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Existence of Fixed Point Theorems in Revised Fuzzy Modular Metric Spaces

A. Muraliraj[1], P. Shanmugavel[2], R. Thangathamizh[3]

[1]PG & Research Department of Mathematics,

Urumu Dhanalakshmi College, Bharathidasan University, Trichy, India.

Email id: karguzali@gmail.com.

[2]Department of Mathematics,

Selvamm arts and science College, Periyar University, Namakkal , India.

Email id: p.sham1988@gmail.com.

[3]Department of Mathematics, KRCE, Trichy, India.

Email id: thamizh1418@gmail.com.

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Abstract:

Aim of this paper, to established existance of new fixed point theorems in revised fuzzy modular space by deriving variants of Banach contraction principle under the class of inner and outer contractions.

Keywords: t-conorm, Revised fuzzy metric, Revised fuzzy modular, fixed point.

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1. Introduction

Alexander sostak [2] present an alternative approach to the concept of a fuzzy metric, calling it a revised fuzzy metric which was based on t-conorm. The theory based between these twos are similar but proving definitions are different. Later, Olga [8] Grigorenko et.al introduce the concept of a fuzzy (pseudo) metric using t-conorms instead of t-norms and call them t-conorm based fuzzy (pseudo) metrics or just CB-fuzzy (pseudo) metrics. In 2020 tarkan oner et.al initiate the concept of metric type spaces besde on the extended t-conorm [22]. In 2021 Muraliraj and Thangathamizh [11] first introduced the concept of revised fuzzy contractive mappings and proved a fixed point theorem for these mappings in revised fuzzy metric spaces. Recently, many authors have also defined various spaces in the fixed point theory [3-8,15-16]. Muraliraj and thangathamizh [13] defined the concept of Revised fuzzy modular space with the help of t-conorms. Which was very interesting to find the distance between nearness of two points. Since then many authors have expansively developed the theory of fuzzy Sets and applications. Especially, Adabitabar Firoja And Firouzian [1], Kidar [17-19], Muraliraj and Thangathamizh [13-14, 24] have introduced the concept of various types of in different ways.

2. Prelimineries

Definition 2.1 (Muraliraj and Thangathamizh [13]).

Let $\mathcal{U}$ be a real or complex vector space with a zero θ, $\diamond$ a continuous t-conorm, and W a revised fuzzy modular on the product $\mathcal{U}\times \mathrm{R}^+$. Suppose that the following properties hold for $u,v\in \mathcal{U}$ and $\varsigma,\tau>0$:

($\mathrm{R_M}$ 1) $\mathrm{W}(u,\tau)<1$;

($\mathrm{R_M}$ 2) $\mathrm{W}(u,\tau)=0$ for all $\tau>0$ if and only if $u=\theta$;

($\mathrm{R_M}$ 3) $\mathrm{W}(u,\tau)=\mathrm{W}(-u,\tau)$;

($\mathrm{R_M}$ 4) $\mathrm{W}(a,\varsigma+\tau)\le \mathrm{W}(u,\varsigma)\diamond \mathrm{W}(y,\tau)$, whenever a is the convex combination between u and v;

($\mathrm{R_M}$ 5) the mapping $\tau\to \mathrm{W}(u,\tau)$ is continuous at each fixed $u\in\mathcal{U}$. (1)

Then, we write $(\mathcal{U},\mathrm{W},\diamond)$ to represent the space with the predefined properties. In particular, we call W a revised fuzzy modular and the triple $(\mathcal{U},\mathrm{M},\diamond)$ a RFM-Space. It is worth noting that every revised fuzzy modular is non-increasing with respect to $\tau>0$.

Example 2.2. (Muraliraj and Thangathamizh [13])

Let $\mathcal{U}$ be a real or complex vector space and W be a modular on $\mathcal{U}$. Take the t-conorm $a\diamond b=\max\{a,b\}$. For every $\tau\in(0,\infty)$, define

$$\mathrm{W}(u,\tau)=\frac{\rho(u)}{\tau+\rho(u)},$$ for all $u\in\mathcal{U}$. Then $(\mathcal{U},\mathrm{W},\diamond)$ is a $RF-$modular space.

Remark 2.3.

Note that the above conclusion still holds even if the t-conorm is replaced by $u\diamond v=u+v-u\cdot v$ and $u\diamond v=min\{u+v,1\}$, respectively. Also studied the topological properties of a RFM-Space with a special property that for every $u\in\mathcal{U}$ and a non-zero real α the equality

$$\mathrm{W}(\beta u,\tau)=\mathrm{W}\left(u,\frac{\tau}{|\beta|^{\alpha}}\right),\ \text{holds for some fixed } \alpha\in(0,1]. \tag{2}$$

If the *RF* Modular W has this property, we shall say that it is $\alpha-$homogeneous. The $\mathrm{W}-$ball in $(\mathcal{U},\mathrm{W},\diamond)$ is the set of the form

$$\mathrm{B}(u,r,\tau):=\{y\in\mathcal{U};\ \mathrm{W}(u-y,\tau)<r\} \tag{3}$$

where $r \in (0,1)$ and $\mathrm{W} > 0$. Now, suppose that σ is $\alpha-$homogeneous for some $\alpha \in (0,1]$. According to Shen and Chen [21] and Muraliraj and Thangathamizh [13], the family B of all $\mathrm{W}-$balls forms a base for a first-countable Hausdorff topology, written as T_{W}. With the notion of the σ- balls, it is easy to see that a sequence (u_n) in U, $\mathrm{W}-$converges (i.e., it converges in the topology T_{W}) to its $\mathrm{W}-$limit $x \in \mathrm{U}$ if and only if $\mathrm{W}-(u-u_n,\tau) \to 0$ as $n \to \infty$ for all $\tau > 0$.

Note here that the $\mathrm{W}-$limit is unique if it does exists after all. It is then natural to say that (x_n) is $\mathrm{W}-$Cauchy if for any given $\varepsilon \in (0,1)$ and $t > 0$, there exists $\mathrm{N} \in$ with $\mathrm{W}(u_m - u_n, \tau) \leq \varepsilon$ whenever $m,n > N$. At this point, let us turn to a typical example of a triangular norm which is defined by $(u \quad v) = max\{u,v\}$.

This triangular norm has a very special property that if ' is an arbitrary triangular conorm, then $(u \quad 'v) \leq (u \quad v)$ for all $u,v \in [0,1]$. With this property, it is suitable to call this a strongest t-conorm. As is claimed by Shen and Chen [21] and Muraliraj and Thangathamizh [13], if U is a real vector space equipped with a $\alpha-$homogeneous revised fuzzy modular $\mathrm{W}-$ and a strongest triangular conorm , then a $\mathrm{W}-$convergent sequence is $\mathrm{W}-$Cauchy.

The authors also mentioned that if is not the strongest one, such implementation is not always true.

The space $(\mathrm{U},\mathrm{W},\ )$ is said to be $\mathrm{W}-$complete if $\mathrm{W}-$Cauchy sequences actually $\mathrm{W}-$converges. We note that it makes more sense if we require a complete RFM-Spaces to be equipped with the strongest triangular conorm.

A mapping g from U into itself is said to be an inner contraction if there exists a positive constant $p < 1$ such that

$$\mathrm{W}(gu - gv, p\tau) \leq \mathrm{W}(u - y, \tau), \text{ for all } u, y \in \mathrm{U} \text{ and } \tau > 0. \quad (4)$$

On the other hand, g is said to be an outer contraction if there exists a positive constant $p < 1$ such that

$$\mathrm{W}(gu - gv, \tau) \leq p(\mathrm{W}(u - y, \tau)) \quad (5)$$

In this paper, we shall be working under the aim of deriving some variants of the Banach contraction principle, by using the classes of mappings defined above, on the structure of RFM-Spaces.

3. Main Results

We divide this section into two parts, discussing independently about the two main categories of our contractions predefined in the previous section. Notice that it is clear from the definition that every RFMS is in turn a RFM-Spaces. Hence, our results are also supplied with corollaries in RFMS. We shall, however, omit such consequences since they are obvious.

3.1 Fixed-Point Theorem for an Inner Contraction

Theorem 3.1. Let $\cup$ be a real vector space equipped with a $\alpha-$homogeneous *RF Modular*, W and the strongest t-conorm $\square$ such that $(\cup, \mathrm{W}, \square)$ is $\mathrm{W}-$complete. Suppose that at each $u \in \cup$, $\mathrm{W}(u,t) \to 0$ as $\tau \to \infty$. If $g : \cup \to \cup$ is an inner contraction with constant $p \in (0,1)$, then g has a unique fixed point.

Proof. Given a point $u_0 \in \cup$, we suppose that $g^n u_0 \neq g^{n+1} u_0$ for all $n \in N$. Let $t > 0$, observe that

$$\mathrm{W}\left(g^n u_0 - g^{n+1} u_0, \tau\right) \le \mathrm{W}\left(g^{n-1} u_0 - g^n u_0, \frac{\tau}{p}\right) \tag{6}$$

As $n \to \infty$, we have $g^n u_0 - g^{n+1} u_0 \to \theta$ for every $t > 0$.

That is, for any given $\tau > 0$ and $\varepsilon \in (0,1)$, there exists $\mathrm{N} \in \square$ such that

$$\mathrm{W}\left(g^N x_0 - g^{N+1} u_0, t\right) \le \mathrm{W}\left(g^N x_0 - g^{N+1} u_0, \frac{\tau}{2^{\alpha+1}}\right) \le \varepsilon . \tag{7}$$

We now claim to show by induction that $\mathrm{W}\left(g^{N+q} u_0 - g^{N+1} u_0, \frac{\tau}{2^{\alpha+1}}\right) < \varepsilon$ for all $q \in \mathrm{N}$.

Let us assume first that $\mathrm{W}\left(g^N u_0 - g^{N+i} u_0, \tau\right) \le \varepsilon$ holds at some $i \in \mathrm{N}$. Observe that

$$\begin{aligned}
\mathrm{W}\left(g^N u_0 - g^{N+1} u_0, t\right) &= \mathrm{W}\left(\frac{1}{2}\left(g^N u_0 - g^{N+1} u_0\right) + \frac{1}{2}\left(g^{N+1} u_0 - g^{N+i+1} u_0, \tau\right), \frac{\tau}{2^{\alpha}}\right) \\
&\le \mathrm{W}\left(g^N u_0 - g^N u_0 - g^{N+1} u_0, \frac{\tau}{2^{\alpha+1}}\right) \square\ \mathrm{W}\left(g^{N+1} u_0 - g^{N+i+1} u_0, \frac{\tau}{2^{\alpha+1}}\right) \\
&\le \mathrm{W}\left(g^N u_0 - g^{N+1} u_0, \frac{\tau}{2^{\alpha+1}}\right) \square\ \mathrm{W}\left(g^N u_0 - g^{N+i} u_0, \frac{\tau}{2^{\alpha+1}.p}\right) \\
&\le \mathrm{W}\left(g^N u_0 - g^{N+1} u_0, \frac{\tau}{2^{\alpha+1}}\right) \square\ \mathrm{W}\left(g^N u_0 - g^{N+i} u_0, \frac{\tau}{2^{\alpha+1}}\right) < \varepsilon \square\ \varepsilon) = \varepsilon .
\end{aligned} \tag{8}$$

We have thus proved our claim.

Next, we shall show that $\left(g^n u_0\right)$ is Cauchy. Let $\tau > 0$ and $\varepsilon \in (0,1)$ be arbitrary, and we choose $\mathrm{N} \in \mathbb{N}$ according to the claim given above. For $n > m > \mathrm{N}$, we may write $m = \mathrm{N} + s$ and $n = \mathrm{N} + \varsigma + \tau$, for some $\varsigma, \tau \in \mathbb{N}$. Now, consider that

$$\mathrm{W}\left(g^m u_0 - g^n u_0, \tau\right) = \mathrm{W}\left(g^{\mathrm{N}+\varsigma} u_0 - g^{\mathrm{N}+\varsigma+\tau} u_0, \tau\right) \le \mathrm{W}\left(g^{\mathrm{N}} u_0 - g^{\mathrm{N}+t} u_0, \frac{\tau}{p^{\varsigma}}\right)$$

$$\le \mathrm{W}\left(g^{\mathrm{N}} u_0 - g^{\mathrm{N}+t} u_0, \tau\right) < \varepsilon. \tag{9}$$

Thus, $\left\{f^n x_0\right\}$ is Cauchy, and so the W -completeness yields that $g^n u_0 \to u^*$ for some $u^* \in \mathrm{U}$. It follows that

$$\lim_{n\to\infty} \mathrm{W}\left(\left(g^{n+1} u - g u^*, \tau\right)\right) \le \lim_{n\to\infty} \mathrm{W}\left(g^n u_0 - u^*, \frac{\tau}{p}\right) \le \lim_{n\to\infty} \mathrm{W}\left(g^n u_0 - u^*, \tau\right) = 0, \forall\ \tau > 0. \tag{10}$$

This means $g u^* = u^*$, since T_β is Hausdorff. To show that the fixed point of g is unique, assume that $v^* \in \mathrm{U}$ is a fixed point of g as well. Finally, we obtain that

$$\mathrm{W}\left(x^* - y^*, \tau\right) = \mathrm{W}\left(g^n u^* - g^n v^*, \tau\right) \le \mathrm{W}\left(u^* - v^*, \frac{\tau}{p^n}\right) \to 0, \forall \tau > 0. \tag{11}$$

Therefore, it must be the case that $u^* = v^*$.

3.2. Fixed-Point Theorem for an Outer Contraction.

For this part, we consider a weaker form of a $\mathrm{W}-$Cauchy sequence, namely, a $\mathrm{W}-G-$Cauchy sequence. This concept has been used all along together with the notion of *RFMS*.

For a RFM-Spaces U, (u_n) is called a $\mathrm{W}-G-$Cauchy sequence if for each fixed $q \in N$ and $\tau > 0$, we have $\lim_{n\to\infty}(u_{n+q}, u_{n,} \tau) = 0$. If every $\mathrm{W}-G-$Cauchy sequence $\mathrm{W}-$converges, U is said to be $\mathrm{W}-G-$complete. It is to be noted that the notion of $\mathrm{W}-G-$completeness is slightly stronger than the ordinary completeness. It is enough to see that every $\mathrm{W}-$Cauchy sequence is also a $\mathrm{W}-G-$Cauchy sequence. For our result, it is still a question whether or not the $\mathrm{W}-G-$completeness assumption can be weakened.

Theorem 3.2. Let U be a real vector space equipped with a $\alpha-$homogeneous RF modular σ and the strongest t-conorm $\diamond$ such that $(\mathrm{U}, \mathrm{W}, \diamond)$ is σ-*G*-complete. If $g : \mathrm{U} \to \mathrm{U}$ is an outer contraction with constant $p \in (0,1)$, then g has a unique fixed point.

Proof. Given a point $u_0 \in \mathrm{U}$, we suppose that $g^n u_0 \ne g^{n+1} u_0$ for all $n \in \mathbb{N}$.

By the definition of an outer contraction, we can rewrite this notion in the following:

$$\mathrm{W}\left(gu - gv, \tau\right) \le p\left(\mathrm{W}\left(u - v, \tau\right)\right) + (p) \tag{12}$$

for all $u,v \in U$ and $\tau > 0$.

Let $t > 0$, observe that

$$W\left(g^n u_0 - g^{n+1}u_0, \tau\right) \le pW\left(g^{n-1}u_0 - g^n u_0, \tau\right) + (p)$$
$$\le p^2 W\left(g^{n-2}u_0 - g^{n-1}u_0, \tau\right) + p(p) + (p) \ldots\ldots$$
$$\le p^n W\left(u_0 - gu_0, \tau\right) + \sum_{k=0}^{n-1} p^k (p) \tag{13}$$

As $n \to \infty$, we have

$$W\left(g^n x_0 - g^{n+1}u_0, \tau\right) < \varepsilon \text{, for } n \text{ being sufficiently large} \tag{14}$$

Next, we shall show that $(f^n x_0)$ is Cauchy. Let $t > 0$ and $\varepsilon \in (0,1)$ be arbitrary, and we choose $N \in$. For $n > N$, $n \in$ and for each $p > 0$. Now, consider that

$$W\left(g^n u_0 - g^{n+q}u_0, t\right) \le W(g^n u_0 - g^{n+1}u_0, \frac{\tau}{2^{\alpha+1}}) \quad W(g^{n+1}u_0 - g^{n+q}u_0, \frac{\tau}{2^{\alpha+1}})$$
$$\le W(g^n u_0 - g^{n+1}u_0, \frac{\alpha}{2^{\alpha+1}}) \quad W(g^{n+1}u_0 - g^{n+2}u_0, \frac{\tau}{2^{2(\alpha+1)}}) \quad W(g^{n+2}u_0 - g^{n+q}u_0, \frac{\tau}{2^{2(\alpha+1)}}) \ldots$$
$$\le W(g^n u_0 - g^{n+1}u_0, \frac{\tau}{2^{\alpha+1}}) \quad W(g^{n+1}u_0 - g^{n+2}u_0, \frac{\tau}{2^{2(\alpha+1)}}) \quad W(g^{n+2}u_0 - g^{n+3}u_0, \frac{\tau}{2^{3(\alpha+1)}})$$
$$\cdots \quad W(g^{n+q-1}x_0 - g^{n+q}x_0, \frac{\tau}{2^{(q-1)(\alpha+1)}}) \}$$
$$< (\varepsilon) \quad (\varepsilon) \quad \ldots \quad (\varepsilon) = \varepsilon . \tag{15}$$

Thus, the sequence $\left(g^n u_0\right)$ is Cauchy, and so the $W-$completeness yields that $g^n u_0 \to u$ for some $u \in U$ It follows that

$$W\left(g^{n+1}u_0 - gu, \tau\right) \le pW\left(g^n u_0 - u, \tau\right) + (p), \ \forall \tau > 0 . \tag{16}$$

Taking $n \to \infty$, we have

$$W\left(g^{n+1}u_0 - gu, \tau\right) \to 0 . \tag{17}$$

This means $gu = u$, since T_W is Hausdorff. To show that the fixed point of g is unique, assume that $u \in U$ is a fixed point of g as well. Finally, we obtain that

$$W\left(u - v, \tau\right) = W\left(gu - gv, \tau\right) \le pW\left(u - v, \tau\right) + (p) . \tag{18}$$

Hence, we have $\mathrm{W}\left(u - v ,\tau\right)(p) \leq (p)$ which implies that $\mathrm{W}\left(u - v ,\tau\right) = 0$. Therefore, it must be the case that $u = v$.

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