, 7]) ensures the existence of such mappings under appropriate conditions.

## 1. Operator Equation Formulation:

To analyse Eq. (1), we define a random operator  $T : \Omega \times C([a, b]; Y) \rightarrow C([a, b]; Y)$  by

$$(T(\omega, x))(t) = f(\omega, t) + \int_a^b K(\omega, t, s, x(s)) ds. \quad (3)$$

If  $x(\omega)$  is a measurable mapping such that  $T(\omega, x(\omega)) = x(\omega)$ , then  $x(\omega)$  is a random fixed point of  $T$ , and the corresponding  $x(\omega, t)$  is a random solution of Eq. (1). Therefore, establishing the existence of a random fixed point for  $T(\omega)$  is equivalent to proving the existence of a random solution for the integral equation. For further analysis, we employ the following result of Xu and Xia (2009), which generalizes earlier random contraction results [4, 2].

## 2. Base Theorem:

**Theorem 5.1** (Xu and Xia, 2009).} Let  $(Y, \|\cdot\|)$  be a separable Banach space and  $(\Omega, \Sigma, \mu)$  be a complete probability space. Let  $T : \Omega \times Y \rightarrow Y$  be a random operator satisfying:

- ❖ For each  $\omega \in \Omega$ ,  $T(\omega, \cdot)$  is continuous;
- ❖ There exists a measurable function  $k : \Omega \rightarrow [0, 1)$  such that

$$\|T(\omega, x) - T(\omega, y)\| \leq k(\omega) \|x - y\|, \quad \forall x, y \in Y;$$

- ❖ There exists a bounded closed convex subset  $B \subseteq Y$  such that  $T(\omega, B) \subseteq B$  for all  $\omega \in \Omega$

Then there exists a measurable mapping  $x^* : \Omega \rightarrow Y$  such that  $T(\omega, x^*(\omega)) = x^*(\omega)$ , for all  $\omega \in \Omega$ . That is,  $T$  admits a random fixed point.

**Remark 5.2.** The above theorem generalizes Banach's classical fixed point principle to random weak contractions, allowing  $k(\omega)$  to vary with  $\omega$ . This flexibility is particularly important for stochastic models where the degree of contraction depends on random parameters [4, 5].

Motivated by these developments, the present paper focuses on applying Xu and Xia's theorem to establish the existence of random solutions for a class of random integral equations in separable Banach spaces. By formulating the integral equation as a random operator equation, we show that the random weak contraction condition guarantees the existence of a measurable random fixed point, which corresponds to a random solution of the original equation. This approach not only provides a unified analytical tool but also offers an elegant generalization applicable to random Volterra and Hammerstein integral equations.

### 3. Main Theorem: Existence of Random Solution

In this section, we extend the random weak contraction theorem of Xu and Xia [4] to random integral equations of the form (1). This generalization covers a large class of random Volterra and Hammerstein-type integral equations and unifies several existing results [2, 3, 7].

**Theorem 6.1 (Main Result).** Let  $(Y, \|\cdot\|)$  be a separable Banach space and  $(\Omega, \Sigma, \mu)$  be a complete probability space. Suppose that

**H1)**  $K : \Omega \times [a, b]^2 \times Y \rightarrow Y$  is measurable in  $(\omega, t, s)$  and continuous in  $x$  for each fixed  $(t, s)$ ;

**H2)** There exists a measurable  $k : \Omega \rightarrow [0, 1)$  such that for all  $x, y \in Y$ ,

$$\|K(\omega, t, s, x) - K(\omega, t, s, y)\| \leq k(\omega) \|x - y\|, \quad \forall t, s \in [a, b];$$

**H3)**  $f : \Omega \times [a, b] \rightarrow Y$  is measurable and bounded, i.e.,

$\|f(\omega, t)\| \leq M$  for some constant  $M > 0$  and all  $(\omega, t)$ ;

**H4)**  $(b - a) \sup_{\omega \in \Omega} k(\omega) < 1$ .

Then the random integral equation (1) has a measurable random solution  $x^*(\omega, \cdot) \in C([a, b]; Y)$ .

**Proof:** Define the random operator  $T : \Omega \times C([a, b]; Y) \rightarrow C([a, b]; Y)$  by Eq():

$$(T(\omega, x))(t) = f(\omega, t) + \int_a^b K(\omega, t, s, x(s)) ds.$$

We verify all the hypotheses of Theorem 5.1 for this operator.

**Step 1: Measurability**

For each  $x \in C([a, b]; Y)$  and fixed  $t \in [a, b]$ , the mapping  $\omega \mapsto (T(\omega, x))(t)$  is measurable because both  $f$  and  $K$  are measurable in  $\omega$ . By the Pettis measurability theorem [6],  $T$  is a measurable random operator.

**Step 2: Continuity in  $x$** 

Let  $\{x_n\}$  be a sequence in  $C([a, b]; Y)$  with  $x_n \rightarrow x$ . By dominated convergence theorem and (H1), we have

$$\begin{aligned} & \| (T(\omega, x_n))(t) - (T(\omega, x))(t) \| \\ & \leq \int_a^b \| K(\omega, t, s, x_n(s)) - K(\omega, t, s, x(s)) \| ds \\ & \leq (b-a) \sup_{s \in [a, b]} \| K(\omega, t, s, x_n(s)) - K(\omega, t, s, x(s)) \| \rightarrow 0. \end{aligned}$$

Hence,  $T(\omega, \cdot)$  is continuous.

**Step 3: Random Weak Contraction Property**

For all  $x, y \in C([a, b]; Y)$ ,

$$\begin{aligned} & \| (T(\omega, x))(t) - (T(\omega, y))(t) \| \\ & = \left\| \int_a^b [K(\omega, t, s, x(s)) - K(\omega, t, s, y(s))] ds \right\| \\ & \leq \int_a^b \| [K(\omega, t, s, x(s)) - K(\omega, t, s, y(s))] \| ds \\ & \leq (b-a)k(\omega) \|x - y\| \end{aligned}$$

Thus,  $T$  is a random weak contraction with measurable contraction constant.  $k'(\omega) = (b-a)k(\omega) < 1$  due to (H4).

**Step 4: Existence of a Bounded Invariant Set**

Define  $B_R = \{x \in C([a, b]; Y) : \|x\| \leq R\}$ . We claim that  $T(\omega, B_R) \subseteq B_R$  for suitable  $R$ . For any  $x \in B_R$ ,

$$\begin{aligned} \| T(\omega, x) \| & = \sup_{t \in [a, b]} \| f(\omega, t) + \int_a^b K(\omega, t, s, x(s)) ds \| \\ & \leq M(b-a) + (b-a)k_{\max} \|x\| \\ & \leq M(b-a) + (b-a)k_{\max} R. \end{aligned}$$

Choose  $R = \frac{M(b-a)}{1-(b-a)k_{\max}}$ , which ensures  $\|T(\omega, x)\| \leq R$  for all  $x \in B_R$ . Hence,

$$T(\omega, B_R) \subseteq B_R.$$

### Step 5: Application of Theorem 5.1

All the assumptions of Xu–Xia’s Theorem (Theorem 5.1) are satisfied. Therefore, there exists a measurable function  $x^*(\omega) \in B_R$  such that

$$T(\omega, x^*(\omega)) = x^*(\omega).$$

Hence,  $x^*(\omega, t)$  is a measurable random solution of the integral equation (1).

**Remark 6.2:** The proof also shows that  $x^*(\omega)$  is unique since the random weak contraction condition guarantees uniqueness of fixed points (see [4, 8, 9]).

## 4. Illustrative Example

We now illustrate Theorem 6.1 with a concrete example. Let  $Y = \square$ ,  $\Omega = [0, 1]$  with the Lebesgue  $\sigma$ -algebra, and  $a = 0$ ,  $b = 1$ .

Define:

$$K(\omega, t, s, x) = \frac{1}{5} e^{-|t-s|} x + \frac{1}{10} \sin(\pi \omega s),$$

$$f(\omega, t) = \frac{1}{4} \sin(\pi \omega t).$$

### Step 1: Verification of hypotheses:

- (i)  $K$  and  $f$  are measurable in  $\omega$ , continuous in  $x$  and  $t$ .
- (ii) For all  $x, y \in \square$ ,

$$|K(\omega, t, s, x) - K(\omega, t, s, y)| = \frac{1}{5} e^{-|t-s|} |x - y| \leq \frac{1}{5} |x - y|.$$

Hence,  $k(\omega) = \frac{1}{5}$  and  $(b-a)k(\omega) = \frac{1}{5} < 1$ . Therefore, (H2) and (H4) hold.

- (iii)  $\|f(\omega, t)\| \leq \frac{1}{4}$ , so (H3) holds with  $M = \frac{1}{4}$ .

### Step 2: Random operator and contraction verification:

Define

$$(T(\omega, x))(t) = \frac{1}{4} \sin(\pi \omega t) + \int_0^1 \left[ \frac{1}{5} e^{-|t-s|} x(s) + \frac{1}{10} \sin(\pi \omega s) \right] ds.$$

Then

$$|(T(\omega, x))(t) - (T(\omega, y))(t)| \leq \frac{1}{5} \int_0^1 e^{-|t-s|} |x(s) - y(s)| ds \leq \frac{1}{5} \|x - y\|.$$

Thus,  $T$  is a random weak contraction with constant  $k'(\omega) = \frac{1}{5}$ .

### Step 3: Bounded invariant set:

Using  $M = \frac{1}{4}$ ,  $k_{\max} = \frac{1}{5}$ , and  $(b - a) = 1$ , we obtain

$$R = \frac{M(b-a)}{1 - (b-a)k_{\max}} = \frac{1/4}{1 - 1/5} = \frac{1/4}{4/5} = \frac{5}{16}.$$

Hence  $T(\omega, B_R) \subseteq B_R$ .

### Step 4: Existence of random solution:

By Theorem 4.1, there exists a measurable function  $x^*(\omega, t) \in B_R$  satisfying

$$x^*(\omega, t) = \frac{1}{4} \sin(\pi\omega t) + \int_0^1 \left[ \frac{1}{5} e^{-|t-s|} x^*(\omega, s) + \frac{1}{10} \sin(\pi\omega s) \right] ds.$$

Using the iteration  $x_{n+1}(\omega) = T(\omega, x_n(\omega))$  with  $x_0(\omega, t) = 0$ , we get uniform convergence  $x_n(\omega, t) \rightarrow x^*(\omega, t)$  for all  $\omega$ . Numerical computation shows convergence rate  $|x_{n+1} - x_n| \leq (1/5)^n$ , confirming contraction property.

### Step 5: Interpretation:

The obtained  $x^*(\omega, t)$  is a measurable random solution bounded by  $|x^*(\omega, t)| \leq \frac{5}{16}$ . It varies smoothly in  $t$  and continuously with respect to  $\omega$ , verifying the practical applicability of the theorem to random integral equations.

## 5. Conclusion

In this paper, we have developed a comprehensive random fixed point framework to study random integral equations in separable Banach spaces. By applying the Xu–Xia random weak contraction theorem, we proved the existence and uniqueness of random solutions under mild measurability and boundedness assumptions. This approach generalizes the classical results of Bharucha–Reid [1], Sehgal–Singh [2], Itoh [3], and Pathak–Singh [7]. The presented framework can be extended to nonlinear random Volterra, Hammerstein, and Fredholm-type integral equations. Future research may explore random coupled integral equations, random hybrid fixed point problems, and numerical approximation methods (see [9–11]).

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