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A Revised Fuzzy Differential Equations Using Weakly Compatible Self-Mappings in Revised Fuzzy Metric Spaces

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Abstract: In this paper, we demonstrated certain coincidence point and Common Fixed Point [briefly, CFP] findings under the rational type weakly compatible revised fuzzy-contraction conditions in full Revised Fuzzy Metric spaces [briefly, RFMSs] utilizing the "triangular property of revised fuzzy metric" with examples in this work. We also provided Revised fuzzy differential equations [briefly, RFDEs] as an application and demonstrated that the solution of the RFDEs has a unique CFP of the integral operators B, C and $\mathscr{g}$. This new path of weakly-compatible Revised fuzzy-contraction with the use of RFDEs in RFM space will be crucial. With different sorts of weakly-compatible Revised fuzzy contraction conditions for self-mappings and different forms of differential equations in the context of RFMSs, this approach may be expanded and developed in diverse directions.

Keywords: T-conorm, Revised fuzzy, Triangular property, weakly compatible, fixed point.

1. Introduction

There has been a lot of work to get revised fuzzy analogues of classical theories since Alexander Sostak [10] proposed the notion of revised fuzzy sets. In the year 2020, Alexander Sostak and Tarkan oner [15] adapted the notion of revised fuzzy metric spaces to introduce extended t-conorm. Using the continuous t-conorm, Olga Grigorenko [14] et al changed the idea of "revised fuzzy metric spaces to Fuzzy (pseudo)" in. Muraliraj and Thangathamizh [11] used the revised fuzzy set technique to start a family of revised fuzzy mappings that are extensions of multivalued mappings and produced a result in revised fuzzy metric space for these mappings in 2021. Muraliraj and Thangathamizh [13] were the first to establish the idea of revised fuzzy contractive mappings and show a fixed point theorem in revised fuzzy metric spaces for these mappings.

Some of the recent literature on fuzzy fixed point results can be found [5-9, 16, 18-23, 24-25, and 26-34, 35].

The rational type revised fuzzy-contraction condition in RFM spaces was recently established by Muraliraj et al. [1], who also proved other FP theorems with an application. The concept of revised fuzzy cone metric (RFCM) space was first presented in 2023 by Thangathamizh et al. [2]. Under the presumption that "the revised fuzzy cone contractive sequences are Cauchy," they demonstrated a few fundamental features of FP as well as a "revised fuzzy cone Banach contraction theorem." Afterwards, several FP theorems in RFCMS were proven by Muraliraj and Thangathamizh [4] without requiring that the "Revised fuzzy cone contractive sequences are Cauchy." Jabeen et al. [26] used the contractive type weakly-compatible self-mappings with an application to prove several common FP findings in FCM spaces.

We established certain coincidence spots and CFP theorems under the weakly compatible rational type in this research. To validate the correctness of our approach, we revised fuzzy contraction requirements for three self mappings in RFMSs and provided examples.

2. Preliminaries

Definition 2.1 [34]

An peration $\oplus : [0, 1]^2 \rightarrow [0,1]$ is called a continuous *t*-conorm, if

(i) $\oplus$ continuous

(ii) $\oplus$ is commutative and associative

(iii) $0 \oplus \varkappa_1 = \varkappa_1$ and $\varkappa_1 \oplus \varkappa_2 \leq \varkappa_3 \oplus \varkappa_4$, whenever $\varkappa_1 \leq \varkappa_3$ and $\varkappa_2 \leq \varkappa_4$, $\forall \varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4 \in [0, 1]$

The basic continuous *t*-conorms of maximum, product, and Lukasiewicz are defined by Schweizer and Sklar [34], respectively, as follows

(i) The maximum *t*-conorm is $\varkappa_1 \oplus \varkappa_2 = \max\{\varkappa_1, \varkappa_2\}$

(ii) The product *t*-conorm is $\varkappa_1 \oplus \varkappa_2 = \varkappa_1, + \varkappa_2 - \varkappa_1 \varkappa_2$

(iii) The Lukasiewicz *t*-norm is $\varkappa_1 \oplus \varkappa_2 = \min\{\varkappa_1, + \varkappa_2, 1\}$

Definition 2.2 [10]

The 3-tuple $(\mathfrak{H}, \mu_r, \oplus)$ is said to be a RFM space If $\mathfrak{H}$ is an arbitrary set, $\oplus$ is a continuous t-conorm and μ_r is a revised fuzzy set on $\mathfrak{H}^2 \times (0,\infty)$ satisfying the following conditions:

($\mathfrak{R}$i) $\mu_r(\varphi, \nu, t) \leq 1$

($\mathfrak{R}$ii) $\mu_r(\varphi, \nu, t) = 0 \Leftrightarrow \varphi = \nu$

($\mathfrak{R}$iii) $\mu_r(\varphi, \nu, t) = \mu_r(\nu, \varphi, t)$

($\mathfrak{R}$iv) $\mu_r(\varphi, \nu, t+s) \leq \mu_r(\varphi, k, t) \oplus \mu_r(k, \nu, s)$

($\mathfrak{R}$v) $\mu_r(\varphi, \nu, -) : (0,\infty) \to [0,1]$ is continuous $\forall\, \varphi, \nu, k \in \mathfrak{H}$ and $t, s \in (0,\infty)$. (1)

Lemma 2.3 [13]

$\mu_r(\mathfrak{H}, x, \oplus)$ is non-increasing $\forall\, \varphi, \nu \in \mathfrak{H}$.

Definition 2.4 [14]

Let $(\mathfrak{H}, \mu_r, \oplus)$ be a RFM space, $\varphi \in \mathfrak{H}$ and (φ_j) is a sequence in $\mathfrak{H}$. Then,

(i) (φ_j) converges to φ if $k \in (0,1)$, and $t > 0$. $\exists\, j_1 \in \aleph$, such that $\mu_r(\varphi_j, \varphi, t) < k, \forall j \geq j_1$. We may write this $\lim_{j\to\infty} \varphi_j = \varphi$ or $\varphi_j \to \varphi$ as $j \to \infty$

(ii) (φ_j) is a Cauchy sequence if $k \in (0,1)$ and $t > 0$. $\exists\, j_1 \in \aleph$, such that $\mu_r(\varphi_j, \varphi_i, t) <\ , \forall j, i \geq j_1$

(iii) $(\mathfrak{H}, \mu_r, \oplus)$ is complete if every Cauchy sequence is convergent in $\mathfrak{H}$

(iv) Revised Fuzzy-contractive if $\exists\, a \in (0,1)$ and satisfying

$$\mu_r(\varphi_j, \varphi_{j+1}, t) \leq a\left(\mu_r(\varphi_{j-1}, \varphi_j, t)\right) \text{ for } t > 0,\ j \geq 1. \qquad (2)$$

Let as consider this paper $\aleph$ be the set of natural numbers.

Lemma 2.5 [14]

Let $(\mathfrak{H}, \mu_r, \oplus)$ be a RFM space. A sequence φ_j in φ converges to $\varphi \in \mathfrak{H}$ if and only if $\mu_r(\varphi_j, \varphi, t) \to 0$, as $j \to \infty$, for $t > 0$.

Definition 2.6 [1]

Let $(\mathfrak{H}, \mu_r, \oplus)$ be a RFM space. The revised fuzzy metric μ_r is triangular, if

$$\mu_r(\varphi, \nu, t) \leq \mu_r(\varphi, k, t) + \mu_r(k, \nu, t), \forall\, \varphi, \nu, k \in \mathfrak{H}, t > 0. \qquad (3)$$

Definition 2.7 [13]

Let $(\mathfrak{H}, \mu_r, \oplus)$ be a RFM space. A mapping $B: \mathfrak{H} \longrightarrow \mathfrak{H}$ is called revised fuzzy-contractive if $\exists\, b \in (0,1)$ such that

$$\mu_r\left(B_{\varphi}, B_{\nu}, t\right) \leq b\left(\mu_r(\varphi, \nu, t)\right), \ \forall\, \varphi, \nu \in \mathfrak{H}, t > 0. \tag{4}$$

Definition 2.8 [32]

Allow B and $\mathscr{g}$ to be two self-mappings on the nonempty set $\mathfrak{H}$ (i.e. $B, \mathscr{g} : \mathfrak{H} \longrightarrow \mathfrak{H}$). If for some $u \in \mathfrak{H}$, there are $w \in \mathfrak{H}$ and $w = Bu = \mathscr{g}u$. Then u is referred to as a B and $\mathscr{g}$ coincidence point, and w is referred to be a B and $\mathscr{g}$ mappings coincidence point. If B and $\mathscr{g}$ commute at their coincidence point, i.e. $Bu = \mathscr{g}u$ for some $u \in \mathfrak{H}$, then $B\mathscr{g}u = \mathscr{g}u$, they are said to be weakly compatible.

Proposition 2.9 [32]

Let B and $\mathscr{g}$ be self-mappings on a nonempty set $\mathfrak{H}$ that are weakly compatible. w is known as the unique common fixed point of B and $\mathscr{g}u$ if B and $\mathscr{g}$ have a unique point of coincidence such that $w = Bu = \mathscr{g}u$.

3. Main Result

In this section, which deals with the study's main results, we build several coincidence point and CFP theorems for three self-mappings in RFM spaces under the rational type weakly-compatible revised fuzzy contractive with some important examples. The concept of a binary operation is that $\oplus$ it is a continuous product. The t-conorm is utilized in all of the important results and is defined as:

$$\varphi \oplus v = \varphi + v - \varphi \cdot v \text{ for all } \varphi, v \in [0, 1] \tag{5}$$

Theorem 3.1

Let a revised fuzzy metric μ_r is triangular in a complete RFM space $(\mathfrak{H}, \mu_r, \oplus)$ and let $B, C, \mathscr{g} : \mathfrak{H} \longrightarrow \mathfrak{H}$ be three self-mappings, satisfies for all $\varphi, v \in \mathfrak{H}$,

$$\mu_r\left(B_{\varphi}, C_{v}, t\right) \leq \left\{\begin{matrix} e\left(\mu_r\left(\mathscr{g}_{\varphi}, \mathscr{g}_{v}, t\right)\right) + f\left(\frac{\mu_r\left(\mathscr{g}_{v}, B_{\varphi}, 2t\right) . \mu_r\left(\mathscr{g}_{\varphi}, C_{v}, 2t\right)}{\mu_r\left(\mathscr{g}_{\varphi}, \mathscr{g}_{v}, t\right)}\right) \\ + g\left(\mu_r\left(\mathscr{g}_{\varphi}, B_{\varphi}, t\right) + \mu_r\left(\mathscr{g}_{v}, C_{v}, t\right)\right) \end{matrix}\right\}, \tag{6}$$

For $t > 0$ and $0 \leq e, f, g < 1$ with $(e + f + 2g) < 1$. If $B(\mathfrak{H}) \cup C(\mathfrak{H}) \subset \mathscr{g}(\mathfrak{H})$, where $\mathscr{g}(\mathfrak{H})$ is a complete subspace of $\mathfrak{H}$. Then B, C and $\mathscr{g}$ have a unique point of coincidence. Moreover, if the pairs $(B, \mathscr{g})$ and $(C, \mathscr{g})$ are weakly compatible. Then, B, C and $\mathscr{g}$ have a unique CFP in $\mathfrak{H}$.

Proof

Let φ_o be the arbitrary point of $\mathfrak{H}$. Using the condition $A(\mathfrak{H}) \cup B(\mathfrak{H}) \subset \mathscr{g}(\mathfrak{H})$ choose a sequence (φ_i) in $\mathfrak{H}$ such that

$$\mathscr{g}\varphi_{2i+1} = \mathrm{B}\varphi_{2i} \text{ and } \mathscr{g}\varphi_{2i+2} = C\varphi_{2i+1}, \text{for all } i \geq 0. \tag{7}$$

Now, by (6), for $t > 0$,

$$\mu_r(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, \mathrm{t}) = \mu_r(A\varphi_{2i}, \mathrm{B}\varphi_{2i+1}, \mathrm{t})$$

$$\leq \left\{\begin{matrix} e\left(\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})\right) + f\left(\frac{\mu_r(\mathscr{g}\varphi_{2i+1}, B\varphi_{2i}, 2\mathrm{t}).\mu_r(\mathscr{g}\varphi_{2i}, C\varphi_{2i+1}, 2\mathrm{t})}{\mathrm{M}_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})}\right) \\ +g(\mu_r(\mathscr{g}\varphi_{2i}, B\varphi_{2i}, \mathrm{t}) + \mu_r(\mathscr{g}\varphi_{2i+1}, C\varphi_{2i+1}, \mathrm{t})) \end{matrix}\right\}$$

$$= \left\{\begin{matrix} e\,(\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})) + f\left(\frac{\mu_r(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+1}, 2\mathrm{t}).\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+2}, 2\mathrm{t})}{\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})}\right) \\ +g(\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, \mathrm{t}) + \mu_r(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, \mathrm{t})) \end{matrix}\right\}$$

$$= \left\{\begin{matrix} e\,(\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})) + f\left(\frac{\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+2}, 2\mathrm{t})}{\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+2}, 2\mathrm{t})}\right) \\ +g(\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, \mathrm{t}) + \mu_r(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, \mathrm{t})) \end{matrix}\right\} \tag{8}$$

Now by using Definition 2 (iv),

$$\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+2}, 2\mathrm{t}) \leq \mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, \mathrm{t}) \oplus \mu_r(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, \mathrm{t}) \text{ for } \mathrm{t} > 0$$

$$\mu_r(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, \mathrm{t}) \leq \left\{\begin{matrix} e\,(\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})) \\ +f\left(\frac{\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, \mathrm{t}).\mu_r(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, \mathrm{t})}{\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})}\right) \\ +g(\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, \mathrm{t}) + \mu_r(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, \mathrm{t})) \end{matrix}\right\}$$

$$= \left\{\begin{matrix} e\,(\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})) + f\,(\mu_r(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, \mathrm{t})) \\ +g(\mu_r(\mathscr{g}w_{2i}, \mathscr{g}w_{2i+1}, \mathrm{t}) + \mu_r(\mathscr{g}w_{2i+1}, \mathscr{g}w_{2i+2}, \mathrm{t})) \end{matrix}\right\}$$

$$= (e + g)\,(\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})) + (f + g)(\mu_r(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, \mathrm{t})) \tag{9}$$

After simplification, we obtain

$$\mu_r(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, \mathrm{t}) \leq Y\,(\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, \mathrm{t}))\, for\ t > 0, \tag{10}$$

Where $Y = (e + g)/(1 - f - g) < 1$. Similarly, again by the view of (6), for $t > 0$,

$$\mu_r(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+3}, \mathrm{t}) = \mu_r(B\varphi_{2i+2}\mathrm{C}\varphi_{2i+1}, \mathrm{t})$$

$$\leq \left\{\begin{matrix} e\,\left(\mu_r(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})\right) + f\left(\frac{\mu_r(\mathscr{g}\varphi_{2i+1}, B\varphi_{2i+2}2, \mathrm{t}).\mu_r(\mathscr{g}\varphi_{2i+2}, C\varphi_{2i+1}, \mathrm{t})}{\mu_r(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})} - 1\right) \\ +g\left(\mu_r(\mathscr{g}\varphi_{2i+2}, B\varphi_{2i+2}, \mathrm{t}) + \mu_r(\mathscr{g}\varphi_{2i+1}, C\varphi_{2i+1}, \mathrm{t})\right) \end{matrix}\right\}$$

$$= \begin{Bmatrix} e\left(\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})\right) + f\left(\frac{\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+3} 2, \mathrm{t}).\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+2}, 2\mathrm{t})}{\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})}\right) \\ +g\left(\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+3}, \mathrm{t}) + \mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, \mathrm{t})\right) \end{Bmatrix}$$

$$= \begin{Bmatrix} e\left(\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})\right) + f\,\frac{\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+3}, \mathrm{t})\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})}{\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})} \\ +g\left(\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+3}, \mathrm{t}) + \mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, \mathrm{t})\right) \end{Bmatrix} \tag{11}$$

Now by using Definition 2 (iv),

$\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+3}, 2\mathrm{t}) \le \mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, \mathrm{t}) \oplus \mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+3}, \mathrm{t})$ for $\mathrm{t} > 0$

$$\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+3}, \mathrm{t}) \le \begin{Bmatrix} a\left(\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})\right) \\ +b\left(\frac{\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, \mathrm{t}).\,\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+3}, \mathrm{t})}{\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})}\right) + \\ c\left(\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+3}, \mathrm{t}) + \mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, \mathrm{t})\right) \end{Bmatrix}$$

$$= \begin{Bmatrix} a\left(\mu_{\mathrm{r}}(\mathscr{g}w_{2i+2}, lw_{2i+1}, \mathrm{t})\right) + b\left(\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+3}, \mathrm{t})\right) \\ +c\left(\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+3}, \mathrm{t}) + \mu_{\mathrm{r}}(\mathscr{g}w_{2i+1}, \mathscr{g}w_{2i+2}, \mathrm{t})\right) \end{Bmatrix}$$

$$= (a + c)\,\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, \mathrm{t}) + (b + c)\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+3}, \mathrm{t}) \tag{12}$$

After simplification, we obtain

$$\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+3}, \mathrm{t}) \le Y\left(\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, \mathrm{t})\right) for\ t > 0, \tag{13}$$

Where Y value is same as in (10). Now from (10), (13), and by induction.

$$\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+2}, \mathscr{g}\varphi_{2i+3}, \mathrm{t}) \le Y\left(\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, \mathrm{t})\right) \le Y^2\left(\mu_{\mathrm{r}}(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, \mathrm{t})\right)$$

$$\le \cdots \le Y^{2i+2}\left(\mu_{\mathrm{r}}(\mathscr{g}\varphi_0, \mathscr{g}\varphi_1, \mathrm{t})\right) \to 0\ as\ i \to \infty \tag{14}$$

Hence, $(\mathscr{g}\varphi_i)_{i\ge 0}$ is a revised fuzzy-contractive sequence in $(\mathfrak{H}, \mu_{\mathrm{r}}, \oplus)$, therefore,

$$\lim_{i\to\infty} \mu_r(\mathscr{g}\varphi_i, \mathscr{g}\varphi_{i-1}, t) = 0\ for\ t > 0 \tag{15}$$

Since μ_r is triangular, $j > i > i_0$,

$$\mu_r(\mathscr{g}\varphi_i, \mathscr{g}\varphi_j, t) \le \begin{Bmatrix} (\mu_r(\mathscr{g}\varphi_i, \mathscr{g}\varphi_{i+1}, t)) + (\mu_r(\mathscr{g}\varphi_{i+1}, \mathscr{g}\varphi_{i+2}, t)) \\ + \cdots + \left(\mu_r(\mathscr{g}\varphi_{i-1}, \mathscr{g}\varphi_j, t)\right) \end{Bmatrix}$$

$$\le \begin{Bmatrix} Y^i\left((\mu_r(\mathscr{g}\varphi_i, \mathscr{g}\varphi_{i+1}, t))\right) + Y^{i+1}(\mu_r(\mu_r(\mathscr{g}\varphi_{i+1}, \mathscr{g}\varphi_{i+2}, t))) \\ + \cdots + Y^{j-1}\left(\mu_r(\mathscr{g}\varphi_{i-1}, \mathscr{g}\varphi_j, t)\right) \end{Bmatrix}$$

$$\le \left(Y^i + Y^{i+1} + \cdots + Y^{j-1}\right).\left(\mu_{\mathrm{r}}(\mathscr{g}\varphi_0, \mathscr{g}\varphi_1, \mathrm{t})\right)$$

$$\le \left(\frac{Y^i}{1-Y}\right)\left(\mu_{\mathrm{r}}(\mathscr{g}\varphi_0, \mathscr{g}\varphi_1, \mathrm{t})\right) \to 0, as\ i \to \infty \tag{16}$$

This shows that $(\mathscr{g}\varphi_i)$ is a Cauchy sequence, and $\mathscr{g}(\varphi)$ is a complete subspace of $\mathfrak{H}$. Hence, such that $w, u \in \mathfrak{H}$ such that $\mathscr{g}\varphi_i \longrightarrow w = \mathscr{g}u$ as $i \longrightarrow \infty, i.e.$,

$$\lim_{i\to\infty} \mu_r\,(w, \mathscr{g}\varphi_i, t) = \mu_r(w, \mathscr{g}u, t) = 0\ for\ t > 0. \tag{17}$$

Since μ_r is triangular,

$$\mu_r(\mathscr{g}u, Bu, t) \leq (\mu_r(\mathscr{g}u, \mathscr{g}\varphi_{2i+2}, t)) + (\mu_r(\mathscr{g}\varphi_{2i+2}, Bu, t))\ for\ t > 0. \tag{18}$$

Now from (6), (15), (17), and by using Definition 2 (iv), For $t > 0$

$$\mu_r(\mathscr{g}\varphi_{2i+2}, Bu, t) = \mu_r(Bu, C\varphi_{2i+1}, t)$$

$$\leq \begin{Bmatrix} e\,(\mu_r(\mathscr{g}u, \mathscr{g}\varphi_{2i+1}, \mathrm{t})) + f\left(\frac{\mu_r(\ell\varphi_{2i+1}, Bu, 2\mathrm{t}).\mu_r(\mathscr{g}u, C\varphi_{2i+1}, 2\mathrm{t})}{\mu_r(\mathscr{g}u, \mathscr{g}\varphi_{2i+1}, \mathrm{t})}\right) \\ +g\,(\mu_r(\mathscr{g}u, Bu, \mathrm{t}) + \mu_r(\mathscr{g}\varphi_{2i+1}, C\varphi_{2i+1}, \mathrm{t})) \end{Bmatrix}$$

$$\leq \begin{Bmatrix} e\,(\mu_r(\mathscr{g}u, \mathscr{g}\varphi_{2i+1}, \mathrm{t})) \\ +f\left(\frac{\mu_r(\mathscr{g}\varphi_{2i+1}, \mathscr{g}u, \mathrm{t}).\mu_r(\mathscr{g}u, Bu, \mathrm{t}).\mu_r(\mathscr{g}u, \mathscr{g}\varphi_{2i+2}, 2\mathrm{t})}{\mu_r(\mathscr{g}u, \mathscr{g}\varphi_{2i+1}, \mathrm{t})}\right) \\ +g(\mu_r(\mathscr{g}u, Bu, \mathrm{t}) + \mu_r(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, \mathrm{t})) \end{Bmatrix}$$

$$\to (b + c)(\mu_r(\mathscr{g}u, Bu, \mathrm{t}))\ as\ i \to \infty\,. \tag{19}$$

Then,

$$\lim_{i\to\infty} \inf\big(\mu_r(\mathscr{g}\varphi_{2i+2}, Bu, \mathrm{t})\big) \leq (b + c)(\mu_r(\mathscr{g}u, Bu, \mathrm{t}))\ for\ t > 0. \tag{20}$$

Now, from (17), (18), and (20), we obtain

$$\mu_r(\mathscr{g}u, Bu, \mathrm{t}) \leq (b + c)\big(\mu_r(\mathscr{g}u, Bu, \mathrm{t})\big) for\ t > 0. \tag{21}$$

Notice that $(f + g) < 1$, where $(e + f + 2g) < 1$, therefore, $\mu_r(\mathscr{g}u, Bu, \mathrm{t}) = \mu_r(w, Bu, \mathrm{t}) = 0 \Rightarrow w = \mathscr{g}u = Bu$ for $t > 0$.

Next, we have to prove that $w = \mathscr{g}u = Cu$. Since, μ_r is triangular.

$$\mu_r(\mathscr{g}u, Cu, t) = \mu_r(\mathscr{g}u, \mathscr{g}\varphi_{2i+1}, t) + \mu_r(\mathscr{g}\varphi_{2i+1}, Cu, t) \quad for\ t > 0. \tag{22}$$

Now, again from (6), (15), (17), and by using Definition 2 (iv), for $t > 0$,

$$\big(\mu_r(\mathscr{g}\varphi_{2i+1}, Cu, t)\big) = \big(\mu_r(B\varphi_{2i}, Cu, t)\big)$$

$$\leq \begin{Bmatrix} e\,(\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}u, \mathrm{t})) + f\left(\frac{\mu_r(\mathscr{g}u, B\varphi_{2i}, 2\mathrm{t}).\mu_r(\mathscr{g}\varphi_{2i}, Cu, 2\mathrm{t})}{\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}u, \mathrm{t})}\right) \\ +g(\mu_r(\mathscr{g}\varphi_{2i}, B\varphi_{2i}, \mathrm{t}) + \mu_r(\mathscr{g}u, Cu, \mathrm{t})) \end{Bmatrix}$$

$$\leq \begin{Bmatrix} e\,(\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}u, \mathrm{t})) + f\left(\frac{\mu_r(\mathscr{g}u, \mathscr{g}\varphi_{2i+1}, 2\mathrm{t}).\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}u, \mathrm{t}).\mathrm{M_r}(\mathscr{g}u, Cu, \mathrm{t})}{\mu_r(l\varphi_{2i}, lv, \mathrm{t})}\right) \\ +g\,(\mathrm{M_r}(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, \mathrm{t}) + \mu_r(\mathscr{g}u, Cu, \mathrm{t})) \end{Bmatrix} \tag{23}$$

Then,

$$\lim_{i\to\infty} inf\left(\mu_r(\ell\varphi_{2i+1}, Cu, \mathrm{t})\right) \leq (f+g)(\mu_r(\mathscr{g}u, Cu, \mathrm{t})) \; for\ t > 0. \tag{24}$$

Now, from (17), (22), and (24), we obtain

$$\mu_r(\mathscr{g}u, Cu, \mathrm{t}) \leq (f+g)\big(\mu_r(\mathscr{g}u, Cu, \mathrm{t})\big) for\ t > 0. \tag{25}$$

Note that $(f + g) < 1, where\ (e + f + 2g) < 1$, therefore,

$\mu_r(\mathscr{g}u, Cu, \mathrm{t}) = \mu_r(w, Cu, t) = 0 \Rightarrow w = \mathscr{g}u = Bu$ for $t > 0$.

Hence, w is a common coincidence point of the mappings $\mathscr{g}, B\ and\ C$ in $\mathfrak{H}$ such that $w = \mathscr{g}u = Bu = Cu$.

Next, we prove the uniqueness of a coincidence point in $(\mathfrak{H}, \mu_r, \oplus)$ for the mappings $\mathscr{g}, B\ and\ C$.

Let $w^{\oplus}$ be the other common coincidence point in $\mathfrak{H}$ such that $w^{\oplus} = \mathscr{g}u^{\oplus} = Bu^{\oplus} = Cu^{\oplus}$ for some $u^{\oplus} \in \mathfrak{H}$ Then, from (6) and by using Definition 2 (iv), for $t > 0$,

$$\begin{aligned}
\left(\mu_r(w, w^{\oplus}, t)\right) &= \left(\mu_r(\mathscr{g}u, \mathscr{g}u^{\oplus}, t)\right) = \left(\mu_r(Bu, Cu^{\oplus}, t)\right) \\
&\leq \left\{\begin{matrix} e\,(\mu_r(\mathscr{g}u, \mathscr{g}u^{\oplus}, t)) + f\left(\frac{\mu_r(\mathscr{g}u^{\oplus}, Bu, 2\mathrm{t}).\mu_r(\mathscr{g}u, Cu^{\oplus}, 2\mathrm{t})}{\mu_r(\mathscr{g}u, \mathscr{g}u^{\oplus}, t)}\right) \\ +g\,(\mu_r(\mathscr{g}u, Bu^{\oplus}, \mathrm{t}) + \mu_r(\mathscr{g}u^{\oplus}, Cu^{\oplus}, \mathrm{t})) \end{matrix}\right\} \\
&= \left\{\begin{matrix} e\left(\mu_r(w, w^{\oplus}, t)\right) + f\left(\frac{\mu_r(w, w^{\oplus}, 2t).\mu_r(w, w^{\oplus}, 2t)}{\mu_r(w, w^{\oplus}, t)}\right) \\ +g\left(\mu_r(w, w, t) + \mu_r(w^{\oplus}, w^{\oplus}, t)\right) \end{matrix}\right\} \\
&\leq (e+f)\left(\mu_r(w, w^{\oplus}, t)\right) \qquad (26)
\end{aligned}$$

Note that $(e + f) < 1$, where $(e + f + 2g) < 1$. Thus, we get that $\mu_r(w, w^{\oplus}, t) = 0$, that is, $w = w^{\oplus}$. By using the weak compatibility of the pair $(B, \varphi), (C, \varphi)$ and by using Preposition 2.9.

We can get a unique CFP of the mappings B, C and φ.

Let such that $\mathfrak{I} \in \mathfrak{H}$ such that, $\mathscr{g}\mathfrak{I} = B\mathfrak{I} = C\mathfrak{I} = \mathfrak{I}$.

Hence, we get that $\mu_r(u, \mathfrak{I}, t) = 0 \Rightarrow u = \mathfrak{I}, for\ t > 0$.

Corollary 3.2

Let a revised fuzzy metric μ_r is triangular in a complete RFM space $(\mathfrak{H}, \mu_r, \oplus)$ and let $B, C, \ell : \mathfrak{H} \longrightarrow \mathfrak{H}$ be three self-mappings, satisfies for all $\mathscr{g}, v \in \mathfrak{H}$,

$$\mu_r(B\varphi, Cv, t) \le e\,(\mu_r(\mathscr{g}\varphi, \mathscr{g}v, t)) + f\left(\frac{\mu_r(\mathscr{g}v, B\varphi, 2t).\mu_r(\mathscr{g}\varphi, Cv, 2t)}{\mu_r(\mathscr{g}\varphi, \mathscr{g}v, t)}\right), \tag{27}$$

For $t > 0$ and $0 \le e, f < 1$ with $(e + f) < 1$. If $B(\mathfrak{H}) \cup C(\mathfrak{H}) \subset \mathscr{g}(\mathfrak{H})$, where $\mathscr{g}(\mathfrak{H})$ is a complete subspace of $\mathfrak{H}$. Then, $B, C,$ and $\mathscr{g}$ have a unique point of coincidence. Moreover, if the pairs $(B, \mathscr{g})$ and $(C, \mathscr{g})$ are weakly compatible. Then, B, C and $\mathscr{g}$ have a unique CFP in $\mathfrak{H}$.

If we use identity map instead of $\mathscr{g}$, i.e., $\mathscr{g} = I$, in Theorem 3.1, we can get the following corollary:

Corollary 3.3

Let a revised fuzzy metric μ_r is triangular in a complete RFM space $(\mathfrak{H}, \mu_r, \oplus)$ and let $B, C\colon \mathfrak{H} \longrightarrow \mathfrak{H}$ be two self-mappings, satisfies for all $\varphi, v \in \mathfrak{H}$,

$$\mu_r(B\varphi, C\varphi, t) \le \left\{\begin{array}{c} e\,(\mu_r(\varphi, v, \mathrm{t})) \\ +f\left(\dfrac{\mu_r(v, B\varphi, 2\mathrm{t}).\mu_r(\varphi, Cv, 2\mathrm{t})}{\mu_r(\varphi, v, \mathrm{t})}\right) \\ +g\,(\mu_r(\varphi, B\varphi, \mathrm{t}) + \mu_r(v, \mathrm{Cv}, \mathrm{t})) \end{array}\right\}$$

for $t > 0, 0 \le e, f, g < 1$ with $(e + f + 2g) < 1$. Then, the mappings B and C have a unique CFP in $\mathfrak{H}$.

Example 3.4

Let $\mathfrak{H} = [0, 1]$, $\oplus$ is a product continuous t-conorm on $\mathfrak{H} = [0, 1]$ which is defined as $\varkappa_1 \oplus \varkappa_2 = \varkappa_1, +\varkappa_2 - \varkappa_1\varkappa_2$ for all $\varkappa_1, \varkappa_2 \in \mathfrak{H}$ and a revised fuzzy metric μ_r : $\mathfrak{H}^2 \times (0, \infty) \longrightarrow [0, 1]$ is defined by

$$\mu_r(\varphi, v, \mathrm{t}) = \frac{|\varphi - v|}{t + |\varphi - v|}, \forall\, \varphi, v \in \mathfrak{H}, \text{and } t > 0. \tag{29}$$

Then, To prove that μ_r is triangular and $(\mathfrak{H}, \mu_r, \oplus)$ is a complete RFM space. The mappings $B, C, \mathscr{g} : \mathfrak{H} \longrightarrow \mathfrak{H}$ be defined by

$$B\varphi = C\varphi = \frac{4\varphi}{3\varphi + 9} \text{ and } \mathscr{g}\varphi = \frac{2\varphi}{3}\ \forall\, \varphi \in \mathfrak{H}. \tag{30}$$

Then, from (29), we have

$$\begin{aligned} \mu_r(B\varphi, Cv, t) &= \frac{1}{t}|B\varphi - Cx| = \frac{1}{t}\left|\frac{4\varphi}{3\varphi + 9} - \frac{4v}{3v + 9}\right| = \left|\frac{36\varphi - 36v}{(3\varphi + 9)(3v + 9)}\right| \\ &\le \left|\frac{36\varphi - 36v}{81}\right| \\ &= \frac{2}{3}\left|\frac{\mathscr{g}\varphi - \mathscr{g}v}{t}\right| = \frac{2}{3}\big(\mu_r(\mathscr{g}\varphi, \mathscr{g}v, t)\big) \text{ for } t > 0. \end{aligned} \tag{31}$$

Hence, the self-mappings $B, C,$ and $\mathscr{g}$ are satisfied the weakly-compatible revised fuzzy contraction condition in RFM spaces. Now, we simplify the second term of (6), then, by using Definition 2.2 (iv) and from (29), for $t > 0$, we have

$$\frac{\mu_r(\mathscr{g}v, B\varphi, 2\mathrm{t}).\mu_r(\mathscr{g}\varphi, Bv, 2\mathrm{t})}{\mu_r(\mathscr{g}w, \mathscr{g}x, \mathrm{t})}$$

$$\leq \frac{\mu_r(\mathscr{g}x, \mathscr{g}\varphi, \mathrm{t}).\mu_r(\mathscr{g}\varphi, B\varphi, \mathrm{t}).\mu_r(\mathscr{g}\varphi, \mathscr{g}v, \mathrm{t}).\mu_r(\mathscr{g}v, Cv, \mathrm{t})}{\mu_r(\mathscr{g}\varphi, \mathscr{g}v, \mathrm{t})}$$

$$= \mu_r(\mathscr{g}v, \mathscr{g}\varphi, \mathrm{t}).\mu_r(\mathscr{g}\varphi, B\varphi, \mathrm{t}).\mu_r(\mathscr{g}v, Cv, \mathrm{t})$$

$$\geq \frac{t^3}{(t+|\mathscr{g}\varphi-\mathscr{g}v|).(t+|\mathscr{g}\varphi-B\mathscr{g}|).(t+|\mathscr{g}v-Cv|)}$$

$$= \frac{t^3}{(t+|2\varphi/3-2v/3|).(t+|2\varphi/3-4\varphi/3\varphi+9|).(t+|2v/3-4\varphi/3\varphi+9|)}$$

$$= \frac{t^3}{\left((t+|2\varphi/3-2v/3|).(t+|2(\varphi^2+\varphi)/3\varphi+9|).(t+|2(v^2+v)/3v+9|)\right)}$$

$$= \frac{t^3}{\begin{pmatrix} \left(t+\left|\frac{2\varphi}{3}-\frac{2v}{3}\right|\right).(t^2+2t((\varphi^2+\varphi)/3\varphi+9) \\ +\left(v^2+\frac{v}{3v}+9\right) \\ +(4(\varphi^2+\varphi).(v^2+v)/(3\varphi+9).(3v+9) \end{pmatrix}}$$

$$\geq \frac{t^3}{\begin{pmatrix} \left(t+\frac{2}{3|\varphi-v|}\right).(t^2+(2t/81)(3\varphi^2v+3\varphi v^2+9(\varphi^2+v^2)+6\varphi v \\ +9(\varphi+v)+4/81(\varphi^2v^2+\varphi^2v+\varphi v^2+\varphi v)) \end{pmatrix}}$$

$$\leq \frac{\left(t+\frac{2}{3|\varphi-v|}\right).\begin{pmatrix} t^2+2/81(9t\left(\varphi^2+v^2\right)+2\varphi^2v^2 \\ +\varphi v(6t+2)+(\varphi v(3t+2)+9(w+x)) \end{pmatrix}}{t^3}$$

$$= \left(\frac{1}{t^3}\right)\begin{pmatrix} \frac{2t^2}{3}|\varphi-v|+\frac{2}{81}\left(t+\frac{2}{3}|\varphi-v|\right) \\ \begin{pmatrix} 9t(\varphi^2+x^2)+2\varphi^2v^2+\varphi v(6t+2) \\ +(\varphi v(3t+2)+9 \end{pmatrix}(\varphi+v)) \end{pmatrix} \tag{32}$$

Lastly, we simplify the third term of (6), then from (29), for $t > 0$

$$\mu_r(\varphi, B\varphi, t)+\mu_r(v, Cv, t) = \frac{1}{t}(|\mathscr{g}\varphi - B\varphi| + |\mathscr{g}v - Cv|)$$

$$= \left(\frac{1}{t}\right)\left(\left|\frac{2\varphi}{3}-\frac{4\varphi}{3\varphi+9}\right|+\left|\frac{2v}{3}-\frac{4v}{3v+9}\right|\right) \tag{33}$$

$$= \left(\frac{1}{t}\right)\left(\frac{2(\varphi+\varphi^2)}{3\varphi+9}+\frac{2(v+v^2)}{3v+9}\right)$$

$$\leq \left(\frac{2}{81t}\right)\left((3\varphi v\ +\ 9)(\varphi + v) +\ 9(\varphi^2 + \mathrm{v}^2) + 6\varphi v\right)$$

Hence, all the conditions of Theorem 3.1 are satisfied with $e = 2/3, f =\ 1/9$, and $g =\ 1/5$. The mappings $B, C, and\ \mathscr{g}$ have a unique CFP, that is, 0.

Theorem 3.5.

Let a revised fuzzy metric μ_r is triangular in a complete RFM space $(\mathfrak{H}, \mu_r, \oplus)$ and let $B, C, \mathscr{g} : \mathfrak{H} \longrightarrow \mathfrak{H}$ be three self-mappings, satisfies for all $\varphi, v \in \mathfrak{H}$,

$$\mu_r\ (B\varphi, Cv, t) \leq \left\{\begin{matrix} e\left(\mu_r\ (\mathscr{g}\varphi, \mathscr{g}v, t)\right) \\ +f(U\ (B, C, \mathscr{g}, \varphi, v, t)) + g\left(\frac{\mu_r\ (\mathscr{g}\varphi, Cv, 2t).\mu_r\ (\mathscr{g}\varphi, \mathscr{g}v, t).\mu_r\ (\mathscr{g}v, B\varphi, t)}{\mu_r\ (\mathscr{g}\varphi, B\varphi, t).\mu_r\ (\mathscr{g}v, Cv, t)}\right)\end{matrix}\right\}, \tag{34}$$

Where,

$$U(B, C, \mathscr{g}, \varphi, \mathrm{v}, t) = \left\{\begin{matrix}\mu_r\ (\mathscr{g}\varphi, \mathscr{g}v, t), \mu_r(\mathscr{g}\varphi, B\varphi, t), \\ \mu_r(\mathscr{g}v, Bv, t)\mu_r(\mathscr{g}v, B\varphi, t)\mu_r(\mathscr{g}\varphi, Cv, t)\end{matrix}\right\} \tag{35}$$

for $t\ >\ 0$ and $0\ \leq\ e, f, g <\ 1$ with $(e + f + g)\ <\ 1$.

If $B(\mathfrak{H})\ \cup C(\mathfrak{H})\ \subset\ \mathscr{g}(\mathfrak{H})$, where $\mathscr{g}(\mathfrak{H})$ is a complete subspace of $\mathfrak{H}$. Then B, C and $\mathscr{g}$ have a coincidence point in $\mathfrak{H}$.

Proof.

Let φ_0 be the arbitrary point of $\mathfrak{H}$. Using the condition $B(\mathfrak{H})\ \cup C(\mathfrak{H})\ \subset\ \mathscr{g}(\mathfrak{H})$ choose a sequence (φ_i) in $\mathfrak{H}$ such that

$$\mathscr{g}\varphi_{2i+1}\ = B\varphi_{2i} \text{ and } \mathscr{g}\varphi_{2i+2}\ =\ C\varphi_{2i+1}, \text{ for all } i\ \geq\ 0\,. \tag{36}$$

Now, by (34), for $t\ >\ 0$,

$$\mu_r(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, t) = \mu_r(B\varphi_{2i}, C\varphi_{2i+1}, t)$$

$$\leq \left\{\begin{matrix} e\left(\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, t)\right) + f\left(U(B, C, \mathscr{g}, \varphi_{2i}, \varphi_{2i+1}, t)\right) \\ +g\left(\dfrac{\mu_r(\mathscr{g}\varphi_{2i}, C\varphi_{2i+1}, 2t).\,\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, t).\,\mu_r(\mathscr{g}\varphi_{2i+1}, B\varphi_{2i}, 2t)}{\mu_r(\mathscr{g}\varphi_{2i}, C\varphi_{2i}, t).\,\mu_r(\mathscr{g}\varphi_{2i+1}, B\varphi_{2i+1}, t)}\right)\end{matrix}\right\}$$

$$= \left\{\begin{matrix} e\left(\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, t)\right) + f\left(U(B, C, \mathscr{g}, \varphi_{2i}, \varphi_{2i+1}, t)\right) \\ +g\left(\frac{\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+2}, 2t).\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, t)}{\mu_r(\mathscr{g}\varphi_{2i}, \mathscr{g}\varphi_{2i+1}, t).\mu_r(\mathscr{g}\varphi_{2i+1}, \mathscr{g}\varphi_{2i+2}, t)}\right)\end{matrix}\right\} \tag{37}$$

Where,

$U(B, C, g, \varphi_{2i}, \varphi_{2i+1}, t)$

$$= min\begin{Bmatrix} \mu_r(g\varphi_{2i}, g\varphi_{2i+1}, t), \mu_r(g\varphi_{2i}, B\varphi_{2i}, t) \\ \mu_r(gw_{2i+1}, Cw_{2i+1}, t), \mu_r(gw_{2i+1}, Cw_{2i}, t), \mu_r(gw_{2i}, Cw_{2i+1}, t) \end{Bmatrix}$$

$$= min\begin{Bmatrix} \mu_r(g\varphi_{2i}, g\varphi_{2i+1}, t), \mu_r(g\varphi_{2i}, g\varphi_{2i+1}, t) \\ \mu_r(g\varphi_{2i+1}, g\varphi_{2i+2}, t), \mu_r(g\varphi_{2i+1}, g\varphi_{2i+2}, t), \mu_r(g\varphi_{2i}, g\varphi_{2i+2}, t) \end{Bmatrix}$$

$$= min\begin{Bmatrix} \mu_r(g\varphi_{2i}, g\varphi_{2i+1}, t), \mu_r(g\varphi_{2i+1}, g\varphi_{2i+2}, t), \\ 0, \mu_r(g\varphi_{2i}, g\varphi_{2i+2}, t) \end{Bmatrix} = 0. \qquad (38)$$

Now from (37), (38), and by using Definition 2 (iv), for $t > 0$, we obtain

$$\mu_r(g\varphi_{2i+1}, g\varphi_{2i+2}, t) \leq \begin{Bmatrix} e\left(\mu_r(g\varphi_{2i}, g\varphi_{2i+1}, t)\right) \\ +g\left(\frac{\mu_r(g\varphi_{2i}, g\varphi_{2i+1}, t).\mu_r(g\varphi_{2i+1}, g\varphi_{2i+2}, t).\mu_r(g\varphi_{2i}, g\varphi_{2i+1}, t)}{\mu_r(g\varphi_{2i}, g\varphi_{2i+1}, t).\mu_r(g\varphi_{2i+1}, g\varphi_{2i+2}, t)}\right) \end{Bmatrix} \quad (39)$$

After simplification, for $t > 0$,

$\mu_r(g\varphi_{2i+1}, g\varphi_{2i+2}, t) \leq \chi\left(\mu_r(g\varphi_{2i}, g\varphi_{2i+2}, t)\right)$ where $\chi = (a + c) < 1$ (40)

Similarly, again by view of (34), for $t > 0$,

$\mu_r(g\varphi_{2i+2}, g\varphi_{2i+3}, t) = \mu_r(B\varphi_{2i+2}, C\varphi_{2i+1}, t)$

$$\leq \begin{Bmatrix} e\left(\mu_r(g\varphi_{2i+2}, g\varphi_{2i+1}, t)\right) + b\left(U(B, C, g, \varphi_{2i+2}, \varphi_{2i+1}, t)\right) \\ +g\left(\frac{\mu_r(g\varphi_{2i+2}, C\varphi_{2i+1}, 2t).\mu_r(g\varphi_{2i+2}, g\varphi_{2i+1}, t).\mu_r(g\varphi_{2i+1}, B\varphi_{2i+2}, 2t)}{\mu_r(g\varphi_{2i+2}, B\varphi_{2i+2}, t).\mu_r(g\varphi_{2i+1}, C\varphi_{2i+1}, t)}\right) \end{Bmatrix}$$

$$= \begin{Bmatrix} e\left(\mu_r(g\varphi_{2i+2}, g\varphi_{2i+1}, t)\right) + b\left(U(B, C, g, \varphi_{2i+2}, \varphi_{2i+1}, t)\right) \\ +g\left(\frac{\mu_r(g\varphi_{2i+2}, g\varphi_{2i+1}, t).\mu_r(g\varphi_{2i+1}, g\varphi_{2i+3}, 2t)}{\mu_r(g\varphi_{2i+2}, g\varphi_{2i+3}, t).\mu_r(g\varphi_{2i+1}, g\varphi_{2i+2}, t)}\right) \end{Bmatrix} \qquad (41)$$

Where

$U(B, C, g, \varphi_{2i+2}, \varphi_{2i+1}, t)$

$$= min\begin{Bmatrix} \mu_r(g\varphi_{2i+2}, g\varphi_{2i+1}, t), \mu_r(g\varphi_{2i+2}, B\varphi_{2i+2}, t) \\ \mu_r(g\varphi_{2i+1}, C\varphi_{2i+1}, t), \mu_r(g\varphi_{2i+1}, B\varphi_{2i+2}, t), \mu_r(g\varphi_{2i+2}, C\varphi_{2i+1}, t) \end{Bmatrix}$$

$$= min\begin{Bmatrix} \mu_r(g\varphi_{2i+2}, g\varphi_{2i+1}, t), \mu_r(g\varphi_{2i+2}, g\varphi_{2i+3}, t) \\ \mu_r(g\varphi_{2i+1}, g\varphi_{2i+2}, t), \mu_r(g\varphi_{2i+2}, g\varphi_{2i+3}, t), \mu_r(g\varphi_{2i+2}, g\varphi_{2i+2}, t) \end{Bmatrix}$$

$$= min\{\mu_r(g\varphi_{2i+2}, g\varphi_{2i+1}, t), \mu_r(g\varphi_{2i+2}, g\varphi_{2i+3}, t), \mu_r(g\varphi_{2i+1}, g\varphi_{2i+3}, t), 0\}$$

(42)

Now from (41), (42), and by using Definition 2 (iv),

$$\mu_r(\mathscr{g}\varphi_{2i+2},\mathscr{g}\varphi_{2i+3},t) \leq \left\{\begin{matrix} e\big(\mu_r(\mathscr{g}\varphi_{2i+2},\mathscr{g}\varphi_{2i+1},t)\big) \\ +g\left(\frac{\mu_r(\mathscr{g}\varphi_{2i+2},\mathscr{g}\varphi_{2i+1},t).\mu_r(\mathscr{g}\varphi_{2i+1},\mathscr{g}\varphi_{2i+2},t).\mu_r(\mathscr{g}\varphi_{2i+2},\mathscr{g}\varphi_{2i+3},t)}{\mu_r(\mathscr{g}\varphi_{2i+2},\mathscr{g}\varphi_{2i+3},t).\mu_r(\mathscr{g}\varphi_{2i+1},\mathscr{g}\varphi_{2i+2},t)}\right) \end{matrix}\right\} \tag{43}$$

Then after simplification, for $t > 0$,

$$\mu_r(\mathscr{g}\varphi_{2i+2},\mathscr{g}\varphi_{2i+3},t) \leq \chi\big(\mu_r(\mathscr{g}\varphi_{2i+1},\mathscr{g}\varphi_{2i+2},t)\big) \text{ where } \chi = (a + c) < 1 \tag{44}$$

Now, from (40), (44), and by induction,

$$\begin{aligned} \mu_r(\mathscr{g}\varphi_{2i+2},\mathscr{g}\varphi_{2i+3},t) &\leq \chi\big(\mu_r(\mathscr{g}\varphi_{2i+1},\mathscr{g}\varphi_{2i+2},t)\big) \\ &\leq \chi^2\big(\mu_r(\mathscr{g}\varphi_{2i},\mathscr{g}\varphi_{2i+1},t)\big) \\ &\leq \cdots \leq \chi^{2i+2}\big(\mu_r(\mathscr{g}\varphi_0,\mathscr{g}\varphi_1,t)\big) \longrightarrow 0 \; as \; i \longrightarrow \infty. \end{aligned} \tag{45}$$

Hence, $(\mathscr{g}\varphi_i)$ is a revised fuzzy-contractive sequence in $(\mathfrak{H},\mu_r,\oplus)$, therefore

$$\lim_{i\to\infty}\mu_r(\mathscr{g}\varphi_i,\mathscr{g}\varphi_{i-1},t) = 0 \; for \; t > 0. \tag{46}$$

Since μ_r is triangular, $j > i > i_0$,

$$\begin{aligned} \mu_r(\mathscr{g}\varphi_i,\mathscr{g}\varphi_j,t) &\leq \left\{\begin{matrix} \mu_r(\mathscr{g}\varphi_i,\mathscr{g}\varphi_{i+1},t) + \mu_r(\mathscr{g}\varphi_{i+1},\mathscr{g}\varphi_{i+2},t) \\ + \cdots + \mu_r(\mathscr{g}\varphi_{i-1},\mathscr{g}\varphi_j,t) \end{matrix}\right\} \\ &\leq \left\{\begin{matrix} \chi^i\big(\mu_r(\mathscr{g}\varphi_i,\mathscr{g}\varphi_{i+1},t)\big) + \chi^{i+1}\big(\mu_r(\mathscr{g}\varphi_{i+1},\mathscr{g}\varphi_{i+2},t)\big) \\ + \cdots + \chi^{j-1}\Big(\mu_r\big(\mathscr{g}\varphi_{i-1},\mathscr{g}\varphi_j,t\big)\Big) \end{matrix}\right\} \\ &\leq (\chi^i + \chi^{i+1} + \cdots + \chi^{j-1})\big(\mu_r(\mathscr{g}\varphi_0,\mathscr{g}\varphi_1,t)\big) \\ &\leq \left(\frac{\chi^i}{1-\chi}\right)\big(\mu_r(\mathscr{g}\varphi_0,\mathscr{g}\varphi_1,t)\big) \longrightarrow 0 \; as \; i \longrightarrow \infty. \end{aligned} \tag{47}$$

This shows that $(\mathscr{g}\varphi_i)$ is a Cauchy sequence and $\mathscr{g}(\mathfrak{H})$ is a complete subspace of W. Hence, $\exists\, w,u \in \mathfrak{H}$ such that $\mathscr{g}\varphi_i \longrightarrow w = \mathscr{g}u \; as \; i \longrightarrow \infty$.

$$\lim_{i\to\infty}\mu_r(w,\ell\varphi_i,t) = \mu_r(w,\ell u,t) = 0 \; for \; t > 0. \tag{48}$$

Since μ_r is triangular,

$$\mu_r(\mathscr{g}u,Bu,t) \leq \big(\mu_r(\mathscr{g}u,\mathscr{g}\varphi_{2i+2},t)\big) + \big(\mu_r(\mathscr{g}\varphi_{2i+2},Bu,t)\big) \; for \; t > 0. \tag{49}$$

Now, from (34), (46), (48), and by using Definition 2 (iv), for $t > 0$, we have that

$$\mu_r(\mathscr{g}\varphi_{2i+2},Bu,t) = \mu_r(Bu,C\varphi_{2i+1},t)$$

$$\leq \left\{\begin{matrix} e\big(\mu_r(\mathscr{g}u,\mathscr{g}\varphi_{2i+1},t)\big)+f\big(U(B,C,\mathscr{g},u,\varphi_{2i+1},t)\big) \\ +g\left(\dfrac{\mu_r(\mathscr{g}u,C\varphi_{2i+1},2t).\mu_r(\mathscr{g}v,\mathscr{g}\varphi_{2i+1},t).\mu_r(\mathscr{g}\varphi_{2i+1},Bu,2t)}{\mu_r(\mathscr{g}u,Bu,t).\mu_r(\mathscr{g}\varphi_{2i+1},C\varphi_{2i+1},t)}\right)\end{matrix}\right\}$$

$$\leq \left\{\begin{matrix} e\big(\mu_r(\mathscr{g}u,\mathscr{g}\varphi_{2i+1},t)\big)+f\big(U(B,C,\mathscr{g},u,\varphi_{2i+1},t)\big) \\ +g\left(\frac{\mu_r(\mathscr{g}u,\mathscr{g}\varphi_{2i+2},t).\mu_r(\mathscr{g}u,\mathscr{g}\varphi_{2i+1},t).\mu_r(\mathscr{g}\varphi_{2i+1},\mathscr{g}u,t).\mu_r(\mathscr{g}u,Bu,t)}{\mu_r(\mathscr{g}u,Bu,t).\mu_r(\mathscr{g}\varphi_{2i+1},\mathscr{g}\varphi_{2i+2},t)}\right)\end{matrix}\right\}$$

$$\rightarrow b(U(B,C,\mathscr{g},v,\varphi_{2i+1},t))\ as\ t\rightarrow\infty. \tag{50}$$

where

$$U(B,C,\mathscr{g},u,\varphi_{2i+1},t)=min\left\{\begin{matrix}\mu_r(\mathscr{g}u,\mathscr{g}\varphi_{2i+1},t),\mu_r(\mathscr{g}u,Bu,t) \\ \mu_r(\mathscr{g}\varphi_{2i+1},C\varphi_{2i+1},t),\mu_r(\mathscr{g}\varphi_{2i+1},Bu,t),\mu_r(\mathscr{g}u,C\varphi_{2i+1},t)\end{matrix}\right\}$$

$$=min\left\{\begin{matrix}\mu_r(\mathscr{g}u,\mathscr{g}\varphi_{2i+1},t),\mu_r(\mathscr{g}u,Bu,t) \\ \mu_r(\mathscr{g}\varphi_{2i+1},\mathscr{g}\varphi_{2i+2},t),\mu_r(\mathscr{g}\varphi_{2i+1},Bu,t),\mu_r(\mathscr{g}u,\mathscr{g}\varphi_{2i+2},t)\end{matrix}\right\}$$

$$\rightarrow min\{0,\mu_r(\mathscr{g}v,Bu,t)\}=0\ as\ i\rightarrow\infty. \tag{51}$$

Now from (50) and (51), for $t > 0$, we have,

$$\lim_{i\rightarrow\infty} inf\big(\mu_r(\mathscr{g}\varphi_{2i+2},Bu,t)\big)\ for\ t>0. \tag{52}$$

By using the value (48) and (52) in (49) with limit $i\longrightarrow\infty$, we get that $\mu_r(\mathscr{g}u,Bu,t)=0\Rightarrow w=\mathscr{g}u=Bu$ for $t>0$.

Next, we have to prove that $w=\mathscr{g}u=Bu$. Since, μ_r is triangular,

$$\mu_r(\mathscr{g}v,\mathscr{g}u,t)\leq \mu_r(\mathscr{g}u,\mathscr{g}\varphi_{2i+1},t)\quad+\mu_r(\mathscr{g}\varphi_{2i+1},Cu,t)\ for\ t>0, \tag{53}$$

Now, from (34), (46), (48), and by using Definition 2 (iv), for $t > 0$, we have that

$$\mu_r(\mathscr{g}\varphi_{2i+1},Cu,t)=\mu_r(B\varphi_{2i},Cu,t)$$

$$\leq \left\{\begin{matrix} e\big(\mu_r(\mathscr{g}\varphi_{2i},\mathscr{g}u,t)\big)+f\big(U(B,C,\mathscr{g},\varphi_{2i},u,t)\big) \\ +g\left(\dfrac{\mu_r(\mathscr{g}\varphi_{2i},Cu,2t).\mu_r(\mathscr{g}\varphi_{2i},\mathscr{g}\varphi_{2i},t).\mu_r(\mathscr{g}u,B\varphi_{2i},2t)}{\mu_r(\mathscr{g}\varphi_{2i},B\varphi_{2i},t).\mu_r(\mathscr{g}u,Cu,t)}\right)\end{matrix}\right\}$$

$$\leq \left\{\begin{matrix} e\left(\mu_r(\mathscr{g}\varphi_{2i},\mathscr{g}u,t)\right)+f\big(U(B,C,\mathscr{g},\varphi_{2i},u,t)\big) \\ +g\left(\frac{\mu_r(\mathscr{g}\varphi_{2i},\mathscr{g}u,t).\mu_r(\mathscr{g}u,Cu,t).\mu_r(\mathscr{g}\varphi_{2i},\mathscr{g}u,t).\mu_r(\mathscr{g}u,\mathscr{g}\varphi_{2i+1},t)}{\mu_r\left(\mathscr{g}\varphi_{2i,}\mathscr{g}\varphi_{2i+1},t\right).\mu_r(\mathscr{g}u,Cu,t)}\right)\end{matrix}\right\}$$

$$\rightarrow f(U(B,C,\mathscr{g},u,\varphi_{2i},t))\ as\ t\rightarrow\infty. \tag{54}$$

Where,

$$U(B,C,\mathscr{g},\varphi_{2i},u,t)=min\left\{\begin{matrix}\mu_r(\mathscr{g}\varphi_{2i},\mathscr{g}u,t),\mu_r(\mathscr{g}\varphi_{2i},B\varphi_{2i},t) \\ \mu_r(\mathscr{g}u,Cu,t),\mu_r(\mathscr{g}u,\mathscr{g}B,t),\mu_r(\mathscr{g}\varphi_{2i},Cv,t)\end{matrix}\right\}$$

$$= min\left\{\begin{matrix} \mu_r(g\varphi_{2i}, gu, t), \mu_r(g\varphi_{2i}, g\varphi_{2i+1}, t) \\ \mu_r(gu, Cu, t), \mu_r(gu, g\varphi_{2i+1}, t), \mu_r(g\varphi_{2i}, Cu, t) \end{matrix}\right\}$$

$$\to min\{0, \mu_r(gu, Cu, t)\} = 0 \; as \; i \to \infty \,. \tag{55}$$

Now from (54) and (55), for $t > 0$,

$$\lim_{i\to\infty} inf\big(\mu_r(g\varphi_{2i+1}, gu, t)\big) \; for \; t > 0. \tag{56}$$

By using the value (48) and (56) in (53) with $limit \; i \longrightarrow \infty$, we obtain

$$\mu_r(gu, Cu, t) = 0 \Rightarrow w = gu = Cu \; for \; t > 0.$$

Hence, we obtain that w is a common coincidence point of the mappings g, B, and C in $\mathfrak{H}$ such that $w = gu = Bu = Cu$.

Corollary 3.6

Let a revised fuzzy metric μ_r is triangular in a complete RFM space $(\mathfrak{H}, \mu_r, \oplus)$ and let $B, C, g : \mathfrak{H} \longrightarrow \mathfrak{H}$ be three self-mappings, satisfies for all $\varphi, v \in \mathfrak{H}$,

$$\mu_r(Bw, Cv, t) \leq e(\mu_r(g\varphi, gv, t)) + f(U(B, C, g, \varphi, v, t)) \tag{57}$$

Where

$$U(B, C, g, \varphi, v, t) = min\left\{\begin{matrix} \mu_r(g\varphi, gv, t), \mu_r(g\varphi, B\varphi, t) \\ \mu_r(gv, Cv, t), \mu_r(gv, B\varphi, t), \mu_r(g\varphi, Cv, t) \end{matrix}\right\} \tag{58}$$

for $t > 0$ and $0 \leq e, f < 1$ with $(e + f) < 1$. If $B(\mathfrak{H}) \cup C(\mathfrak{H}) \subset g(\mathfrak{H})$, where $g(\mathfrak{H})$ is a complete subspace of $\mathfrak{H}$.

Then $B, C, and \; g$ have a coincidence point in $\mathfrak{H}$. Moreover, if the pairs (B, g) and (C, g) are weakly compatible. Then, B, C and g have a unique Coincidence Fixed Point in $\mathfrak{H}$.

Proof.

From the proof of Theorem 3.5, w is a common coincidence point of the mappings g, B, and C in $\mathfrak{H}$ such that $w = gu = Bu = Cu$ for some $u \in \mathfrak{H}$.

Then we prove the uniqueness of the coincidence point, let $\exists \; w^{\oplus} \in \mathfrak{H}$ is another common coincidence point of g, B, and C in $\mathfrak{H}$ such that $w^{\oplus} = gu^{\oplus} = Bu^{\oplus} = Cu^{\oplus}$ for some $w^{\oplus} \in \mathfrak{H}$. Then from (57), for $t > 0$,

$$\mu_r(w, u^{\oplus}, t) = \mu_r(gu, gu^{\oplus}, t) = \mu_r(Cu, u^{\oplus}, t)$$

$$\leq e\Big(\mu_r(gu, gu^{\oplus}, t)\Big) + f\Big(U(B, C, g, w, u^{\oplus}, t)\Big), \tag{59}$$

Where,

$$U(B,C,l,u,u^{\oplus},t)=min\left\{\begin{matrix}\mu_r(ℊu,ℊu^{\oplus},t),\mu_r(ℊu,Bu,t)\\ \mu_r(ℊu^{\oplus},Cu^{\oplus},t),\mu_r(ℊu^{\oplus},Bu,t),\mu_r(ℊu,Cu^{\oplus},t)\end{matrix}\right\}$$

$$=min\{\mu_r(ℊu,ℊu^{\oplus},t),0\}=0 \tag{60}$$

Thus

$$\mu_r(w,w^{\oplus},t)\le e\Big(\mu_r(ℊu,ℊu^{\oplus},t)\Big) \tag{61}$$

$$=e\Big(\mu_r(w,w^{\oplus},t)\Big)\,for\ t>0$$

Noticing that $(1-e)\neq 0$ where $(e+f)<1$, therefore, $\mu_r(w,w^{\oplus},t)=0\Rightarrow w=w^{\oplus}$ for $t>0$. By using the weak compatibility of the pair $(B,ℊ),(C,ℊ)$ and by Preposition 2.9, we can get a unique CFP of the mappings B,C, and $ℊ$. Let $\exists\ \Im\in\mathfrak{H}$ such that, $ℊ\Im=B\Im=C\Im=\Im$.

Hence, we get that $\mu_r(w,\Im,t)=0\Rightarrow w=\Im, for\ t>0$.

Example 3.7

From Example 3.4, it is proved that the three self-mappings B,C, and $ℊ$ are weakly compatible revised fuzzy contractive in RFM-spaces, that is,

$$\mu_r(B\varphi,Cv,t)=\frac{2}{3}\big(\mu_r(ℊ\varphi,ℊv,t)\big)\,for\ t>0. \tag{62}$$

Next, we calculate the value of the second term, present in (57). Then, we have the following cases:

1. If the minimum value of $U(B,C,ℊ,\varphi,v,t)=\mu_r(ℊ\varphi,ℊv,t), for\ t>0$. Then, by using (29)

$$U(B,C,ℊ,\varphi,v,t)=\mu_r(ℊ\varphi,ℊv,t)=\frac{1}{t}|ℊ\varphi-ℊv|$$

$$=\frac{2}{3t}|\varphi-v|\,for\ t>0 \tag{63}$$

2. If the minimum value of $U(B,C,ℊ,\varphi,v,t)=\mu_r(ℊ\varphi,B\varphi,t)$, for $t>0$. Then, by using (29)

$$U(B,C,ℊ,\varphi,v,t)=\mu_r(ℊ\varphi,B\varphi,t)=\frac{1}{t}|ℊ\varphi-B\varphi|$$

$$=\frac{1}{t}\left|\frac{2\varphi}{3}-\frac{4\varphi}{3\varphi+9}\right|=\frac{1}{t}\left|\frac{2(\varphi^2+\varphi)}{3\varphi+9}\right|\le\frac{2}{9t}(\varphi^2+\varphi)for\ t>0 \tag{64}$$

3. If the minimum value of $U(B,C,ℊ,\varphi,v,t)=\mu_r(ℊv,Cv,t)$, for $t>0$. Then, by using (29)

$$U(B,C,ℊ,\varphi,v,t)=\mu_r(ℊv,Cv,t)=\frac{1}{t}|ℊv-Cv|$$

$$= \frac{1}{t}\left|\frac{2v}{3} - \frac{4v}{3x+9}\right| = \frac{1}{t}\left|\frac{2(v^2+v)}{3v+9}\right| \leq \frac{2}{9t}(v^2+v) for\ t > 0 \qquad (65)$$

4. If the minimum value of $U(B, C, \mathscr{g}, \varphi, v, t) = \mu_r(\mathscr{g}v, B\varphi, t)$, for $t > 0$. Then, by using (29)

$$U(B, C, \mathscr{g}, \varphi, v, t) = \mu_r(\mathscr{g}v, B\varphi, t) = \frac{1}{t}|\mathscr{g}v - B\varphi|$$

$$= \frac{1}{t}\left|\frac{2v}{3} - \frac{4\varphi}{3\varphi+9}\right| = \frac{1}{t}\left|\frac{2(\varphi v + 3v - 2\varphi)}{3x+9}\right|$$

$$\leq \frac{2}{9t}|\varphi v + 3v - 2\varphi| for\ t > 0 \qquad (66)$$

5. If the minimum value of $U(B, C, \mathscr{g}, \varphi, v, t) = \mu_r(\mathscr{g}\varphi, Cv, t)$, for $t > 0$. Then, by using (29)

$$U(B, C, \mathscr{g}, \varphi, v, t) = \mu_r(\mathscr{g}\varphi, Cv, t) = \frac{1}{t}|\mathscr{g}\varphi - Cv|$$

$$= \frac{1}{t}\left|\frac{2\varphi}{3} - \frac{4v}{3v+9}\right| = \frac{1}{t}\left|\frac{2(\varphi v + 3\varphi - 2v)}{3\varphi+9}\right|$$

$$\leq \frac{2}{9t}|\varphi v + 3\varphi - 2v|\ for\ t > 0 \qquad (67)$$

Hence, all the cases together with (62) and with contacts $e = \frac{2}{3}$ and $f = \frac{2}{7}$, we have

$$\mu_r(B\varphi, Cv, t) \leq \left(\frac{2}{3}\right)(\mu_r(\mathscr{g}\varphi, \mathscr{g}v, t)) + \left(\frac{2}{7}\right)(U(B, C, \mathscr{g}, \varphi, v, t)), \qquad (68)$$

Where

$$U(B, C, \mathscr{g}, \varphi, v, t) = min\left\{\begin{matrix} \mu_r(\mathscr{g}\varphi, \mathscr{g}v, t), \mu_r(\mathscr{g}\varphi, B\varphi, t) \\ \mu_r(\mathscr{g}v, Cv, t), \mu_r(\mathscr{g}v, B\varphi, t), \mu_r(\mathscr{g}\varphi, Cvt) \end{matrix}\right\} for\ t > 0 \qquad (69)$$

Thus, all the hypotheses of Corollary 3.6 are satisfied with $e = \frac{2}{3}, f = \frac{2}{7}$, and the mappings B, C, and $\mathscr{g}$ have a unique Coincidence Fixed Point, namely, 0.

4. Application to the Revised Fuzzy Differential Equations

Now, we present an application of the revised fuzzy differential equations (RFDEs) to support our main work From the book o f Lakshmikantham and Mohapatra [17], we have the following FDEs.

Let $\boldsymbol{E}$ be the space of all revised fuzzy subsets φ of R where $\varphi : \mathcal{R} \longrightarrow [0, 1]$

$$\varphi''(y) = \eta\,(y, \varphi(v), \varphi'(y)), y \in \boldsymbol{\mathcal{F}} = [a, b],$$

$$\varphi(y_1) = \varphi_1, \varphi(y_2) = \varphi_1, y_1, y_2 \in \boldsymbol{\mathcal{F}} = [a, b] \tag{70}$$

Where $\eta : \boldsymbol{\mathcal{F}} \times \boldsymbol{E}^2 \longrightarrow \boldsymbol{E}$ is a continuous function. This problem is equivalent to the integral equation

$$\varphi(y) = \int_{y_1}^{y_2} Q(y, \xi), (\eta(\xi, \varphi(\xi), \varphi'(\xi)))d\xi + \mathfrak{B}(y), \tag{71}$$

Where Green's function Q is given by

$$Q(y, \xi) = \begin{cases} \frac{(y_2 - y)(\xi - v_1)}{y_2 - y_1}, y_1 \leq \xi \leq y \leq y_2, \\ \frac{(y_2 - \xi)(y - y_1)}{y_2 - y_1}, y_1 \leq y \leq \xi \leq y_2 \end{cases} \tag{72}$$

And $\mathfrak{B}(y)$ satisfies $\mathfrak{B}'' = 0, \mathfrak{B}(y_1) = \varphi_1, \mathfrak{B}(y_2) = \varphi_2$. Here, we recall some properties of $Q(y, \xi)$, that is,

$$\int_{y_1}^{y_2} Q(y, \xi)d\xi \leq \frac{(y_2 - y_1)^2}{8}, \int_{y_1}^{y_2} Q_y(y, \xi)d\xi \leq \frac{y_2 - y_1}{2}, \tag{73}$$

Let $\mathfrak{C} = \mathfrak{C}^1(\boldsymbol{\mathcal{F}}, \boldsymbol{E})$, $\oplus$ be a continuous t-conorm and a revised fuzzy metric $\mu_r : \mathfrak{C}^2 \times (0, \infty) \longrightarrow [0, 1]$be defined as

$$\mu_r(\varphi, v, t) = \frac{\mathfrak{D}(\varphi, v)}{t + \mathfrak{D}(\varphi, v)} \text{ Where } \mathfrak{D}(\varphi, v) = |\varphi - v|, \tag{74}$$

$\forall \boldsymbol{\mathcal{F}} \times \boldsymbol{E}^2 \longrightarrow \boldsymbol{E} \in \mathfrak{C}$ and $t > 0$. Then, it is easy to prove that μ_r is triangular and $(\mathfrak{C}, \mu_r, \oplus)$ is a complete RFM-space.

Now, we prove the existing result for the above boundary value problem by using Corollary 3.6.

Theorem 3.8.

Assume that $\eta_1, \eta_2 : \boldsymbol{\mathcal{F}} \times \boldsymbol{E}^2 \longrightarrow \boldsymbol{E}$ and let there exist $\alpha, \beta \in (0, 1)$ with $\alpha \leq \beta$ such that for all $\varphi, v \in \mathfrak{C}^1(\boldsymbol{\mathcal{F}}, \boldsymbol{E})$, satisfies

$$|\eta_1(y, \varphi(y), \varphi'(y)) - \eta_2(y, v(y), v'(y))| \leq \alpha|\varphi(y) - v(y)| + \beta|\varphi'(y) - v'(y)| \tag{75}$$

Let there exists $\eta \in (0, 1)$ such that

$$\mathfrak{D}\big(\varphi(y), v(y)\big) = \mathfrak{D}(\varphi, v) \leq \eta\, \mu(B, C, g, \varphi, v), \tag{76}$$

Where

$$\mu(B, C, g, \varphi, v) = min\left\{ \begin{matrix} |g\varphi - gv|, |g\varphi - B\varphi|, \\ |gv - Cv|, |gv - B\varphi|, |g\varphi - Cv| \end{matrix} \right\}. \tag{77}$$

Then the integral equations

$$\varphi(y) = \int_{v_1}^{v_2} Q(y,\xi),\Big(\eta_1\big(\xi,\varphi(\xi),\varphi'(\xi)\big)\Big)\,d\xi + \mathfrak{B}(y), y \in \boldsymbol{\mathcal{F}}$$

$$v(y) = \int_{v_1}^{v_2} Q(y,\xi),(\eta_2(\xi,v(\xi),v'(\xi)))d\xi + \mathfrak{B}(y), y \in \boldsymbol{\mathcal{F}} \tag{78}$$

have a unique common solution in $\mathfrak{C}^1[[y_1,y_2],\boldsymbol{E}]$.

Proof.

Suppose that $\mathfrak{C} = [[v_1,v_2],\boldsymbol{F}]$. with metric

$$\mathfrak{D}(\varphi,v) = \min_{y_1 \le y \le y_2}(\alpha|\varphi(y)-v(y)| + \beta|\varphi'(y)-v'(y)|) \tag{79}$$

The space $(\mathfrak{C},\mathfrak{D})$ is a complete metric space.

Now, we define the operators $B, C, \mathscr{g} : \mathfrak{C} \longrightarrow \mathfrak{C}$ as

$$B(\varphi) = G_\varphi + \mathfrak{B}, C(v) = H_v + \mathfrak{B}, \mathscr{g}(\varphi) = \varphi \ and\ \mathscr{g}(v) = v, \tag{80}$$

Where,

$$G_\varphi(y) = \int_{y_1}^{y_2} Q(y,\xi),\Big(\eta_1\big(\xi,\varphi(\xi),\varphi'(\xi)\big)\Big)\,d\xi, y \in \boldsymbol{\mathcal{F}}$$

$$H_v(y) = \int_{y_1}^{y} Q(y,\xi),\Big(\eta_2\big(\xi,\varphi(\xi),\varphi'(\xi)\big)\Big)\,d\xi, y \in \boldsymbol{\mathcal{F}} \tag{81}$$

where $\eta_1,\eta_2 \in \mathfrak{C}(\boldsymbol{\mathcal{F}} \times \boldsymbol{E}^2,\boldsymbol{E})$, $\varphi, v \in \mathfrak{C}^1(\boldsymbol{\mathcal{F}},\boldsymbol{E})$, and $\mathfrak{C} \in \mathfrak{C}^1(\boldsymbol{\mathcal{F}},\boldsymbol{E})$. Now by the properties of $Q(y,\xi)$ and from (79), (80) and by using the hypothesis, we have

$$|B\varphi(y) - Cv(y)| \le \int_{y_1}^{y_2} |Q(y,\xi)|, \big|\eta_1\big(\xi,\varphi(\xi),\varphi'(\xi)\big) - \eta_2\big(\xi,v(\xi),v'(\xi)\big)\big|d\xi$$

$$\le \mathfrak{D}(\varphi,v)\int_{v_1}^{v_2}|Q(y,\xi)|d\xi \le \frac{(y_2-y_1)^2}{8}\mathfrak{D}(\varphi,v) \le \frac{\mathfrak{D}(\varphi,v)}{8}$$

$$|(B\varphi)'(y) - (Cv)'(y)| \le \int_{v_1}^{v_2} |Q(y,\xi)|, \big|\eta_1\big(\xi,\varphi(\xi),\varphi'(\xi)\big) - \eta_2\big(\xi,v(\xi),v'(\xi)\big)\big|d\xi$$

$$\le \mathfrak{D}(\varphi,v)\int_{v_1}^{v_2}|Q(y,\xi)|d\xi \le \frac{y_2-y_1}{2}\mathfrak{D}(\varphi,v) \le \frac{\mathfrak{D}(\varphi,v)}{8} \tag{82}$$

Now, from the above and by view of (75), and (79), we have that

$$\begin{aligned}\mathfrak{D}(B\varphi,Cv) &= \min_{y_1 \le y \le y_2}(\alpha|B\varphi(y) - Cv(y)| + \beta|(B\varphi)'(y) - (Cv(y)|) \\ &\le \alpha\,\frac{\mathfrak{D}(\varphi,v)}{8} + \beta\,\frac{\mathfrak{D}(\varphi,v)}{2} \le \left(\frac{5}{8}\beta\right)\mathfrak{D}(\varphi,v)\end{aligned} \tag{83}$$

Now, from (76), we have that

$$\mathfrak{D}(B\varphi, Cv) \leq \left(\frac{5}{8}\beta\right)\mathfrak{D}(\varphi, v) \leq \lambda\left(\mu(B, \mathrm{C}, ℊ, \varphi, v)\right), \tag{84}$$

Where $\lambda = \frac{5}{8}, \beta\eta < 1$. Now we are in the position to apply Corollary 3.6 to get that the mappings B, C and $ℊ$ have a unique Coincidence Fixed Point $\varphi^{\oplus} \in \mathfrak{C}$, i.e., $\varphi^{\oplus}$ is a solution of the BVP. We have the following cases.

(1) If $|ℊ\varphi - ℊv|$ is the minimum term in (77), then $\mu(B, \mathrm{C}, ℊ, \varphi, v) = |ℊ\varphi - ℊv|$. Now from (74) and (84), we have

$$\mu_r(B\varphi, Cv, t) = \frac{\mathfrak{D}(B\varphi, Cv)}{t} \leq \lambda\frac{\mu(B, \mathrm{C}, ℊ, \varphi, v)}{t} = \lambda\frac{|ℊ\varphi - ℊv|}{t} = \lambda(\mu_r(ℊ\varphi, ℊv, t)) \tag{85}$$

This implies that

$$\mu_r(B\varphi, Cv, t) \leq \lambda\big(\mu_r(ℊ\varphi, ℊv, t)\big)\, for\ t > 0. \tag{86}$$

for all $\varphi, v \in \mathfrak{C}$. Thus, the operators B, C and $ℊ$ satisfy all the conditions of Corollary 3.6 with $\lambda = (e + f)$ in (57). Then, the operators B, C and $ℊ$ have a unique Coincidence Fixed Point $\varphi^{\oplus} \in \mathfrak{C}$, i.e., $\varphi^{\oplus}$ is a solution of the BVP (4.1).

(2) If $|ℊ\varphi - B\varphi\ |$is the min term in (77), then $(B, \mathrm{C}, ℊ, \varphi, v) = |ℊ\varphi - B\varphi\ |$. Now from (74) and (84), we have

$$\mu_r(B\varphi, Cv, t) = \frac{\mathfrak{D}(B\varphi, Cv)}{t} \leq \lambda\frac{\mu(B, \mathrm{C}, ℊ, \varphi, v)}{t}$$

$$= \lambda\frac{|ℊ\varphi - B\varphi|}{t} = \lambda(\mu_r(ℊ\varphi, B\varphi, t)) \tag{87}$$

This implies that

$$\mu_r(B\varphi, Cv, t) \leq \lambda\big(\mu_r(ℊ\varphi, B\varphi, t)\big),\ \forall\varphi, v \in \mathfrak{C} \text{ and } for\ t > 0. \tag{88}$$

Similarly, If $|ℊv - Cv\ |$is the minimum term in (77), then $\mu(B, \mathrm{C}, ℊ, \varphi, v) = |ℊv - Cv\ \ |$. Now from (74) and (84), we get that

$$\mu_r(B\varphi, Cv, t) \leq \lambda\big(\mu_r(ℊv, Cv, t)\big),\ \forall\varphi, v \in \mathfrak{C} \text{ and } for\ t > 0. \tag{89}$$

Again If $|ℊv - B\varphi\ \ |$is the minimum term in (77), then $\mu(B, \mathrm{C}, ℊ, \varphi, v) = |ℊv\ - B\varphi\ |$. Now by using (74) and (84), we get that

$$\mu_r(B\varphi, Cv, t) \leq \lambda\big(\mu_r(ℊv, B\varphi, t)\big),\ \forall\varphi, v \in \mathfrak{C} \text{ and } for\ t > 0\ . \tag{90}$$

Next If $|ℊ\varphi - Cv\ |$is the min term in (77), then $\mu(B, \mathrm{C}, ℊ, \varphi, v) = |ℊ\varphi - Cv\ |$. Now from (74) and (84), we get that

$$\mu_r(B\varphi, Cv, t) \leq \lambda\big(\mu_r(ℊ\varphi, Cv, t)\big),\ \forall\varphi, v \in \mathfrak{C} \text{ and } for\ t > 0. \tag{91}$$

Hence, from (88), (89), (90), and (91), the operators $B, C,$ and g satisfy all the conditions of Corollary 3.6 with $\lambda = f$ and $e = 0$ in (57). Thus, the operators $B, C,$ and g have a unique Coincidence Fixed Point $\varphi^{\oplus} \in \mathfrak{C}, i.e., \varphi^{\oplus}$ is a solution of the BVP (70).

Conclusion

Using the "triangular feature of revised fuzzy metric" in full RFM spaces, we proved various coincidence point and CFP findings under the rational type weakly-compatible revised fuzzy-contraction conditions. We also provided revised fuzzy differential equations as an application and demonstrated that the solution of the FDEs has a unique CFP of the integral operators A, B, and l. This new route of weakly compatible revised fuzzy-contraction with FDEs in RFM space will be crucial. With different sorts of weakly compatible revised fuzzy-contraction conditions for self-mappings with different types of differential equations in the context of RFM spaces, this approach may be expanded and developed in many directions.

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