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Banach Contraction Principle its Generalizations and Applications

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Abstract: The Banach Contraction Principle (BCP), a cornerstone of fixed-point theory, employs the method of successive approximations to determine fixed points of operator equations. These fixed points often represent solutions to complex mathematical problems, making the principle highly valuable in a wide range of scientific and technological disciplines. Over time, numerous extensions and modifications of the classical BCP have been developed to broaden its scope and adapt it to more complex systems. This paper aims to explore these generalizations and highlight their significance in various mathematical contexts. In particular, it focuses on their application within iterated function systems (IFS), where repeated function application results in the formation of self-similar patterns. These patterns, known as fractals, possess unique geometric structures and have applications in modeling natural phenomena and solving real-world problems. By analyzing these generalizations, the study underscores how BCP continues to evolve as a powerful mathematical tool, offering insights into both abstract theory and practical implementations. Through this discussion, the paper emphasizes the enduring relevance of fixed point methods and their expanding role in the study of dynamic systems and geometric constructions.

Key Words and Phrases: Fixed point, *T*-orbital complete space, iterative function system, fractal.

AMS Subject Classification: 47H10, 54H25

1. Introduction

The classical Banach contraction principle (BCP) is a fundamental tool in fixed-point theory, widely applied across mathematics, science, and engineering. It employs an iterative method, where a function is repeatedly applied to an initial value, generating a sequence that progressively approaches a stable solution. This sequence operates like a feedback loop—each output becomes the next input—leading to convergence at a single, unchanging value known as the fixed point.

What makes BCP particularly powerful is its ability to guarantee both the existence and uniqueness of this fixed point under specific conditions, especially in complete metric spaces. The iterative process ensures that, regardless of the starting point, the sequence will draw closer to the fixed point, provided the mapping satisfies a contraction condition—typically, that it brings points closer together by a constant factor less than one.

Beyond its mathematical elegance, BCP finds real-world applications in solving equations that arise in various fields, such as differential equations in physics, optimization problems in engineering, and models in biological systems. Its simplicity and reliability have made it a cornerstone of modern analysis and a foundational concept in nonlinear functional analysis and computational methods. (see for instance, Barnsley [2] and Zeidler [31]).

Let (X, d) be a metric space. The Bcp states that a self-map T of complete metric space X admits a unique fixed point if T is a Banach contraction, i.e. if T satisfies

$$d(Tx,Ty)\leq kd(x,y),\ x,y\in X, \quad (1.1)$$

where $0 \le k < 1$. One of the earliest generalizations of condition (1.1) is due to Edelstein [8] he proved that the map T satisfying the following condition has a unique fixed point provided that space X is compact.

$$d(Tx,Ty)<d(x,y), \quad x,y\in X \text{ with } x\neq y. \quad (1.2)$$

For some fundamental generalizations of condition (1.1) and their comparison, one may refer to Rhoades [25], Jachymski [11], Kumar et al. [16] and Kumar [17] (see also Kirk and Sims [15]).

Rakotch [24] gave a new direction to the study of fixed-point theory by replacing constant k in (1.1) by some real-valued function. He considered:

$$d(Tx,Ty)\le\varphi(d(x,y))d(x,y),\ x,\ y\in X, \quad (1.3)$$

where $\varphi:[0,+\infty)\to[0,+\infty)$ is a monotonically decreasing function.

Given a function $\varphi:[0,+\infty)\to[0,+\infty)$ such that $\varphi(t)<t$ for $t>0$, and a self-map T of X. Then we say that $T$is φ-contractive if

$$d(Tx,Ty)\le\varphi\, d(x,y),\ x,y\in X. \quad (1.4)$$

In general, when condition (1.4) is satisfied, the associated function is referred to as a *contractive gauge function* (see [10]). Several types of such functions have been explored to extend the foundational result originally proposed by Rakotch [24]. One notable advancement came from Browder [5], who formulated a fixed point theorem for mappings defined on complete and bounded metric spaces, where the contractive function is assumed to be nondecreasing and right-continuous.

Subsequently, Boyd and Wong [4] broadened Browder's result by removing the requirement of the metric space being bounded. Instead, they considered functions that are upper semi-continuous from the right, without requiring monotonicity. Under this framework, they established a fixed point theorem for so-called φ\varphiφ-contractive mappings (also discussed by Kirk and Sims [15]).

Further developments were made by Matkowski [19], who extended Browder's result in a different direction. He allowed the contractive function to be nondecreasing but not necessarily upper semi-continuous. Under this approach, the function must satisfy both condition (1.4) and an additional compatibility condition, which ensures the convergence required for fixed point existence. These generalizations have played a significant role in advancing fixed point theory in metric spaces.

$$\lim_{n\to\infty}\varphi^n(t)=0, \text{ for } t>0. \quad (1.5)$$

We remark that the classes of contractive guage functions studied by Boyd and Wong [4] and Matowski [19] are independent (see Jachymski [12, p. 2328 and p. 2334] and Jachymski [13, p.151].

Recently, Ri [27] replaced the condition of upper semi continuity of the guage function φ by the following condition and generalized the result of Boyd and Wong [4].

$$\lim_{s\to t^+}\sup\varphi(s)<t \text{ for all } t>0. \quad (1.6)$$

We remark that the condition (1.6) on φ implies the upper semi continuity of φ (see Suzuki [28]).

The following theorem is essentially due to Ri [27].

Theorem 1.1 [27]. Let T be a self-map of a complete metric space X satisfying the condition (1.4) where $\varphi:[0,+\infty)\to[0,+\infty)$ with $\varphi(t)<t$ satisfies condition (1.6) then T has a unique fixed point in X.

We remark that the guage function $\varphi:[0,+\infty)\to[0,+\infty)$ with $\varphi(t)<t$ satisfying condition (1.6) was perhaps first studied by Jotic [14]. He proved the following theorem.

Theorem 1.2 [14]. Let *T* be a self-map of complete metric space *X* satisfying condition

$$d(Tx,T^2x)\le \varphi\, d(x,Tx),\quad \text{for every } x\in X.$$

where $\varphi:[0,+\infty)\to[0,+\infty)$ with $\varphi(t)<t$ satisfies condition (1.6). Then $\{T^n(x)\}$ is a Cauchy sequence. Further, if a map G(x) = d (x, Tx) is lower semi continuous at a limit point of $\{T^n(x)\}$, say *x**, then *x** is a fixed point of *T*.

Babu and Kameswari [1] gave the following counterexample to the result of Jotic [14] and demonstrated that with the present hypothesis Theorem 1.2 is not valid.

Example 1.1. [1]. Let $X=\{\sum_{k=1}^{n}\frac{1}{k}:n=1,2,3,\ldots\}$ with the usual metric. Define self-map *T* on *X* by

$$T\left(\sum_{k=1}^{n}\frac{1}{k}\right)=\sum_{k=1}^{n+1}\frac{1}{k}.$$

Further define $\varphi:[0,+\infty)\to[0,+\infty)$ by $\varphi=t(1+t)^{-1}$ for $t>0$. Then *T* and φ satisfy all the conditions of Theorem 1.2 above and the sequence $\{x_n\}$ defined by $Tx_n=x_{n+1}$ with $x_0=1$ is not Cauchy in *X*, It may also be observed that *T* is asymptotic regular at x_0.

Inspired by the above Example 1.1 of Babu and Kameswari [1], in 2019 Bisht [3] utilized the same example to disprove the result proved by Ri [27]. Bisht [3] assumed that *T* and φ satisfy the condition (1.4) of Theorem 1.1 but sequence $\{x_n\}$ is not a Cauchy sequence for $x_0=1$.

Dung and Sintunavarat [7] recalculated the Example 1.1 and concluded that the observation of Bisht [3] for Theorem of Ri [27] (cf. Theorem (1.1) in light of Example 1.1 is incorrect. In particular if *x*=1 and $y=1+\frac{1}{2}+\frac{1}{3}$ in Example 1.1 then condition (1.4) of Theorem 1.1 is not satisfied. Showing the validity of Theorem 1.1.

Let $\Phi=\{\varphi:(0,\infty)\to(0,\infty):\varphi(t)<t \text{ and } \lim\sup_{t\to r^+}\varphi(t)<r \text{ for all } t>0\}$. We note the following definitions.

For $x_0\in X$, the set $O(x_0,T)=\{T^nx_0:n=0,1,2,\ldots\}$ is called the orbit of *T* at x_0.

Definition 1.1. [1], [3].A self-map *T* of *X* is said to be orbitally continuous at a point $z\in X$ iffor any sequence $\{x_n\}\subset O(x,T)$ for some $x\in X, x_n\to z$ implies $Tx_n\to Tz$ as $n\to\infty$.

Every continuous self-map in a metric space is orbitally continuous however the converse of this is not true.

Definition 1.2. [1], [3]. A space *X* is said to be *T*-orbitally complete iff every Cauchy sequence contained in $O(x,T)$ for some $x\in X$ converges in *X*.

We remark that every complete metric space is *T*-orbitally complete for any *T* but an *T*- orbitally complete metric space need not be a complete space.

Definition 1.3. [1]. A self-map *T* of metric space *X* is said to be asymptotically regular if

$$\lim_{n\to\infty} d(T^nx,T^{n+1}x)=0 \text{ for all } x,y\in X.$$

The following result is precisely due to Babu and Kameswari [1].

Theorem 1.3. [1]. Let (X, d) be T-orbitally complete and T be a self-map of X. Let there exists $\varphi_{x_0} \in \Phi$ for some $x_0 \in X$, such that

$$d(Tx,Ty)\le\varphi_{x_0}(d(x,y)), \quad \text{for every } x\neq y \text{ and } x,y\in \overline{0(x_0,T)}. \tag{1.7}$$

Then the sequence $\{x_n\}$ is Cauchy in X and $\lim_{n\to\infty} x_n = z, z\in X$. If T is orbitally continuous at z then z is a fixed point of T and z is unique in the sense that $\overline{0(x_0,T)}$ contains one and only one fixed point of T.

Pant [21] considered the following Suzuki type generalized φ-contractive condition to generalize the result obtained by Ri [27].

$$\frac{1}{2}d(x,Tx)\le d(x,y) \text{ implies } d(Tx,Ty)\le\varphi(\max\{d(x,y),d(x,Tx),d(y,Ty)\} \text{ for all } x,y\in X, \text{where } \varphi\in\Phi. \tag{1.8}$$

This paper presents new fixed point theorems involving Suzuki-type ϕ-contractions, which extend and generalize several well-known results in fixed point theory. Notably, the findings encompass and build upon the work of researchers such as Jotic, Mukharjee, Babu and Kameswari, Ri, Bisht, and Boyd and Wong. A key feature of this approach is the use of **T-orbital complete metric spaces** rather than traditional complete metric spaces, allowing for broader applicability. Importantly, the existence and uniqueness of a fixed point are established without imposing any continuity conditions on the mapping, which marks a significant relaxation of traditional assumptions. In the final section of the paper, attention is given to the concept of *Iterated Function Systems* (IFS) and their relationship with fractals. The study demonstrates how a fractal set can be generated as the unique fixed point of an IFS using Suzuki-type ϕ-contractions. This connection highlights the practical relevance of the theoretical results in modeling self-similar structures. Overall, the work offers a meaningful contribution to fixed point theory by expanding the class of contraction mappings and by applying these findings to the construction of fractals through IFS.

2. Fixed Point Theorems.

Throughout the paper, let N denotes the set of natural numbers,

$$M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}[d(x,Ty)+d(y,Tx)]\},$$

$$m(x,y)=\max\{d(x,y),ad(x,Tx)+(1-a)d(y,Ty),(1-a)d(x,Tx)+ad(y,Ty)\},0<a<1.$$

Theorem 2.1. Let (X, d) be a T- orbitally complete metric space and T be a self-map of X such that for all $x,y\in X$,

$$\frac{1}{2}d(x,Tx)\le d(x,y) \text{ implies } d(Tx,Ty)\le\varphi(M(x,y)), \tag{2.1}$$

where $\varphi\in\Phi$. Then T has a unique fixed point in X.

Proof. Choose an arbitrary element $x_0\in X$ and define a sequence $\{x_n\}$ by $x_n=T^n x_0$ for all $n\in N$. Let $a_n=(T^n x_0,T^{n+1}x_0)=(x_n, x_{n+1})$. We show that a_n is a convergent sequence. Notice that for any $n\in N$, $\frac{1}{2}d(x_n,x_{n+1})\le d(x_n,x_{n+1})$ using (2.1), we have

$$d(x_{n+1},x_{n+2}) =d(Tx_n,Tx_{n+1}) \le\varphi(M(x_n, x_{n+1}))$$

$$=\varphi(\max\{d(x_n, x_{n+1}),d(x_n, x_{n+1}),d(x_{n+1}, x_{n+2}),\frac{1}{2}[d(x_n,x_{n+2})+d(x_{n+1},x_{n+1})]\})$$

$$=\varphi(\max\{d(x_n, x_{n+1}),\frac{1}{2}[d(x_n,x_{n+2})+d(x_{n+1},x_{n+1})]\})=\varphi(d(x_n, x_{n+1})<d(x_n, x_{n+1}) \quad (2.2)$$

Hence $a_{n+1}<a_n$ and $\{a_n\}$ is strictly decreasing sequence of real numbers. We therefore assert that $\lim_{n\to\infty} a_n$ exists and tends to a limit $a\geq 0,$ i.e. $\lim_{n\to\infty} a_n = a > 0.$ If a> 0 then using (2.2) we have,

$$a_{n+1}\leq\varphi(a_n),$$

and

$$a\leq\lim\sup_{t\to a^+}\varphi(t)\leq\varphi(a)<a,$$

a contradiction. Therefore, for each $x\in X$,

$$\lim_{n\to\infty} a_n=\lim_{n\to\infty} d(T^n x_0,T^{n+1}x_0)=\lim_{n\to\infty} d(x_n,x_{n+1})=0. \quad (2.3)$$

This shows that map T is asymptotically regular at some point $x_0\in X$.

Now we show that $\{x_n\}\subset o(x_0,T)$ is a Cauchy sequence. If not, there exists an $\varepsilon>0$ and integers m_k, n_k with $m_k>n_k>k$ such that $d(x_{nk}, x_{mk})\geq\varepsilon$ and $d(x_{nk},x_{mk-1})<\varepsilon.$

Hence

$$\varepsilon\leq d(x_{nk},x_{mk})\leq d(x_{nk},x_{mk-1})+d(x_{mk-1},x_{mk})<\varepsilon+d(x_{mk-1},x_{mk}).$$

Now, $\lim_{k\to\infty} d(x_{mk-1},x_{mk})=0$ implies $\lim_{k\to\infty} d(x_{nk},x_{mk})=\varepsilon.$ (2.4)

Notice that $\lim_{n\to\infty} a_n=\lim_{n\to\infty} d(x_n,x_{n+1})=0.$ So there exists some integer k such that

$\frac{1}{2}d(x_{nk},x_{nk+1})\leq d(x_{nk},x_{mk})$ for $m_k>n_k>k.$ Using (2.1), we have

$$d(x_{nk+1},x_{mk+1})=d(Tx_{nk},Tx_{mk})\leq\varphi(M(x_{nk},x_{mk})).$$

Using triangle inequality

$$\varepsilon\leq d(x_{nk},x_{mk})\leq d(x_{nk},x_{nk+1})+d(x_{nk+1},x_{mk+1})+d(x_{mk+1},x_{mk})\leq a_{nk}+a_{mk}+\varphi(M(x_{nk},x_{mk}))$$

$$=a_{nk}+a_{mk}+\varphi(\max\{d(x_{nk},x_{mk}),d(x_{nk},x_{nk+1}),d(x_{mk},x_{mk+1}),\frac{1}{2}[d(x_{nk}, x_{mk+1})+d(x_{mk},x_{nk+1})]\}). \quad (2.5)$$

Consider the inequality

$$d(x_{nk},x_{mk})\leq d(x_{nk},x_{nk+1})+d(x_{nk+1},x_{mk}),$$

letting $k\to\infty$ and using (2.3) and (2.4), we have

$$\varepsilon\leq\lim_{k\to\infty} d(x_{mk},x_{nk+1}). \quad (2.6)$$

Now using again (2.3) and the inequality

$$d(x_{mk},x_{nk+1}) \le d(x_{nk+1},x_{nk})+d(x_{nk},x_{mk})$$

yields $\lim_{k\to\infty} d(x_{mk},x_{nk+1})\ge\varepsilon.$ (2.7)

From (2.6) and (2.7)

$$\lim_{k\to\infty} d(x_{mk},x_{nk+1})=\varepsilon. \quad (2.8)$$

Similarly, it can be shown that

$$\lim_{k\to\infty} d(x_{nk},x_{mk+1})=\varepsilon. \quad (2.9)$$

Hence using (2.3), (2.4), (2.8) and (2.9) in (2.5) and by the property of φ, we have

$$\varepsilon=\lim_{k\to\infty} d(x_{nk},x_{mk}) \le \lim\sup_{t\to\varepsilon^+} \varphi(t)<\varepsilon,$$

a contradiction. Hence $\{x_n\}\subset o(x_0,A)$ is a Cauchy sequence. Since X is T-orbitally complete, there exists a point $z\in X$ such that $x_n \to z$ as $n\to\infty$.

We claim that

$$\frac{1}{2}d(x_n,x_{n+1})\le d(x_n,z) \text{ or } \frac{1}{2}d(x_{n+1},x_{n+2})\le d(x_{n+1},z). \quad (2.10)$$

Otherwise, we have

$$d(x_n,z)<\frac{1}{2}d(x_n,x_{n+1}) \text{ and } d(x_{n+1},z)<\frac{1}{2}d(x_{n+1},x_{n+2}). \quad (2.11)$$

Using (2.11),

$$d(x_n,x_{n+1})\le d(x_n,z)+d(x_{n+1},z)<\frac{1}{2}d(x_n,x_{n+1})+\frac{1}{2}d(x_{n+1},x_{n+2})$$

$$<\frac{1}{2}d(x_n,x_{n+1})+\frac{1}{2}d(x_n,x_{n+1})=d(x_n,x_{n+1}),$$

a contradiction. Therefore (2.10) is true.

Considering $\frac{1}{2}d(x_n,x_{n+1})\le d(x_n,z)$ we get

$$\frac{1}{2}d(x_n,x_{n+1})=\frac{1}{2}d(x_n,Tx_n)\le d(x_n,z), \text{by (2.1) we have}$$

$$d(x_{n+1},Tz)=d(Tx_n,Tz)\le\varphi(M(x_n,z)).$$

Making $n\to\infty$ we obtain

$$d(z,Tz)=\lim_{n\to\infty} d(x_{n+1},Tz)\le\lim_{n\to\infty}\varphi(d(z,Tz))<d(z,Tz),$$

a contradiction. This yields $Tz=z$. Uniqueness of fixed point follows easily.

To appreciate the wide-ranging applicability of Theorem 2.1, it is useful to examine the comparative study by Rhoades [25], where numerous contractive conditions are systematically analyzed. In particular, condition (2.1) of Theorem 2.1 serves as a Suzuki-type modification of the condition labeled as (21′) in Rhoades [25, p. 267]. This modified form reflects a significant generalization, showcasing how Theorem 2.1 extends beyond classical contractive frameworks. By incorporating elements of Suzuki-type contractions, the theorem enables the exploration of fixed-point results in more generalized settings, thus accommodating mappings that may not satisfy the stricter conditions of traditional fixed-point theorems. This generality enhances its utility in various mathematical and applied contexts, particularly in problems involving iterative procedures and convergence analysis. The comparison with earlier contractive conditions not only emphasizes the strength and flexibility of Theorem 2.1 but also situates it within a broader spectrum of fixed-point results. As such, it serves as a valuable tool in the ongoing development of fixed-point theory, particularly in spaces where standard contractive requirements are either too restrictive or not directly applicable.

$$d(Tx,Ty)\le a\,M(x,y), 0\le a<1. \tag{C}$$

We remark that condition (C) is general than many well-known conditions in the ambit of metric fixed point theory.

The following theorem can immediately be obtained by replacing *T*-orbital complete metric space by complete metric space.

Theorem 2.2. Let *T* be a self-map of a complete metric space satisfying the condition (2.1). Then *T* has a unique fixed point in *X*.

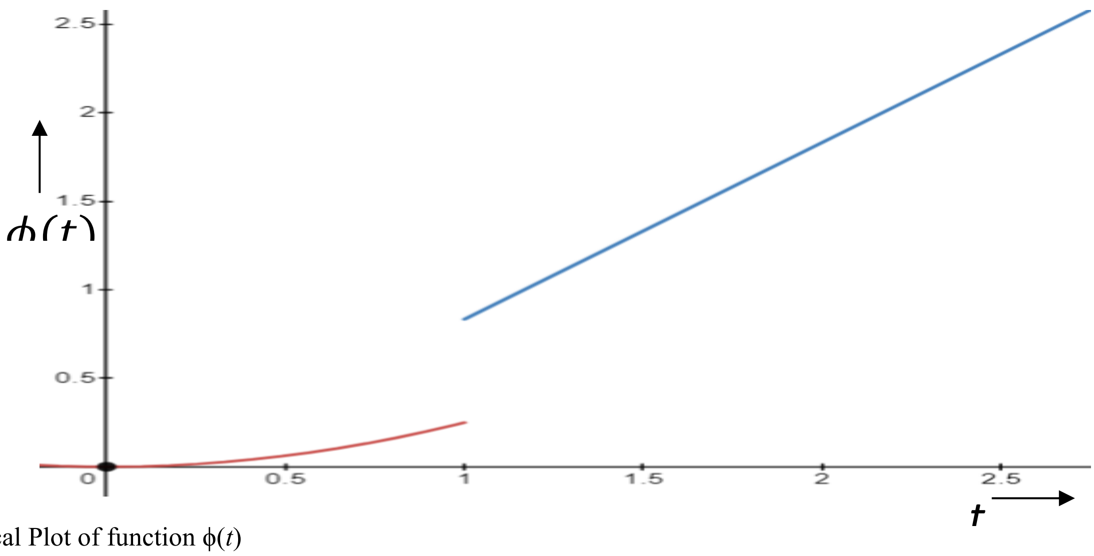
The following example vindicates the generality of above Theorem as compared to Theorem 2.1 [3], Theorem 2.1 [27] and the theorems proved in [1, 4, 14, 20].

Example 2.1. Let *X* = {(0, 0), (0, 4), (4, 0), (4, 5), (5, 4)} and the metric on *X* be defined by

$$d[(x_1,x_2),(y_1,y_2)]=|x_1-y_1|+|x_2-y_2|.$$

Further, suppose self-map *T* on *X* and function $\varphi:(0,\infty)\to(0,\infty)$ be such that

$$T(x_1,x_2)=\begin{cases}(x_1,0)\, if\ x_1\le x_2\\ (0,x_2)\, if\ x_1>x_2\end{cases} \quad \text{and} \quad \varphi(t)=\begin{cases}\frac{t^2}{4} & if\ t\le 1\\ t-\frac{1}{6} & if\ t>1.\end{cases}$$



Graphical Plot of function ϕ(*t*)

Then it can be verified that all the conditions of Theorem 2.2 are satisfied and $x = (0, 0)$ is the unique fixed point of T. Further, notice that map T does not satisfy the condition (i) of Theorem 2.1 of [3] for $(x, y) = ((4, 5), (5, 4))$ as well as for $(x, y) = ((5, 4), (4, 5))$.

Following Theorem is just another version of Theorem 2.1 which includes the results of Babu and Kameswari [1] and Bisht [3].

Theorem 2.3. Let (X, d) be a T- orbitally complete metric space and the self-map T of X satisfies the condition (2.1). Then the sequence $\{x_n\}$ defined by $x_n = T^n x_0$ is Cauchy in X and $\lim_{n\to\infty} x_n = z, z\in X$. If T is orbitally continuous at z then z is unique fixed point of T.

Following theorem generalizes the main result of Bisht [3], it can be proved in the same manner as Theorem 2.1 is proved.

Theorem 2.4. Let (X, d) be a T- orbitally complete metric space and self map T of X be such that for all $x, y\in X$,

$$\frac{1}{2}d(x,Tx)\le d(x,y) \text{ implies } d(Tx,Ty)\le\varphi(m(x,y)), \tag{2.12}$$

where $\varphi\in\Phi$. Then T has a unique fixed point in X.

From Theorem 2.1 we can derive some corollaries.

Corollary2.1. Let T be a self-map of a complete metric space satisfying the condition (2.1) where $\varphi:(0,\infty)\to(0,\infty):\varphi(t)<t \text{ for all } t>0$ is an increasing and right continuous function. Then T has a unique fixed point in X.

Corollary2.2. Let T be a self-map of a complete metric space satisfying the condition (2.1) where $\varphi:(0,\infty)\to(0,\infty):\varphi(t)<t \text{ for all } t>0$ is upper semi-continuous from the right on $(0,\infty)$. Then T has a unique fixed point in X.

3. Application to Iterated Function Systems.

The Banach Contraction Principle (BCP) finds a significant application in the theory of Iterated Function Systems (IFS) and the study of fractals. Fractals, a term introduced by Benoît Mandelbrot, describe complex geometric structures that exhibit self-similarity across different scales. In other words, when these structures are magnified, their smaller components appear similar or identical to the entire object, maintaining the same pattern regardless of the level of zoom.

The mathematical foundation for self-similar fractal sets was established by Hutchinson, who introduced the formal concept of IFS—a collection of contraction mappings on a complete metric space. This framework allows for the construction of a unique compact set, known as the attractor or fractal, which remains invariant under the action of these functions. Barnsley later expanded Hutchinson's work by applying it more broadly, illustrating how fractals can be generated using simple mathematical rules with repeated iteration.

By applying BCP within the context of IFS, it is possible to guarantee the existence and uniqueness of such attractors. This connection not only strengthens the theoretical understanding of fractals but also provides practical methods for generating them, with applications spanning computer graphics, natural modeling, and various scientific simulations.

Let (X, d) be a metric space and $H(X)$ be the class of all non-empty compact subsets of X. We recall the following distance functions.

$D(A,B)=\sup\{d(x,B):x\in A\}$ where $d(x,B)=\inf\{d(a,b):b\in B\}$.

The Hausdorff - Pompeiu metric h on $H(X)$ is defined as

$h(A,B)==\max\{D(A,B),D(B,A)\}$ for all $A,\ B\in H(X)$.

The fundamental result on which the basic theory of IFS is based stems from Banach contraction principle (Bcp). Indeed an IFS consists of a complete metric space (*X*, *d*) together with a finite set of contraction maps.

The basic result of Huchinson and Barnsley says that if all the self-maps $T_1, T_2, \ldots, T_n$

Of *X* are Banach contractions then the operator defined as

$$T(A) = T_1(A) \bigcup T_2(A) \bigcup \ldots T_n(A) = \bigcup_{i=1}^{n} T_i(A), \quad A \subseteq X,$$

is a Banach contraction on *H*(*X*) endowed with metric *H* Consequently, $\widehat{T}$ has a unique fixed point in *H*(*X*), which is called fractal in the sense of Barnsley.

The theory developed by Hutchinson and Barnsley has been generalized and extended by various researchers through the replacement of the classical Banach contraction principle (BCP) with other fixed-point theorems in metric spaces (see, for example, [6], [21–23], [26–27], [29–30]). These efforts have broadened the applicability of fractal construction methods by allowing for more generalized contractive conditions. In this context, we make use of Theorem 1.2 to construct a fractal set via an iterated function system (IFS). The application of this theorem enables the generation of self-similar structures beyond the scope of traditional contraction mappings. By doing so, it not only preserves the essence of the original approach introduced by Hutchinson and Barnsley but also offers a more flexible framework for analyzing fractal geometry within a generalized metric setting. This highlights the significance of alternative fixed-point results in advancing the theory and applications of iterated function systems.

We will use following lemmas to obtain the main Theorem of the section.

Lemma 3. 1. [2]. If (*X*, *d*) is a complete metric space then (*H*(*X*), *h*) is also a complete metric space.

Lemma 3.2. Let $T : X \to X$ be a continuous self-map of a metric space *X* satisfying the condition (2.1). Then for all $A, B \in H(X)$, the map $T : H(X) \to H(X)$ defined by $T(B) = \{T(x) : x \in B\}$ satisfies the following condition on $(H(X), h)$

$$\frac{1}{2} h(A, T(A)) \leq h(A, B) \text{ implies } h(T(A), T(B)) \leq \varphi(M_h(A, B)), \quad (3.1)$$

where $M_h(A, B)) = \max\{h(A, B), h(A, T(A), h(B, T(B), \frac{1}{2}[h(A, T(B) + h(B, T(A)]\}$.

Proof. Let $A, B \in H(X)$ and $x_0 \in A$. By compactness of *A* and *B*, there exists $y_0 \in B$ such that

$$d(x_0, y_0) = \inf_{y \in B} d(x_0, y).$$

These yields

$$\inf_{y \in B} \varphi(d(x_0, y)) \leq \varphi(d(x_0, y_0)) = \varphi(\inf_{y \in B} d(x_0, y)).$$

Using the definitions of Haus Dorff distance and supremum we have,

$$\inf_{y \in B}(d(x_0, y)) \leq \sup_{x \in A} \inf_{y \in B}(d(x, y)) \leq M_h(A, B).$$

By the monotone non-decreasing property of φ, it follows that

$$\varphi(\inf_{y \in B}(d(x_0, y))) \leq \varphi(\sup_{x \in A} \inf_{y \in B}(d(x, y))) \leq \varphi(M_h(A, B)).$$

x_0 being arbitrary in the above inequality, it is that

$\sup_{x\in A}\varphi(\inf_{y\in B}(d(x,y))\leq\varphi(M_h(A,B)).$

Therefore,

$\sup_{x\in A}\inf_{y\in B}\varphi(d(x,y))\leq\sup_{x\in A}\varphi(\inf_{y\in B}d(x,y))\leq\varphi(M_h(A,B)),$ and

$\sup_{y\in B}\inf_{x\in A}\varphi(d(x,y))\leq\sup_{y\in B}\varphi(\inf_{x\in A}d(x,y))\leq\varphi(M_h(A,B)).$

Now for all $x,\in A, y\in B,$

$\frac{1}{2}d(x,Tx)\leq d(x,y)$ implies $\frac{1}{2}D(A,T(A))\leq\frac{1}{2}h(A,T(A))\leq h(A,B),$

also

$\frac{1}{2}d(y,Ty)\leq d(x,y)$ implies $\frac{1}{2}D(B,T(B))\leq\frac{1}{2}h(B,T(B))\leq h(A,B).$

Further,

$D(T(A),T(B)=\sup_{Tx\in T(A)}\inf_{Ty\in T(B)}d(Tx,Ty)=\sup_{x\in A}\inf_{y\in B}d(Tx,Ty).$

As $T:X\to X$ satisfies (2.1), $\frac{1}{2}D(A,T(A))\leq\frac{1}{2}h(A,T(A))$ implies

$D(T(A),T(B))\leq\sup_{x\in A}\inf_{y\in B}\varphi(d(x,y))\leq\varphi(M_h(A,B)).$

Analogously, $\frac{1}{2}D(B,T(B))\leq\frac{1}{2}h(B,T(B)$ implies

$D(T(B),T(B))\leq\sup_{y\in B}\inf_{x\in A}\varphi(d(x,y))\leq\varphi(M_h(A,B)).$

Moreover, $h(T(A),T(B))=\max\{D(T(A),T(B),D(T(B),T(A))\}$ and symmetry of $h(A,B)$ leads us to conclude that for all $A,B\in H(X),$

$\frac{1}{2}h(A,T(A))\leq h(A,B)$ implies $h(T(A),T(B))\leq\varphi(M_h(A,B)).$ This completes the proof.

Lemma 3.3. Let (X, d) be a metric space and $T_n:H(X)\to H(X), n\in N,$ be continuous maps on $(H(X),h)$ satisfying the following condition for all $A,B\in H(X),$

$\frac{1}{2}h(A,T_n(A))\leq h(A,B)$ implies $h(T_n(A),T_n(B))\leq\varphi_n(M_{h,T_n}(A,B)),$ (3.2)

where $M_{h,T_n}(A,B)=\max\{h(A,B),h(A,T_n(A)),h(B,T_n(B)),\frac{1}{2}[h(A,T_n(B))+h(B,T_n(A))]\}.$

Define $\widehat{T}:H(X)\to H(X)$ by

$\widehat{T}(A)=T_1(A)\cup T_2(A)\cup\ldots\cup T_n(A)=\bigcup_{i=1}^{n}T_i(A),$ for each $A\in H(X).$ (3.3)

Then $\widehat{T}$ satisfies the following condition for all $A,B\in H(X),$

$\frac{1}{2}h(A,\widehat{T}(A))\le h(A,B)$ implies $h(\widehat{T}(A),\widehat{T}(B))\le \kappa(M_{h,\widehat{T}}(A,B))$, (3.4)

where $M_{h,\widehat{T}}(A,B)=\max\{h(A,B),h(A,\widehat{T}(A)),h(B,\widehat{T}(B)),\frac{1}{2}[h(A,\widehat{T}(B))+h(B,\widehat{T}(A))]\}$ and

$\kappa=\max(\varphi_n : n=1,2 3,\ldots,n)$.

Proof. We shall relay on the mathematical induction and property of metric h to prove the lemma. For $N=1$, the statement is evident. For $N=2$,

$$h(\widehat{T}(A),\widehat{T}(B)) = h(T_1(A)\bigcup T_2(A),T_1(B)\bigcup T_2(B)) \le \max\{ h(T_1(A),T_1(B)),T_2(A),T_2(B))\}.$$

As T_1 and T_2 satisfy condition (3.2). Therefore,

$\frac{1}{2}h(A,T_1(A))\le h(A, B)$ implies $h(T_1(A),T_1(B))\le \varphi_n(M_{h,T_1}(A,B))$ and

$\frac{1}{2}h(A,T_2(A))\le h(A, B)$ implies $h(T_2(A),T_2(B))\le \varphi_n(M_{h,T_2}(A,B))$.

Hence, we get

$$h(T(A),T(B))\le \max(\varphi_1(M_{h,T_1}(A,B),\varphi_2(M_{h,T_2}(A,B))$$
$$=\kappa(\max\{h(A,B),h(A,T_1(A)\bigcup T_2(A)),h(B,T_1(B)\bigcup T_2(B)),\frac{1}{2}[h(A,T_1(B)\bigcup T_2(B))+h(B,T_1(A)\bigcup T_2(A))]\}$$
$$=\max\{h(A,B),h(A,\widehat{T}(A)),h(B,\widehat{T}(B)),\frac{1}{2}[h(A,\widehat{T}(B))+h(B,\widehat{T}(A))]\}$$
$$=\kappa(M_{h,\widehat{T}}(A,B)),\text{ where } \kappa=\max\{\varphi_1,\varphi_2\}.$$

As a consequence of all the above results, we present the following theorem by means of which we construct a fractal set as a unique fixed point of IFS using condition (2.1). Our result generalizes the several results, among other, [21], [26-27].

Theorem 3.1. Let (X, d) be a complete metric space, $T_n : H(X)\to H(X), n\in N$, be continuous maps on $(H(X),h)$ and satisfy the condition (3.2) for all $A,B\in H(X)$. Further, let the map $\widehat{T}:H(X)\to H(X)$ defined by (3.3) satisfies the condition (3.4). Then the map $\widehat{T}$ has a unique fixed-point A in $H(X)$ which is also called an attractor or a fractal. Moreover $\lim_{n\to\infty}\widehat{T}^n(B)=A$ for all $B\in H(X)$.

Proof. Since (X, d) is a complete metric space therefore by lemma 3.1 $(H(X), h)$ is also a complete metric space. Further, by lemma 3.3 the map $\widehat{T}$ satisfies the condition (3.4). Therefore, by the application of theorem 2.1, $\widehat{T}$ has a unique fixed point also called an attractor or a fractal.

We finally pose the following questions.

Question 1. Can we obtain Theorem 2.1 by replacing the condition (2.1) by the following condition:

For all $x,y\in X$,

$\frac{1}{2}d(x,Tx)\le d(x,y)$ implies $d(Tx,Ty)\le \varphi\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}$, (3.5) where $\varphi \in \Phi$.

Precisely, we conjecture the following.

Theorem 3.2. Let (*X*, *d*) be a *T*- orbitally complete metric space and *T* be a self-map of *X* such that condition (3.5) is satisfied then *T* has a unique fixed point in *X*.

Question 2. Can we obtain Theorem 3.1 by replacing condition (3.2) by a more general condition.

Indeed, it means whether the following is valid.

Theorem3.3. Let (*X*, *d*) be a complete metric space, $T_n : H(X) \to H(X), n \in N,$ be continuous maps on $(H(X), h)$ and satisfy the following condition, for all $A, B \in H(X)$,

$$\frac{1}{2} h(A, T_n(A)) \leq h(A, B) \text{ implies}$$

$$h(T_n(A), T_n(B)) \leq \varphi_n \max\{ h(A,B), h(A,T_n(A), h(B,T_n(B), h(A,T_n(B), h(B,T_n(A))\}.$$

Further, let the map $\widehat{T} : H(X) \to H(X)$ defined by (3.3) satisfies the following condition

$$\frac{1}{2} h(A, \widehat{T}(A)) \leq h(A, B) \text{ implies}$$

$$h(\widehat{T}(A), \widehat{T}(B)) \leq \kappa \max\{ h(A,B), h(A,\widehat{T}(A)), h(B,\widehat{T}(B)), h(A,\widehat{T}(B)), h(B, \widehat{T}(A))\},$$

where $\kappa = \max\ (\varphi_n : n = 1, 2 3, \ldots, n)$.

Then the map $\widehat{T}$ has a unique fixed-point *A* in *H*(*X*). Moreover $\lim_{n\to\infty} \widehat{T}^n(B) = A$ for all $B \in H(X)$.

Conclusion:

In this paper, we have explored the classical Banach Contraction Principle (BCP), its various generalizations, and significant applications, particularly in the context of fixed-point theory and iterated function systems (IFS). The generalizations studied, including those by Edelstein, Rakotch, Boyd and Wong, Browder, Matkowski, Ri, and others, demonstrate the evolution of contractive conditions beyond simple linear constraints to more flexible gauge functions. Central to our discussion is the introduction and validation of Suzuki-type φ-contractions within T-orbitally complete metric spaces. The results presented in Theorems 2.1 through 2.4 not only encompass existing theorems in the literature but also expand their applicability by eliminating the need for continuity or completeness conditions in some cases. These extensions allow the identification of unique fixed points under broader settings, enhancing the utility of fixed-point theory in abstract and applied mathematics. Further, the application of these generalized contraction principles to IFS provides a novel approach to generating fractal sets, reaffirming the relevance of BCP-based results in mathematical modeling of self-similar structures. By employing Hausdorff–Pompeiu metrics and leveraging compactness, we establish the existence of unique attractors in fractal spaces through Theorem 3.1. The paper concludes by proposing conjectures for even broader generalizations of the current theorems. These open questions aim to stimulate further research into relaxing contractive constraints and expanding the scope of fixed-point results. Overall, this work bridges classical theory with modern extensions, contributing to both theoretical enrichment and practical applications in fields where recursive mappings and invariance play a critical role.

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