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Linear Evolution Equations: Between the Heat and Wave Equations

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Abstract: In this paper we study the Cauchy problem for linear evolutive equation. Using the Kernel operator. First, we give the representative formula of solutions for the considered Cauchy problem. Then we establish the space-time estimates for the corresponding equation.

Keywords: kerenal operator, fractional derivative, evolution equation, space time estimate.

1. Introduction

In this paper we consider a class of linear evolution equations which interpolates the heat equation and wave equation:
$D_{0+}^{\alpha}u + (-\Delta)^{\beta}u = f(t,x),\ 0 \le \alpha \le 2, 0 \le \beta \le 1,$
where $D_{0+}^{\alpha}u,\ 0 \le \alpha \le 2,$ is the Riemann-Liouville fractional derivative, $(-\Delta)^{\beta}u;\ 0 < \beta < 1$, is the fractional Laplacian.

The study of this problem is very important in view of recent developments in medical and health science see to this e_ect [17],[18] used fractional differential to describe human diseases and how to control them. [19] studied the issue of robust stability for fractional order Hopeld neural networks with parameter uncertainties using the fractional order Lyapunov direct method.

When $0 < \alpha \le 2$ and $\beta = 1$, F.Huang and F.Liu [4] solved the time fractional difusion equation and its solution has been obtained in terms of Green functions by Schneider and Whss [12].

The case $\alpha = 1, \beta = 1$, 1, corresponds to the heat equation and was investigated in [13],[6],[7],[8].The case α=2, β=1 corresponds to the wave equation and in [9] was established the local well-posedness and small-posedness of the Cauchy problem for semi-linear parabolic equations [11]. In the case $\alpha = 1, 0 \le \beta \le 1$ (i.e., in the case of nonlinear fractional power dissipative equation), many of mathematicians studied this equation. In [15] Zhichun Zhai obtained Strichartz estimates for the fractional heat equation by using both the abstract Strichartz estimates of Keel-Tao and the Hardy-Littlewood inequality. In [10] was made a detailed analysis of the kernel function of the fractional power operator semigroup $S_{\alpha}(t) = e^{t(-\Delta)^{\alpha}}$ and the authors established the space-time estimates for the corresponding fractional power dissipative equation. Bai and Gu studied the case of $\alpha \ge 1, \beta = 1$, using the integrodifferencial equation, in [1].

The aim of this paper is to generalize the fractional power equation in the paper [10]. Using the Kernel operator, we give the representative formula of the solution for the considered

Cauchy problem. Then we establish the space-time estimates for the corresponding fractional power evolutive equation.

The paper is organazid as follows. In Section 3, we give a detailed analysis of the kernel function of the fractional power operator $S_{\alpha,\beta}$. Then in section 4, we derive point-wise estimates of the kernel function of the operator $S_{\alpha,\beta}(t)$ by an invariant derivative technique. In Section 5, using the point-wise estimates of the kernel function obtained in Section 4, we establish space- time estimates for the corresponding fractional power equation.

2. Preliminaries

We denote by F and F^{-} the Fourier transform in $\mathbb{R}^n$ respectively, $\mathcal{L}$ and $\mathcal{L}^{-}$ the Laplace transform and Laplace inverse transform in $\mathbb{R}^+$, respectively, dened by

$F(f) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix.\xi} f(x)dx,$

$F^{-}(g) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix.\xi} g(\xi)d\xi,$

$\mathcal{L}(h) = \int_0^{+\infty} e^{-ts} h(s)ds,$

and

$$\mathcal{L}^{-}(k) = \int_0^{+\infty} e^{ts} k(t)dt.$$

Definition 1: Let $s \in (0,1)$ and $\Delta= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$. The fractional Laplacian of a function u: $\mathbb{R}^n \to \mathbb{R}$ is defined as:

$$(-\Delta)^s u(x) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(\xi)}{|x-\xi|^{n+2s}} d\xi,$$

where $P.V.$ stands for the Cauchy principal value, and the constant $C_{n,s}$ is given by:

$C_{n,s} = \left(\int_{\mathbb{R}^n} \frac{1-cosy}{|y|^{n+2s}} dy \right)^{-1} = \pi^{-\frac{n}{2}} 2^{2s} \frac{\Gamma\left(\frac{n+2s}{2}\right)}{\Gamma(1-s)} s$

and $y = (y_1, y_2, y_3, \ldots, y_n) \in \mathbb{R}^n$.

On the other hand, the fractional Laplacians in $\mathbb{R}^n$ can be written by the Fourier transform:

$$(-\Delta)^s u(x) = F^{-}\left(|y|^{2s} F(u)(\xi)\right)(x).$$

Proposition 1 Let $s \in (0,1)$ and let $(-\Delta)^s: S \to L^2(\mathbb{R}^n)$ is the fractional laplacian.

$E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ denote Mittag-Leffer's function and the generalized Mittag-Leffler's function, respectively, defined by:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \alpha > 0,$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \alpha > 0,$$

Assume that $u: (0, \infty) \to \mathbb{R}^n$. Let $0 < \alpha < 1$; we define the fractional integral of u of order α as:

$$I_t^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds \,,$$

and the Riemann-Liouville fractional derivative of u of order α as:

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha} u(s) ds \,,$$

where Γ is the Gamma function.

Let $\Phi_{\alpha}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$ and $\Phi_{\alpha}(t) = 0$ for $t \leq 0$; then

$$I_t^{\alpha} u(t) = (\Phi_{\alpha} * u)(t),$$

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial}{\partial t} (\Phi_{\alpha} * u)(t),$$

where the symbol $*$ stands for the convolution operation.

Definition 2 The Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ is defined from functions such that $u \in L^1$ and

$\Phi_{\alpha} * u \in W^{1,1}[0, t]$ as:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial}{\partial t} (\Phi_{\alpha} * u)(t),$$

where $W^{1,1}[0, t] = \{f \in L^1([0, t]) / Df \in L^1([0, t])\}$, denotes a Sobolev space.

Let $\alpha \notin \mathbb{N}$, we define function spaces $C^{\alpha}(I, X)$, which are the set of (set of "α-times continuously differentiable functions" on I as follows: We say that a continuous X-valued function u on an interval I is α-continuously differentiable at the point $t_0 \in I$, if there exist a continuous X-valued function ω on I such that ω is $[\alpha]$-times continuously differentiable at the point $t \in I$ and

$$u(t) = u(0) + \int_0^t \phi_{\alpha-[\alpha]} \omega(s) ds \,,$$

where $[\alpha]$ is the integer part of α. Then, the α-th derivative of u at the point t is defined by

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} u(t) = \omega^{([\alpha])}(t).$$

If u is α-times continuously differentiable for all $t \in I$, u is called α-times continuously differentiable on I.

Definition 3. Let T be an operator from L^p to L^p. We recall that T is to be sub additive if $|T(f+g)| \leq |Tf| + |Tg|$. This includes all linear operators. We say that T is weakly bounded from L^p to L^p if

$$|x: f(x) > t| \lesssim \frac{|f|_p^p}{t^p}.$$

On the other, T is said to be strongly bounded from L^p to L^p if

$$|Tf|_p \lesssim |f|_p.$$

Theorem 1. (Marcinkiewicz,Riesz). Let $1 \leq p < q < \infty$, let T be a subadditive operator on $L^p \cup L^q$. Then, if T is weakly bounded from L^q to L^q, it strongly bounded from L^r to L^r for all $p < r < q$.

3. **Solutions of the Cauchy problem of fractional evolution equation**
   Consider the two Cauchy problem for the homogeneous fractional power dissipative equation:

$$\begin{cases} D_{0+}^{\alpha} u + (-\Delta)^{\beta} u = f(t,x), \quad t > 0, x \in \mathbb{R}^n, 0 < \alpha < 1, \quad 0 < \beta < 1 \\ u(0) = \varphi(x), \ \ x \in \mathbb{R}^n. \end{cases} \tag{3.1}$$

and

$$\begin{cases} D_{0+}^{\alpha} u + (-\Delta)^{\beta} u = f(t,x), \quad t > 0, x \in \mathbb{R}^n, 1 < \alpha < 2, \quad 0 < \beta < 1 \\ u(0) = \varphi(x), u^{[\alpha-1]}(0,x) = \psi(x), \ \ x \in \mathbb{R}^n. \end{cases} \tag{3.2}$$

Where $D_{0+}^{\alpha} u$, $0 < \alpha < 2$, is the Riemann-Liouville fractional derivative, $(-\Delta)^{\beta} u$, $0 < \beta < 1$, is the fractional Laplacian with respect to the space variable $x, \varphi \in S(\mathbb{R}^n)$, $n \geq 1$. When $1 \leq \alpha < 2$, we inderstand $\varphi(x) = \psi(x)$.

It is easily shown that the equation above interpolates the heat equation and the wave equation. Here $S(\mathbb{R}^n)$ is the Schwartz space.

Theorem 2. Let $\varphi \in S(\mathbb{R}^n)$. Then the problem (3.1) admits a unique solution $u \in C^{\alpha}([0,\infty), S(\mathbb{R}^n)\ )$ *and*

$$u(t,x) = F^{-}\left(E_{\alpha}\left(-t^{\alpha}|\xi|^{2\beta}\right)\right) * \varphi(x) + \int_0^t (t-s)^{\alpha-1} F^{-}\left(E_{\alpha,\alpha}\left(-(t-s)^{\alpha}|\xi|^{2\beta}\right)\right) * f(x,s) ds,$$

where F is the Fourier transform in the direction of the space variable, F^{-} is its inverse and E_{α} is the Mittag-Leffler function.

Proof: We take the Laplace transform $\mathcal{L}$ of both sides of (3.1) with respect to $t \in [0,\infty)$. We get:

$$s^{\alpha}\mathcal{L}(u) - s^{\alpha-1}u(0) + (-\Delta)^{\beta}\mathcal{L}(u) = \mathcal{L}(f),$$

or

$$\mathcal{L}(u) = s^{-1}\varphi(x) - s^{-\alpha}(-\Delta)^{\beta}\mathcal{L}(u) + s^{-\alpha}\mathcal{L}(f).$$

Now we take the Fourier transform of both sides of the last equation. We find:

$$F\mathcal{L}(u) = s^{-1}F\varphi(x) - s^{-\alpha}|\xi|^{2\beta}F\mathcal{L}(u) + F\left(s^{-\alpha}\mathcal{L}(f)\right),$$

or

$$\left(1 + \frac{|\xi|^{2\beta}}{s^{\alpha}}\right)F\mathcal{L}(u) = s^{-1}F\varphi(x) + s^{-\alpha}F\left(\mathcal{L}(f)\right),$$

or

$$F\mathcal{L}(u) = \frac{1}{s}\frac{1}{1 + \frac{|\xi|^{2\beta}}{s^{\alpha}}}F\varphi(x) + \frac{s^{-\alpha}}{1 + \frac{|\xi|^{2\beta}}{s^{\alpha}}}F\left(\mathcal{L}(f)\right),$$

or

$$F\mathcal{L}(u) = \mathcal{L}\left(E_{\alpha}\left(-t^{\alpha}|\xi|^{2\beta}\right)\right)F\varphi(x) + \mathcal{L}\left(t^{\alpha-1}E_{\alpha,\alpha}\left(-t^{\alpha}|\xi|^{2\beta}\right)\right)F\left(\mathcal{L}(f)\right),$$

Hence,

$$F(u(t,x)) = E_{\alpha}\left(-t^{\alpha}|\xi|^{2\beta}\right)F\varphi(x) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-(t-s)^{\alpha}|\xi|^{2\beta}\right)F\left(f(x,s)\right)ds,$$

and

$$u(t,x) = F^{-}\left(E_{\alpha}\left(-t^{\alpha}|\xi|^{2\beta}\right)\right) * \varphi(x) + \int_0^t (t-s)^{\alpha-1}F^{-}\left(E_{\alpha,\alpha}\left(-(t-s)^{\alpha}|\xi|^{2\beta}\right)\right) * f(x,s)ds.$$

We need only to show that $F^{-}\left(E_{\alpha}\left(-t^{\alpha}|\xi|^{2\beta}\right)\right) * \varphi(x) \in C^{\alpha}([0,\infty), S(\mathbb{R}^n))$ for $0 < \alpha < 1$.

In fact, we see:

$$\int_0^t \phi_{\alpha}(t-s)(t-s)^{\alpha(k-1)}ds = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha(k-1)}ds$$

$$= \frac{t^{\alpha k}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}s^{\alpha(k-1)}ds$$

$$= \frac{t^{\alpha k}}{\Gamma(\alpha)}B(\alpha,\ \alpha(k-1)+1)$$

$$= \frac{\Gamma(\alpha(k-1)+1)}{\Gamma(\alpha k+1)}t^{\alpha k}$$

This means that

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(\frac{t^{\alpha k}}{\Gamma(\alpha k+1)}\right)=\frac{t^{\alpha(k-1)}}{\Gamma(\alpha(k-1)+1)}.$$

Hence, using differenciation under integral convergence theorem, we have:

$$\frac{\partial^\alpha}{\partial t^\alpha}E_\alpha\left(-t^\alpha|\xi|^{2\beta}\right)=\frac{\partial^\alpha}{\partial t^\alpha}\left(\sum_{k=0}^{\infty}\frac{\left(-t^\alpha|\xi|^{2\beta}\right)^k}{\Gamma(\alpha k+1)}\right)$$

$$=\frac{\partial^\alpha}{\partial t^\alpha}\left(\sum_{k=0}^{\infty}\frac{t^{\alpha k}\left(-|\xi|^{2\beta}\right)^k}{\Gamma(\alpha k+1)}\right)$$

$$=\sum_{k=0}^{\infty}\frac{t^{\alpha(k-1)}\left(-|\xi|^{2\beta}\right)^k}{\Gamma(\alpha(k-1)+1)}$$

$$=\sum_{k=0}^{\infty}\frac{t^{\alpha(k-1)}\left(-|\xi|^{2\beta}\right)^{k-1}\left(-|\xi|^{2\beta}\right)}{\Gamma(\alpha(k-1)+1)}$$

$$=\left(-|\xi|^{2\beta}\right)E_\alpha\left(-t^\alpha|\xi|^{2\beta}\right),$$

this implies :

$F^-\left(E_\alpha\left(-t^\alpha|\xi|^{2\beta}\right)\right)*\varphi(x)\in C^\alpha([0,\infty),S(\mathbb{R}^n)),$

for $\varphi(x)\in S(\mathbb{R}^n)$.This completes the proof.

The problem (3.2) is equivalent to the folowing problem:

$$\begin{cases}D_{0+}^{\upsilon+1}u+(-\Delta)^\beta u=f, \quad t>0, x\in\mathbb{R}^n, 0<\upsilon<1, \quad 0<\beta<1\\ u(0)=\varphi(x), u^{[\upsilon]}(0,x)=\psi(x),\ \ x\in\mathbb{R}^n.\end{cases}\tag{3.3}$$

Proposition 2. If $0<\upsilon<1$, the above problem is equivalent to the following problem:

$$\begin{cases}\dfrac{\partial u(t)}{\partial t}+\displaystyle\int_0^t\phi_\upsilon(t-s)(-\Delta)^\beta u(s)ds=\delta_{1,\upsilon}\psi(x)+\int_0^t\phi_\upsilon(t-s)f(s,x)ds\ ,\\ u(0,x)=\varphi(x), u^{[\upsilon]}(0,x)=\psi(x),\ \ x\in\mathbb{R}^n.\end{cases}\tag{3.4}$$

Theorem 3. Let $\varphi\in S(\mathbb{R}^n)$, $\psi\in S(\mathbb{R}^n)$ and $0<\upsilon<1$. Then, (3.3) admits a unique solution $u\in C^{1+\upsilon}([0,\infty);S(\mathbb{R}^n))$, $f\in L^1([0,\infty);S(\mathbb{R}^n))$ such that:

$$u(t,x)=F^-\left(E_{1+\upsilon}\left(-t^{\upsilon+1}|\xi|^{2\beta}\right)\right)*\varphi(x)+\delta_{\upsilon,\beta}\int_0^t F^-\left(E_{1+\upsilon}\left(-s^{1+\upsilon}|\xi|^{2\beta}\right)\right)*\psi(x)ds$$

$$+F^-\left(s^{1+\upsilon}E_{\alpha,\alpha}\left(-t^\alpha|\xi|^{2\beta}\right)\right)*f(t,x)$$

$$=S_{\alpha,\beta}\varphi(x)+(Gf)(t,x).$$

Proof. We take Laplace transform on both sides of (3.4) with respect to $t\in\mathbb{R}^+$. It is easy to get:

$s\mathcal{L}(u)-u(0,x)=\delta_{1,\upsilon}\psi(x)\delta(s)-s^{-\upsilon}(-\Delta)^{\beta}\mathcal{L}(u)+s^{-\upsilon}\mathcal{L}\big(f(t,x)\big)$, (3.6)

Now we take Fourier transform on both sides of (3.6) with respect to $x\in\mathbb{R}^n$; we have:

$$sF\big(\mathcal{L}(u)\big)-F\big(u(0,x)\big)=\delta_{1,\upsilon}F\big(\psi(x)\big)\delta(s)-s^{-\upsilon}\ \ |\xi|^{2\beta}F\big(\mathcal{L}(u)\big)+s^{-\upsilon}\mathcal{L}\big(f(t,x)\big),$$

that is,

$$F\mathcal{L}(u)=\frac{1}{s+s^{-\upsilon}|\xi|^{2\beta}}\left(\left(s^{-\upsilon}F\left(\mathcal{L}\big(f(t,x)\big)\right)\right)+F\big(\varphi(x)\big)\right)+\frac{1}{s+s^{-\upsilon}|\xi|^{2\beta}}\delta_{1,\upsilon}F\big(\psi(x)\big)\delta(s)$$

$$=\frac{s^{\upsilon}}{s^{1+\upsilon}+|\xi|^{2\beta}}\ F\big(\varphi(x)\big)+\frac{s^{-\upsilon}}{s+s^{-\upsilon}|\xi|^{2\beta}}\ F\left(\mathcal{L}\big(f(t,x)\big)\right)+\frac{1}{s+s^{-\upsilon}|\xi|^{2\beta}}\delta_{1,\upsilon}F\big(\psi(x)\big)\delta(s)$$

$$=\mathcal{L}\left(E_{1+\upsilon}\big(-t^{\upsilon+1}|\xi|^{2\beta}\big)\right)F\big(\varphi(x)\big)+\frac{1}{s+s^{-\upsilon}|\xi|^{2\beta}}\delta_{1,\upsilon}F\big(\psi(x)\big)\delta(s)+\mathcal{L}\left(s^{\upsilon}E_{1+\upsilon,1+\upsilon}\big(-s^{1+\upsilon}|\xi|^{2\beta}\big)\right)*F\left(\mathcal{L}\big(f(t,x)\big)\right).$$

Then:

$$u(t,x)=F^{-}\left(E_{1+\upsilon}\big(-t^{\upsilon+1}|\xi|^{2\beta}\big)\right)*\varphi(x)+\delta_{1,\upsilon}\int_0^t F^{-}\left(E_{1+\upsilon}\big(-s^{1+\upsilon}|\xi|^{2\beta}\big)\right)ds*\psi(x)+t^{\upsilon}E_{1+\upsilon,1+\upsilon}\big(-t^{1+\upsilon}|\xi|^{2\beta}\big)*f(t,x).$$

We need only to show that $F^{-}\left(E_{1+\upsilon}\big(-t^{\upsilon+1}|\xi|^{2\beta}\big)\right)*\varphi(x)\in C^{1+\upsilon}([0,\infty);S(\mathbb{R}^n))$ *for* $0<\upsilon<1$. *In fact, we see:*

$$\int_0^t\phi_{\upsilon+1}(t-s)(t-s)^{(1+\upsilon)(k-1)}ds=\frac{1}{\Gamma(1+\upsilon)}\int_0^t(t-s)^{\upsilon}s^{(1+\upsilon)(k-1)}ds$$

$$=\frac{t^{(1+\upsilon)k}}{\Gamma(1+\upsilon)}\int_0^1(t-s)^{\upsilon}s^{(1+\upsilon)(k-1)}ds,\text{ we take } s=\tau t$$

$$=\frac{t^{(1+\upsilon)k}}{\Gamma(1+\upsilon)}B(\upsilon+1,(1+\upsilon)(k-1)+1)$$

$$=\frac{\Gamma\big((1+\upsilon)(k-1)+1\big)}{\Gamma\big((1+\upsilon)k+1\big)}t^{(1+\upsilon)k}$$

This means that:

$$\frac{\partial^{\upsilon+1}}{\partial t^{\upsilon+1}}\left(\frac{t^{(1+\upsilon)k}}{\Gamma\big((1+\upsilon)k+1\big)}\right)=\frac{t^{(1+\upsilon)(k-1)}}{\Gamma\big((1+\upsilon)(k-1)+1\big)}.$$

Hence, using differenciation under integral theorem, we have:

$$\frac{\partial^{\upsilon+1}}{\partial t^{\upsilon+1}}E_{\upsilon+1}\left(-t^{\upsilon+1}|\xi|^{2\beta}\right)=\frac{\partial^{\upsilon+1}}{\partial t^{\upsilon+1}}\left(\sum_{k=0}^{\infty}\frac{\left(-t^{\upsilon+1}|\xi|^{2\beta}\right)^k}{\Gamma\left((\upsilon+1)k+1\right)}\right)$$

$$=\frac{\partial^{\upsilon+1}}{\partial t^{\upsilon+1}}\left(\sum_{k=0}^{\infty}\frac{t^{(\upsilon+1)k}\left(-|\xi|^{2\beta}\right)^k}{\Gamma\left((\upsilon+1)k+1\right)}\right)$$

$$=\sum_{k=0}^{\infty}\frac{t^{(1+\upsilon)(k-1)}\left(-|\xi|^{2\beta}\right)^k}{\Gamma\left((\upsilon+1)(k-1)+1\right)}$$

$$=\sum_{k=0}^{\infty}\frac{t^{(1+\upsilon)(k-1)}\left(-|\xi|^{2\beta}\right)^{k-1}\left(-|\xi|^{2\beta}\right)}{\Gamma\left((\upsilon+1)(k-1)+1\right)},$$

Then, we find:

$$\frac{\partial^{\upsilon+1}}{\partial t^{\upsilon+1}}E_{\upsilon+1}\left(-t^{\upsilon+1}|\xi|^{2\beta}\right)=\left(-|\xi|^{2\beta}\right)E_{\upsilon+1}\left(-t^{\upsilon+1}|\xi|^{2\beta}\right),$$

this implies $F^{-}\left(E_{1+\upsilon}\left(-t^{\upsilon+1}|\xi|^{2\beta}\right)\right)*\varphi(x)\in C^{1+\upsilon}([0,\infty);S(\mathbb{R}^n))$ *for* $\varphi\in S(\mathbb{R}^n)$. *Uniqueness is an immidiate result of Gronwall's inequality; see [16] for details.*

4. Analysis of the operator $S_{\alpha,\beta}$

Now, we will show that the kernel function of the operator

$S_{\alpha,\beta}(t,x)=F^{-}\left(E_\alpha\left(-t^\alpha|\xi|^{2\beta}\right)\right)*\varphi(x),\ 0<\alpha<1, 0<\beta<1$

generates a bounded operator on $L^p(\mathbb{R}^n)$ for $[1,\infty)$ (for the case $\mathbf{0<\alpha<2,0<\beta<1}$ *we can apply the same method*). Note that:

$$K_t(x)=F^{-}\left(E_\alpha\left(-t^\alpha|\xi|^{2\beta}\right)\right)=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{ix.\xi}\,E_\alpha\left(-t^\alpha|\xi|^{2\beta}\right)d\xi,$$

Let $\eta=t^{\frac{\alpha}{2\beta}}\xi$. Then $d\xi=t^{-\frac{\alpha n}{2\beta}}d\eta$ and

$$K_t(x)=\frac{1}{(2\pi)^{\frac{n}{2}}}t^{-\frac{\alpha n}{2\beta}}\int_{\mathbb{R}^n}e^{ix.\frac{\eta}{t^{\frac{\alpha}{2\beta}}}}E_\alpha\left(-|\eta|^{2\beta}\right)d\eta.$$

In this way we find that:

$$K_t(x)=t^{-\frac{\alpha n}{2\beta}}K\left(\frac{x}{t^{\frac{\alpha}{2\beta}}}\right).\qquad(4.1)$$

Therefore, it is enough to consider the kernel function :

$$K(x)=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{ix.\xi}\,E_\alpha\left(-|\xi|^{2\beta}\right)d\xi,$$

Whith $C_0(\mathbb{R}^n)$ we will denote the space of all function $f \in C(\mathbb{R}^n)$ satisfying $\lim_{|x|\to\infty} f(x) = 0$. By Lemma 2.2 and the Riemann-Lebesgue Lemma, we get the following corollary.

Corollary 1. We have

$$(-\Delta)^{\frac{v}{2}}K \in L^\infty(\mathbb{R}^n)\cap C_0(\mathbb{R}^n), \nabla K \in \left(L^\infty(\mathbb{R}^n)\cap C_0(\mathbb{R}^n)\right)^n, i\xi \boldsymbol{E_\alpha}\left(-|\boldsymbol{\xi}|^{2\boldsymbol{\beta}}\right) \in \left((\boldsymbol{L}^1(\mathbb{R}^n)^n)\right),$$

For any v and for any $\xi \in \mathbb{R}^n$.

Lemma 1. The kernel function $K(x)$ has the following point wise estimate:

$$|K(x)| \le C|x|^{-n-2\beta},$$

for $\beta > 0$. Hence,

$$K \in L^p(\mathbb{R}^n), K_t \in L^p(\mathbb{R}^n), 0 < t < \infty,$$

For any $1 \le p \le \infty$.

Proof Define the invariant differential operator: $L(x, D) = \frac{x.\nabla\xi}{i|x|^2}$. Then $L(x, D)\boldsymbol{e^{ix.\xi}} = \boldsymbol{e^{ix.\xi}}$. The conjugate operator is $L^*(x, D) = -\frac{x.\nabla\xi}{i|x|^2}$. Thus we can rewrite $K(x)$ as follows:

$$K(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix.\xi} L^*\left(E_\alpha\left(-|\xi|^{2\beta}\right)\right) d\xi$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix.\xi} \rho\left(\frac{\xi}{\delta}\right) L^*\left(E_\alpha\left(-|\xi|^{2\beta}\right)\right) d\xi$$

$$+ \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix.\xi} \rho\left(1 - \frac{\xi}{\delta}\right) L^*\left(E_\alpha\left(-|\xi|^{2\beta}\right)\right) d\xi$$

$$= I + II.$$

where $\delta > 0$ will be determined later and $\rho(x)$ is $C_0^\infty(\mathbb{R}^n)$ function satisfying:

$$\rho(x) = \begin{cases} 1 & if\ |\xi| \le 1, \\ 0 & if\ |\xi| > 2. \end{cases}$$

We have:

$$|I(x)| \le \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{|\xi|\le 2\delta} \left|L^*\left(E_\alpha\left(-|\xi|^{2\beta}\right)\right)\right| d\xi$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{|\xi|\le 2\delta} 2\beta|\xi||\xi|^{2\beta-2} \left|\nabla\xi\left(E_\alpha\left(-|\xi|^{2\beta}\right)\right)\right| d\xi$$

$$= \frac{C}{(2\pi)^{\frac{n}{2}}|x|} \delta^{2\beta+n-1}.$$

To estimate II, we take a sufficiently large natural number $N > 2\beta + n$ and integrating by parts we get :

$$|II(x)| = \frac{1}{(2\pi)^{\frac{n}{2}}}\left|\int_{\mathbb{R}^n} e^{ix.\xi}\ \rho\left(1-\frac{\xi}{\delta}\right)L^*\left(E_\alpha\left(-|\xi|^{2\beta}\right)\right)d\xi\right|$$

$$\leq \frac{1}{(2\pi)^{\frac{n}{2}}}\left|\int_{\mathbb{R}^n} e^{ix.\xi}\ (L^*)^{N-1}\left(1-\left(\frac{\xi}{\delta}\right)\right)L^*\left(E_\alpha\left(-|\xi|^{2\beta}\right)\right)d\xi\right|$$

$$\leq C|x|^{-N}\left|\int_{|\xi|\geq\delta}\sum_{l=1}^{N}|\xi|^{2l\beta-N}\ \left(E_\alpha\left(-|\xi|^{2\beta}\right)\right)d\xi\right|$$

$$+C|x|^{-N}\sum_{k=1}^{N-1}C_k\delta^{-k}\int_{\delta\leq|\xi|\leq2\delta}\sum_{l=1}^{N-k}C_l|\xi|^{2l\beta-(N-k)}\ E_\alpha\left(-|\xi|^{2\beta}\right)d\xi$$

Then:

$$\begin{aligned}|\boldsymbol{II}(\boldsymbol{x})| \ \leq\ & \boldsymbol{C}|\boldsymbol{x}|^{-\boldsymbol{N}}\left|\int_{|\xi|\geq\delta}|\boldsymbol{\xi}|^{2\boldsymbol{\beta}-\boldsymbol{N}}\boldsymbol{E}_{\boldsymbol{\alpha}}\left(-|\boldsymbol{\xi}|^{2\boldsymbol{\beta}}\right)\boldsymbol{d\xi}\right| \\ +\ & \boldsymbol{C}|\boldsymbol{x}|^{-\boldsymbol{N}}\left|\int_{|\xi|\geq\delta}|\boldsymbol{\xi}|^{2\boldsymbol{\beta}-\boldsymbol{N}}|\boldsymbol{\xi}|^{2\boldsymbol{\beta}(\boldsymbol{N}-1)}\boldsymbol{E}_{\boldsymbol{\alpha}}\left(-|\boldsymbol{\xi}|^{2\boldsymbol{\beta}}\right)\boldsymbol{d\xi}\right| \qquad (\mathbf{4.3}) \\ +\ & \boldsymbol{C}|\boldsymbol{x}|^{-\boldsymbol{N}}\sum_{k=1}^{N-1}\boldsymbol{C}_{\boldsymbol{k}}\boldsymbol{\delta}^{-\boldsymbol{k}}\int_{\delta\leq|\xi|\leq2\delta}|\boldsymbol{\xi}|^{2\boldsymbol{\beta}-\boldsymbol{N}}\boldsymbol{E}_{\boldsymbol{\alpha}}\left(-|\boldsymbol{\xi}|^{2\boldsymbol{\beta}}\right)+|\boldsymbol{\xi}|^{2\boldsymbol{\beta}(\boldsymbol{N}-\boldsymbol{k})-\boldsymbol{N}}\boldsymbol{E}_{\boldsymbol{\alpha}}\left(-|\boldsymbol{\xi}|^{2\boldsymbol{\beta}}\right)\boldsymbol{d\xi}\end{aligned}$$

Using Corollary 3.7 and 3.8, we have that:

$|\boldsymbol{\xi}|^{2\boldsymbol{\beta}(\boldsymbol{N}-1)}\boldsymbol{E}_{\boldsymbol{\alpha}}\left(-|\boldsymbol{\xi}|^{2\boldsymbol{\beta}}\right)\leq\boldsymbol{C},\ |\boldsymbol{\xi}|^{2\boldsymbol{\beta}(\boldsymbol{N}-\boldsymbol{k}-1)}\boldsymbol{E}_{\boldsymbol{\alpha}}\left(-|\boldsymbol{\xi}|^{2\boldsymbol{\beta}}\right)\leq\boldsymbol{C},$

for $k = 1,2,\ldots,N-1$. Therefore

$$|\boldsymbol{II}(\boldsymbol{x})| \leq \boldsymbol{C}|\boldsymbol{x}|^{-\boldsymbol{N}}\left(\int_{|\xi|\geq\delta}|\boldsymbol{\xi}|^{2\boldsymbol{\beta}-\boldsymbol{N}}\boldsymbol{d\xi}+\int_{\delta\leq|\xi|\leq2\delta}|\boldsymbol{\xi}|^{2\boldsymbol{\beta}-\boldsymbol{N}}\boldsymbol{d\xi}\right)\leq\boldsymbol{C}|\boldsymbol{x}|^{-\boldsymbol{N}}\boldsymbol{\delta}^{2\boldsymbol{\beta}+\boldsymbol{N}-\boldsymbol{n}}$$

Thus we get:

$$|K(x)| \leq C|x|^{-1}\delta^{2\beta+n-1} + C|x|^{-N}\delta^{2\beta+N-n}$$

Taking $\delta = |\boldsymbol{x}|^{-\mathbf{1}}$, we find:

$$|K(x)| \leq C|x|^{-1}|x|^{-2\beta+n-1} + C|x|^{-N}\delta^{2\beta+N-n},$$

Consequently

$|K(x)| \leq C|x|^{-n-2\beta}$.

This completes the proof.

Let $K^{v}(x) = (-\Delta)^{\frac{v}{2}}K(x),\ K_t^{v}(x) = (-\Delta)^{\frac{v}{2}}K_t^{v}(x),$

Lemma 2. The kernel function $K^{v}(x)$ has the following point wise estimate

$|K^{v}(x)| \leq C|x|^{-v-n}$ (4.4)

$x \in \mathbb{R}^{\boldsymbol{n}}$, for $\upsilon > 0$. Hence,

$$K^{\upsilon}(x) \in L^p(\mathbb{R}^{\boldsymbol{n}}), K_t^{\upsilon}(x) \in L^p(\mathbb{R}^{\boldsymbol{n}}), 0 < t < \infty,$$

For any $1 \leq p \leq \infty$.

Proof: We can split $K^{\upsilon}(x)$ in the following way:

$$|K^{\upsilon}(x)| = \frac{1}{(2\pi)^{\frac{n}{2}}}\left|\int_{\mathbb{R}^n} |\xi|^{\upsilon} e^{ix.\xi}\ \rho\left(\frac{\xi}{\delta}\right)\left(E_{\alpha}\left(-|\xi|^{2\beta}\right)\right) d\xi\right|$$

$$+\frac{1}{(2\pi)^{\frac{n}{2}}}\left|\int_{\mathbb{R}^n}|\xi|^{\upsilon} e^{ix.\xi}\ \left(1-\left(\frac{\xi}{\delta}\right)\right)\left(E_{\alpha}\left(-|\xi|^{2\beta}\right)\right) d\xi\right|$$

$$= I + II.$$

We have:

$$\begin{aligned} |I| &= \frac{1}{(2\pi)^{\frac{n}{2}}}\left|\int_{\mathbb{R}^n} |\xi|^{\upsilon} e^{ix.\xi}\ \rho\left(\frac{\xi}{\delta}\right) L^*\left(E_{\alpha}\left(-|\xi|^{2\beta}\right)\right) d\xi\right| \\ &\leq \int_{|\xi|\leq 2\delta} \delta^{\upsilon} d\xi \\ &\leq C\delta^{\upsilon+n}\ . \end{aligned}$$

To estimate II we use the technique of invariant derivatives together with integration by parts. In this way we get:

$$|II| \leq C \int_{\mathbb{R}^n} \left|e^{ix.\xi}(L^*)^N\left(1-\left(\frac{\xi}{\delta}\right)\right)|\xi|^{\upsilon}\left(E_{\alpha}\left(-|\xi|^{2\beta}\right)\right)\right| d\xi$$

As we have proved Lemma 2.4, we get :

$$|II| \leq C|x|^{-N}\left(\left(\int_{|\xi|\geq\delta} |\xi|^{\upsilon-N} d\xi + \int_{\delta\leq|\xi|\leq 2\delta} \delta^{\upsilon-N} d\xi\right)\right)$$

$$\leq C|x|^{-N}\delta^{\upsilon-N+n}.$$

Taking $\delta = |\boldsymbol{x}|^{-1}$, we obtain $|K^{\upsilon}(x)| \leq C|\boldsymbol{x}|^{-\upsilon-n}$. This completes the proof.

Remark 1. By Lemma 2.5 we obtain

$$|\nabla K(x)| \leq C(1+|x|)^{-n-1}.$$

Hence,

$$\nabla K(x), \nabla_t K(x) \in L^p(\mathbb{R}^{\boldsymbol{n}}), 0 < t < \infty,$$

5. Space-time estimates

In this section we consider the Cauchy problem for the fractional power evolutive equation:

$$\begin{cases} D_{0+}^{\alpha}u + (-\Delta)^{\beta}u = f(t,x), \quad t > 0, x \in \mathbb{R}^n, t > 0, \quad 0 < \alpha < 1, \quad 0 < \beta < 1 \\ u(0) = \varphi(x), \ \ x \in \mathbb{R}^n \end{cases} \tag{5.1}$$

The solution to the problem (5.1) can be written in the following integral form:

$$\begin{aligned} u(t,x) &= F^{-}\left(E_{\alpha}\left(-t^{\alpha}|\xi|^{2\beta}\right)\right) * \varphi(x) \\ &\quad + \int_0^t (t-s)^{\alpha-1} F^{-}\left(E_{\alpha,\alpha}\left(-(t-s)^{\alpha}|\xi|^{2\beta}\right)\right) * f(x,s)ds \qquad (5.2) \\ &= F^{-}\left(E_{\alpha}\left(-t^{\alpha}|\xi|^{2\beta}\right)\right) * \varphi(x) + (Gf)(t,x). \end{aligned}$$

We first consider the space-time estimates for the homogeneous part of the solution u given in the integral form (5.2).

Lemma 3. Let $1 \le r \le p \le \infty$ and $\varphi \in L^p(\mathbb{R}^n)$. Then the homogeneous part of the solution (5.2) satisfies the estimates:

$$\left\| S_{\alpha,\beta}(t,x)\varphi(x) \right\|_p \le C t^{-\frac{n\alpha}{2\beta}\left(\frac{1}{r}-\frac{1}{p}\right)} \|\varphi(x)\|_{L^r},$$

and

$$\left\| (-\Delta)^{\frac{v}{2}} S_{\alpha,\beta}(t,x)\varphi(x) \right\|_p \le C t^{\frac{v\alpha}{2\beta}-\frac{n\alpha}{2\beta}\left(\frac{1}{r}-\frac{1}{p}\right)} \|\varphi(x)\|_{L^r},$$

for $\alpha > 0$ and $v > 0$.

Proof:

The solution of the problem (5.1) can be written as:

$$u(t,x) = F^{-}\left(E_{\alpha}\left(-t^{\alpha}|\xi|^{2\beta}\right)\right) * \varphi(x) = K_t(x) * \varphi(x),$$

hence:

$$K_t(x) = F^{-}\left(E_{\alpha}\left(-t^{\alpha}|\xi|^{2\beta}\right)\right)(x) = t^{-\frac{n\alpha}{2\beta}} K\left(\frac{x}{t^{\frac{\alpha}{2\beta}}}\right),$$

then,

$$K(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix.\xi} E_{\alpha}\left(-|\xi|^{2\beta}\right) d\xi = F^{-}\left(E_{\alpha}\left(-|\xi|^{2\beta}\right)\right)(x).$$

Using the Young inequality for convolutions, we find:

$$\left\| S_{\alpha,\beta}\varphi \right\|_p \le \|K_t\|_q \|\varphi\|_{L^r},$$

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where $\frac{1}{q} = \frac{1}{p} - \frac{1}{r} + 1$. To estimate $\|K_t\|_q$, we will use the scaling property:

$$K_t(x) = t^{-\frac{n\alpha}{2\beta}} K\left(\frac{x}{t^{\frac{\alpha}{2\beta}}}\right)$$

and the inequality:

$$|K(x)| \le C(1 + |x|)^{-n-2\beta},$$

Then

$$\|K_t(x)\|_q^q = t^{-\frac{n\alpha q}{2\beta}} \left\| K\left(\frac{x}{t^{\frac{\alpha}{2\beta}}}\right)\right\|_q^q, C = (2\beta)^{-\frac{n\alpha q}{2\beta}},$$

where $\|K_t(x)\|_q^q = \int_{\mathbb{R}^n} |K_t(x)|^q dx$**.** Hence,

$$I_q = \int_{\mathbb{R}^n} \left| K\left(\frac{x}{t^{\frac{\alpha}{2\beta}}}\right)\right|^q dx \le C \int \left(1 + \frac{|x|}{t^{\frac{\alpha}{2\beta}}}\right)^{-q(n+2\beta)} dx.$$

Let $x = t^{\frac{\alpha}{2\beta}} y$. Then $dx = t^{\frac{\alpha n}{2\beta}} dy$ and

$$I_q \le C t^{\frac{\alpha n}{2\beta}} \int (1 + |y|)^{-q(n+2\beta)} dx$$

$$\le C t^{\frac{\alpha n}{2\beta}}.$$

It follows this estimates :

$$\|K_t(x)\|_q^q \le C_0 t^{-\frac{n\alpha q}{2\beta}} t^{\frac{\alpha n}{2\beta}},$$

$$\|K_t(x)\|_q \le C_1 t^{-\frac{n\alpha}{2\beta}} t^{\frac{\alpha n}{2\beta q}}$$

$$\le C_1 t^{-\frac{n\alpha}{2\beta}\left(1-\frac{1}{q}\right)},$$

$$\|K_t(x)\|_q \le C_1 t^{-\frac{n\alpha}{2\beta}\left(\frac{1}{r}-\frac{1}{p}\right)}.$$

Therefore

$$\left\|S_{\alpha,\beta}\varphi\right\|_p \le C t^{-\frac{n\alpha}{2\beta}\left(\frac{1}{r}-\frac{1}{p}\right)} \|\varphi(x)\|_{L^r}.$$

Now , we will prove the inequality

$$\left\|(-\Delta)^{\frac{v}{2}} S_{\alpha,\beta}(t,x)\varphi(x)\right\|_p \le C t^{\frac{v\alpha}{2\beta} - \frac{n\alpha}{2\beta}\left(\frac{1}{r}-\frac{1}{p}\right)} \|\varphi(x)\|_{L^r}.$$

By the Young inequality for convolutions, we have:

$$\left\|(-\Delta)^{\frac{v}{2}}S_{\alpha,\beta}(t,x)\varphi(x)\right\|_p \leq C\left\|(-\Delta)^{\frac{v}{2}}S_{\alpha,\beta}(t,x)\right\|_q \|\varphi(x)\|_{L^r}$$
$$= C\|K_t^v(x)\|_q\|\varphi(x)\|_{L^r},$$

and

$$\|K_t^v(x)\|_q^q = Ct^{-\frac{n\alpha q}{2\beta}}t^{v\frac{\alpha q}{2\beta}}\int_{\mathbb{R}^n}\left|K\left(\frac{x}{t^{\frac{\alpha}{2\beta}}}\right)\right|^q dx.$$

By Lemma 2.5, we get:

$$II_q \leq \int_{\mathbb{R}^n}\left|K\left(\frac{x}{t^{\frac{\alpha}{2\beta}}}\right)\right|^q dx$$
$$\leq \boldsymbol{C}\int_{\mathbb{R}^n}\left|\frac{x}{t^{\frac{\alpha}{2\beta}}}\right|^{-\boldsymbol{q}(\boldsymbol{v}+\boldsymbol{n})} dx.$$

Let $x = t^{\frac{\alpha}{2\beta}}y$. Then $dx = t^{\frac{\alpha n}{2\beta}}dy$ and

$$II_q \leq Ct^{\frac{\alpha n}{2\beta}}\int_{\mathbb{R}^n}|y|^{-\boldsymbol{q}(\boldsymbol{v}+\boldsymbol{n})}dx$$
$$\leq C't^{\frac{\alpha n}{2\beta}}.$$

Using the inequelities above, it follows that:

$$\|K_t^v(x)\|_q^q \leq C_0 t^{-\frac{v\alpha q}{2\beta}}t^{\frac{\alpha n}{2\beta}}t^{-\frac{n\alpha q}{2\beta}}$$

and

$$\|K_t^v(x)\|_q \leq C_0 t^{-\frac{v\alpha}{2\beta}}t^{\frac{\alpha n}{2\beta q}}t^{-\frac{n\alpha}{2\beta}}.$$

Therefore

$$\|K_t^v(x)\|_q \leq C_1 t^{-\frac{v\alpha}{2\beta}-\frac{n\alpha}{2\beta}\left(1-\frac{1}{q}\right)},$$

where upen

$$\|K_t^v(x)\|_q \leq C_1 t^{-\frac{v\alpha}{2\beta}-\frac{n\alpha}{2\beta}\left(\frac{1}{r}-\frac{1}{p}\right)}.$$

Therefore

$$\left\|(-\Delta)^{\frac{v}{2}}S_{\alpha,\beta}(t,x)\varphi(x)\right\|_p \leq Ct^{\frac{v\alpha}{2\beta}-\frac{n\alpha}{2\beta}\left(\frac{1}{r}-\frac{1}{p}\right)}\|\varphi(x)\|_{L^r}.$$

This completes the proof.

Definition 4. We call (q,p',p) the admissible triplet (for the Cauchy problem) with respect to α and β if:

$$\frac{1}{q}=\frac{n\alpha}{2\beta}\left(\frac{1}{p}-\frac{1}{p'}\right),$$

$$1<p\le p'<\begin{cases}\dfrac{n\alpha p}{n\alpha-2\beta}, & si\ n>2\beta\\ \infty & si\ n\le 2\beta.\end{cases}$$

Definition 5. Let B be a Banach space and let $I=[0,T)$. We define the time-weighted space time Banach space $C_\sigma(I;B)$ and the correponding homogenous space: $\dot{C}_\sigma(I;B)$ as follows:

$$C_\sigma(I;B)=\left\{f\in C(I;B)/\|f;C_\sigma(I;B)\|=\sup_{t\in I}t^{\frac{1}{\sigma}}\|f\|_B<\infty\right\},$$

$$\dot{C}_\sigma(I;B)=\left\{f\in C(I;B)/\|f;C_\sigma(I;B)\|=\sup_{t\in I}t^{\frac{1}{\sigma}}\|f\|_B<\infty\right\},$$

In this paper the Banach space B is taken to be $L^p(\mathbb{R}^n)$ with $1<p<\infty$.

Theorem 4.

i) Let $I=[0,\infty)$ or $I=[0,T]$ with $T>0$, (p,q,r) be any admissible triplet and $\varphi\in L^r$. Then $F^-\left(E_\alpha(-t^\alpha|\xi|^{2\beta})\right)*\varphi(x)\in L^q(I,L^r)\cap C_b(I;L^p)$ with the estimate:

$$\left\|F^-\left(E_\alpha(-t^\alpha|\xi|^{2\beta})\right)*\varphi(x)\right\|_{L^q(I,L^p)}\le C\|\varphi(x)\|_{L^r}\qquad(5.6)$$

where $C>0$ is a positive constant independent of φ and T.

ii) Let $I=[0,\infty)$ or $I=[0,T]$ with $T>0$, (p,q,r) be any generalized admissible triplet and $\varphi\in L^p$.

Then $F^-\left(E_\alpha(-t^\alpha|\xi|^{2\beta})\right)*\varphi(x)\in C_q(I;L^p)\cap C_b(I;L^r)$ and

$$\left\|F^-\left(E_\alpha(-t^\alpha|\xi|^{2\beta})\right)*\varphi(x)\right\|_{C_q(I;L^p)}\le C\|\varphi(x)\|_{L^r}\qquad(5.7)$$

Proof. The statement (i) follows easily from Lemma 5.1. So we only need to prove (5.6). For the case $p=r,q=\infty$, the estimate (5.6) is true from Lemma 5.1. We now consider the case $p\neq r$. Let:

$$U(t)\varphi=\left\|E_\alpha(-t^\alpha|\xi|^{2\beta})*\varphi(x)\right\|_p,$$

and regard U as a operator from the space of mesurable functions on $\mathbb{R}^n$ to the one for $t\in[0,\infty)$: $U:\varphi\mapsto U(.)\varphi$. *Obviously, U is a subadditive operator; that is,*

$$U(t)(\varphi_1+\varphi_2)=\left\|E_\alpha(-t^\alpha|\xi|^{2\beta})*(\varphi_1+\varphi_2)(x)\right\|_p$$

$$=\left\|E_\alpha(-t^\alpha|\xi|^{2\beta})*(\varphi_1)(x)+E_\alpha(-t^\alpha|\xi|^{2\beta})*(\varphi_2)(x)\right\|_p$$

$$\leq U(t)(\varphi_1) + U(t)(\varphi_2).$$

Then, and since (p,q,r) is an admissible triplet , we deduce by Young's inequality that:

$$U(t)\varphi \leq Ct^{-\frac{1}{q}}\|\varphi(x)\|_p.$$

It is easy to see that:

$$\mu\{t: |U(t)\varphi| > \tau\} \leq \mu\left\{t: Ct^{-\frac{1}{q}}\|\varphi(x)\|_r > \tau\right\}$$

$$= \mu\left\{t: t < \left(\frac{C\|\varphi(x)\|_r}{\tau}\right)^q\right\},$$

$$\leq \left(\frac{C\|\varphi(x)\|_r}{\tau}\right)^q,$$

where $m(A)$ is a Lebesgue measure of A in $[0,\infty)$, which implies that $U(t)$ is a weak type (r,q) operator.

On the other hand, by proposition 1 $U(t)$ is a sub-additive and satisfies that:

$$\|U(t)\varphi\|_\infty = \sup_t U(t)\varphi = \sup_t \left\|S_{\alpha,\beta}\varphi\right\|_r \leq \sup_t Ct^{-\frac{1}{q}}\|\varphi(x)\|_r \leq C\|\varphi(x)\|_{L^r},$$

for $r \leq p \leq \infty$, which means that $U(t)$ is also a weak type (p,∞) operator . Since for any (p,q,r) we can find another (p_1,q_1,r_1) such that: $q_1 < q < \infty, r_1 < r < p$ and $\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{\infty}$, $\frac{1}{p} = \frac{\theta}{r_1} + \frac{1-\theta}{p}$, then the Marcinkiewicz interpolation theorem implies that $U(t)$ is a strong (r,q)-type operator. The estimate (5.6).thus follows, and the proof of Theorem 5.4 is complete.

Proposition 3. Let (p,q,r) be any admissible triplet, $T > 0$ and $\lambda > 0$ satisfy:

$max(n\lambda, n\alpha\lambda) < 2\beta p$ and $n\lambda < 2\beta r$.

Then, for any $f \in L^{\frac{q}{\lambda+1}}\left(I; L^{\frac{p}{\lambda+1}}(\mathbb{R}^n)\right)$, $Gf \in L^q(0,T;L^p)$ and

$$\|Gf\|_{L^q(I;L^p)} \leq T^{\alpha\left(1-\frac{n\alpha\lambda}{2r\beta}\right)}\|f\|_{L^{\frac{q}{\lambda+1}}\left(I;L^{\frac{p}{\lambda+1}}(\mathbb{R}^n)\right)},$$

where C is a constant independent of f and T.

Proof. We can apply Lemma (5.1) and $r = \frac{p}{1+\lambda}$:

$$\|Gf\|_{L^q(I;L^p(\mathbb{R}^n))} = \left\|\int_0^t (t-s)^{\alpha-1}F^-\left(E_{\alpha,\alpha}\left(-(t-s)^\alpha|\xi|^{2\beta}\right)\right) * f(x,s)ds\right\|_{L^q(I;L^p(\mathbb{R}^n))}$$

$$= \left\|\left(\left\|\int_0^t (t-s)^{\alpha-1}F^-\left(E_{\alpha,\alpha}\left(-(t-s)^\alpha|\xi|^{2\beta}\right)\right) * f(x,s)ds\right\|_{L^P}\right)\right\|_{L^q(I)}$$

$$\leq \left\| \int_0^t (t-s)^{\alpha-1} C(t-s)^{\frac{-n\alpha}{2\beta}\left(\frac{1+\lambda}{p}-\frac{1}{p}\right)} \|f(x,t)\|_{L^{\frac{p}{\lambda+1}}} ds \right\|_{L^q(I)}$$

$$\leq \left\| C t^{\frac{-n\alpha}{2\beta}\left(\frac{1+\lambda}{p}-\frac{1}{p}\right)+\alpha} \|f(x,t)\|_{L^{\frac{p}{\lambda+1}}} \right\|_{L^q(I)}$$

$$= \left\| C t^{\frac{-n\alpha\lambda}{2\beta p}+\alpha} \|f(x,t)\|_{L^{\frac{p}{\lambda+1}}} \right\|_{L^q(I)}$$

$$\leq \left\| C T^{\frac{-n\alpha\lambda}{2\beta p}+\alpha} \|f(x,t)\|_{L^{\frac{p}{\lambda+1}}} \right\|_{L^q(I)}$$

$$\leq C T^{-\frac{n\alpha\lambda}{2\beta p}+\alpha} \|f(x,t)\|_{L^{\frac{q}{\lambda+1}}\left(0,T;L^{\frac{p}{\lambda+1}}\right)}$$

By the condition $n\lambda < 2\beta r$, we have:

$$\|Gf\|_{L^q(I;L^p(\mathbb{R}^n))} \leq T^{\alpha\left(1-\frac{n\alpha\lambda}{2r\beta}\right)} \|f\|_{L^{\frac{q}{\lambda+1}}\left(I;L^{\frac{p}{\lambda+1}}(\mathbb{R}^n)\right)}.$$

This implies the statement of the proposition.

Theorem 5. *Let $p \geq p_0 = \frac{n\alpha}{2\beta}\lambda$. Then, for any $\varphi \in L^p(\mathbb{R}^n)$ where $p \geq p_0$, there is a maximal existence time $T^* > 0$ and a function u satisfying (5.1), the following properties: (We can apply Lemma (5.1) and $r = \frac{p}{1+\lambda}$).*

a) *$u \in C\big((0,T^*); L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$. In particular, for any $p \geq r$ and $t \in (0,T^*)$, $u(t) \in L^p(\mathbb{R}^n)$.*
b) *This solution u is unique in the class*

$$\left\{ v \in C\big((0,T^*), L^p(\mathbb{R}^n)\big) : \sup_{t\in(0,T^*)} t^{\frac{1}{q}} \|v\|_p < \infty \right\},$$

for any $p \geq r$. Here, q is determined as (q,p,r) form an admissble triplet.

c) *For any $p \geq r$, u satisfies $t^{\frac{1}{q}}\|v(t)\|_p \to 0$ as $t \to 0$. Again, q is determined as (q,p,r) form an admissble triplet.*
d) *If $T^* < \infty$ for any $p \geq r$, $\lim_{t\to T^*} \|u(t)\|_p = \infty$, $r \leq p \leq \infty$.*

In addition, we have $\|u(t)\|_p \geq C(T^*-t)^{\alpha\left(\frac{n}{2\beta p}-\frac{1}{\lambda}\right)}$.

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