

## Periodic Nature in General System and Cellular Automata

L Praveen Kumar<sup>1</sup>, Dr Vajha Srinivasa Kumar<sup>2</sup>

<sup>1</sup> Research scholar in Mathematics, JNTUH College of Engineering, JNTU, Kukatpalle, Hyderabad - 500085  
Telangana State, India,

Email: praveenkumar155@gmail.com

<sup>2</sup> Assistant Professor Mathematics JNTUH College of Engineering Hyderabad, JNTU, Kukatpalle, Hyderabad -  
500085, Telangana State, India, Email: vajhasrinu@gmail.com

---

### Article History:

**Received:** 26-01-2024

**Revised:** 28-03-2024

**Accepted:** 16-04-2024

### Abstract:

**Introduction:** In this research paper we have discussed the various periodic points in general systems and established relation among them by adopting strong conditions to them so we can relate smoothly. Additionally, we developed the relationship with the dynamical systems and derived periodic points in Tent maps and circular maps in general systems. We established the fundamental characteristics of cellular automata in general systems, such as continuity. Furthermore, we covered periodic points in additive CAs with radius zero and the inverse of a periodic point in additive cellular automata.

**Objectives:** To be familiar with the relationships between different periodic points in a general system, general system examples, and the contrast between the periodic behavior of a dynamical system and a new system. Explaining periodic nature in additive form and defining cellular automata in a new framework.

**Methods:** We employed fundamental notions of dynamical systems and the definition of cellular automata in a new system.

**Results and Conclusion:** The relationships between different periodic points in a general system, periodic points in a new system, the periodic nature of additive cellular automata in the new system, and its basic characteristics were all revealed to us.

---

### 1. Introduction

One variation on a dynamical system is a General system, where  $X$  is a topological or metric space along with the continuous function is defined on  $X \times X \rightarrow X$  instead of  $X \rightarrow X$  as in a dynamical system.

The author [1] first introduced it in 2008 over recurrent iterated function systems. Then [2] applied it to topological dynamical systems in 2022 and talked about chaos and shadowing properties in it. Since general systems is a novel idea, many of its proofs are still pending. Since each element in an orbit of general system depends on the two elements before it makes stronger conditions than those in dynamical systems as it depends upon only one element. It yields all of the concepts, including uniform continuity, sensitivity, and equi-continuity. Every concept in dynamical systems must be defined in accordance with general systems, and

occasionally we must create new definitions such as strong periodic instead of periodic to satisfy the requirements. Moving backwards in the orbit is not possible in general system unless pre assumption met, rendering this system as more complex. The many kinds of periodic points, such as quasi-periodic, almost-periodic, and recurrent points [3], as well as their relationships, have all been discussed in this research article. Additionally, we discovered the periodic and attracting periodic points for the linear polynomial. We discovered the periodic relationship between the original tent map [4] and circle map in the dynamical system and its definition in the general system.

Cellular Automata are introduced by Ulam [5] and mathematical theory initiated by [6]. Shift map and local rule [3] definition are the two definitions of cellular automata in dynamic systems and [6] had shown that these two definitions are equivalent. However, since general systems lack a shift map, shifting map definition is not feasible. Thus, we only apply the local rule. The lack of shift map forces us to rewrite many results that hold true for dynamical systems. In this research article we found the periodic points and periods of cellular automata in simplest form.

## 2. Objectives

Studying the new dynamical system, called general system, is our goal in this. We have established the relationships between different kinds of periodic points and developed the ideas in accordance with the new system. We will compare the periodic characteristics of the old and new systems for maps like tent map and circular map and determine the stability in various scenarios. In the end, we have demonstrated the cellular automata's essential properties and defined it within the new system without changing its true definition. We discovered the periods and periodic spots for additive cellular automata in their simplified version, providing room for more study.

## 3. Methods

A Continuous function  $f: X \times X \rightarrow X$  is called the General system where  $(X, d)$  compact space or  $X$  is a topological space. (1)

An orbit of any element  $x \in X$  defined as  $O(x) = \{x_n\}_0^\infty = \{x_0, x_1, x_2, \dots, x_n, \dots\}$  (2)

Whereas  $x_0 = x_1 = x$  and for  $n \geq 2, f(x_{n-2}, x_{n-1}) = x_n$ .

In any orbit  $x_n$  denotes the  $n^{th}$  term in it. It is similar to the  $f^n(x)$  in the orbit of  $x$  in dynamical system.

We call  $x \in X$  is a fixed point if  $f(x, x) = x$ . (3)

If  $x_{kn+i} = x_i$  for all  $k \in \mathbb{N}$  and  $0 \leq i < n$  then We say that  $x \in X$  is *periodic* of periodic *period*  $n$ . A fixed point is of period one. (4)

We define  $P_n(f) = \{x: x_n = x\}$  which means the points whose period divides  $n$ . (5)

As the general system is stronger where every element in the periodic orbit is need not be periodic. This forces us to define stronger definition like strong periodic, which is periodic and every element in the orbit also periodic.

For every open set  $U$  of  $x$ , there exist a natural number  $p$  and for every  $n \geq 0$  such that  $x_{np} \in U$  is called Quasi periodic and if it is quasi periodic and every element in the orbit also quasi periodic then it called strong quasi periodic. (6)

A point  $x \in X$  is called an almost periodic if for every open set  $U$  of  $x$  there exist natural number  $m$  such that.  $\{x_{n+i} : i = 0, 1, \dots, m\} \cap U \neq \emptyset$  for all naturals  $n$ . An Almost periodic and every element in the orbit also almost periodic then it called strong almost periodic. (7)

A point  $x \in X$  is recurrent if for every open set  $U$  of  $x$ , there exist natural  $n$  such that  $x_n \in U$ . An recurrent point and every element in the orbit also recurrent then it is called strong recurrent. (8)

Now we prove the relation between these periodic points in general system which is true in dynamical system.

#### 4. Results

Theorem 3.1: If  $(X, f)$  is a general system and  $f: X \times X \rightarrow X$  be a general system as defined in Eq. (1)

- A. Strong Periodic point
- B. Strong Quasi periodic
- C. Strong Almost periodic
- D. Strong Recurrent

Then  $A \Rightarrow B \Rightarrow C \Rightarrow D$ .

Proof for  $A \Rightarrow B$ :

Let  $x \in X$  be a strong periodic point with period  $k$  then

$$x_{kn+i} = x_i \text{ for all } n \geq 0 \text{ and } i = 1, 2 \dots k$$

Then for every open set  $U$  of  $x$  then we have  $x_{kn} \in U$  as  $x_{kn} = x_0 = x$  by the definition of orbit. So  $x$  is quasi periodic and every element of orbit also periodic makes  $x$  also strong quasi periodic point.

Proof for  $B \Rightarrow C$

Let  $x \in X$  be a strong Quasi periodic point with period  $k$ .

Choose  $m = k$  in the definition of strong almost periodic then

$$\{x_{n+i} : i = 0, 1, 2 \dots kn - n\} \cap U \neq \emptyset$$

So  $x$  is almost periodic and by the strong condition it is strong almost periodic.

Proof for  $C \Rightarrow D$

Let  $x \in X$  be a strong almost periodic then

$$\{x_{n+i} : i = 0,1,2 \dots n\} \cap U \neq \emptyset$$

Then  $x_n \cap U \neq \emptyset$ . Which means  $x_n \in U$  for every open set  $U$  of  $x$ .

Which means  $x$  is recurrent and strong recurrent by every element of orbit also almost periodic.

Definition 3.2: A fixed point  $k \in X$  is stable if for every  $\epsilon > 0$  there exist  $\delta > 0$  and for all  $y \in B_\delta(k)$  and for all  $n \geq 0$ ,  $d(y_n, k) < \epsilon$ .

Definition 3.3: A fixed point is attracting, if it is stable and there exist  $\delta > 0$  and for every  $y \in B_\delta(k)$  and  $\lim_{n \rightarrow \infty} d(y_n, k) = 0$

Theorem 3.3: If the  $(X, f)$  is general system and  $f: X \times X \rightarrow X$  is a continuous function as and  $X = \mathbb{R}$  or  $\mathbb{C}$  and defined as

$$f(x, y): ax + ay$$

Then zero is attracting fixed point for  $|a| < 1$  and zero is not stable for  $|a| > 1$

Proof: For the continuous function  $f(x, y) = ax + ay$  for  $x, y \in X$ .

Clearly 0 is the fixed point as  $f(0,0) = 0$

For any  $k \in X$  the orbit is  $\mathcal{O}(x) = \{x_0, x_0, x_1, \dots x_n \dots\}$  then

$$\Rightarrow x_0 = x_0 = k$$

$$\Rightarrow x_1 = f(k, k) = ak + ak = k(a + a) = 2ak$$

$$\Rightarrow x_2 = f(x_0, x_1) = a(k) + a(2ak) = ak(1 + 2a)$$

$$\Rightarrow x_3 = f(x_1, x_2) = a(2ak) + a(ak(1 + 2a)) = a^2k(3 + 2a)$$

By proceeding like that we get coefficient of  $a^n$  for the large values of  $n$  in  $k_n$ . So  $\lim_{n \rightarrow \infty} k_n = 0$  for  $|a| < 1$ . Hence we say 0 is attracting fixed point.

For  $|a| > 1$ ,  $\lim_{n \rightarrow \infty} k_n$  is divergent so 0 is not stable.

Lemma 3.2: : If the  $(X, f)$  is general system and  $f: X \times X \rightarrow X$  is a continuous function and  $X = \mathbb{R}$  or  $\mathbb{C}$  and defined as

$$f(x, y): ax^n + ay^n$$

then it has attracting fixed point for  $|a| < 1$  and not stable for  $|a| > 1$ .

Proof: By the theorem 3.1, For any  $k_n$  is the coefficient of larger values of power of  $a$ , So  $\lim_{n \rightarrow \infty} k_n = 0$  for  $|a| < 1$  and not stable for  $|a| > 1$ .

Lemma 3.3: : If the  $(X, f)$  is general system and  $f: X \times X \rightarrow X$  is a continuous function as defined in (1.1) and  $X = \mathbb{R}$  or  $\mathbb{C}$  and defined as

$$f(x, y): ax^n y^n$$

then it has attracting fixed point for  $|a| < 1$  and not stable for  $|a| > 1$ .

Proof: By the theorem 3.1, For any  $k_n$  is the coefficient of larger values of power of  $a$ , So  $\lim_{n \rightarrow \infty} k_n = 0$  for  $|a| < 1$  and not stable for  $|a| > 1$ .

Definition 3.4: Let  $k$  be a fixed point. The Basin of attraction  $w^s(k) = \{x: \lim_{n \rightarrow \infty} x_n = k\}$ . The strong basin of attraction of a fixed point is

$w^s(k) = \{x: \lim_{n \rightarrow \infty} (x_m)_n = k, x_m \in O(x)\}$ . That is every point in the orbit also approaches to  $k$ .

Definition 3.5: A subset  $U$  of  $X$  of the general system with  $f: X \times X \rightarrow X$  is invariant under  $f$  if  $f(U, U) \in U$

Theorem 3.5:  $X$  be a general system with  $f: X \times X \rightarrow X$ . Let  $k$  be the asymptotically stable fixed point of  $f$  then strong basin of attraction  $w^s(k)$  is an invariant set of  $f$ .

Proof: As  $k$  is fixed point then  $k \in w^s(k)$  implies  $w^s(k) \neq \emptyset$ .

To show strong  $w^s(k)$  is invariant under  $f$ , we prove  $y_1 \in w^s(k)$  for every  $y \in w^s(k)$ , strong basin of attraction of  $k$ . Here  $y_1 = f(y, y)$ .

Consider  $y_1 = f(y, y)$

Here  $w^s(k)$  is strong basin of attraction and  $f$  is continuous makes

$\Rightarrow (y_1)_n = f(y_n, y_n)$ ,  $(y_1)_n$  is the  $n^{th}$  element in the orbit of  $y_1$ .

$\Rightarrow \lim_{n \rightarrow \infty} (y_1)_n = f(\lim_{n \rightarrow \infty} y_n, \lim_{n \rightarrow \infty} y_n)$

$\Rightarrow \lim_{n \rightarrow \infty} (y_1)_n = f(k, k)$  as  $y \in w^s(k)$ ,

$\Rightarrow \lim_{n \rightarrow \infty} (y_1)_n = k$ ,

We can write  $y_1 \in w^s(k)$  which shows Strong basin of attraction is invariant under  $f$ .

Example 3.6: We can define some of the dynamical systems in the general systems also and make the relation between them by defining in suitable form like Tent map

We define it as  $T: [0,1] \times [0,1] \rightarrow [0,1]$  as

$T(x, y) = x + y$  for  $x + y \leq 1$  and  $2 - (x + y)$  for  $x + y \geq 1$ .

For any  $x \in [0,1]$  the orbit of  $x$  is  $O(x) = \{x, x, x_1, x_2, x_3 \dots\}$

If  $2x \leq 1$  then  $x_1 = 2x$  then fixed point is  $f(x, x) = x$  implies  $2x = x$  implies  $x = 0$ .

If  $2x \geq 1$  then  $x_1 = 2 - 2x$  and fixed point is  $2 - 2x = x$  implies  $x = \frac{2}{3}$ .

Which are the same fixed points for the tent map in dynamical system. By defining the dynamical systems in suitable way we can reduce the burden of finding fixed points in general systems.

Example 3.7: We define circular map in general system as  $T_a: [0,1) \times [0,1) \rightarrow [0,1)(mod 1)$  by  $T_a(x, y) = \frac{x}{2} + \frac{y}{2} + a(mod 1)$ .

For any  $x \in [0,1)$  the orbit of  $x$  is  $O(x) = \{x_0, x_1, x_2, x_3, x_4 \dots\}$  as  $x_0 = x_1 = x$

$$\text{Here } x_2 = T_a(x, x) = \frac{x}{2} + \frac{x}{2} + a = x + a(\text{mod } 1)$$

$$\Rightarrow x_3 = T_a(x, x + a) = x + \frac{a}{2}(\text{mod } 1)$$

$$\Rightarrow x_4 = T_a\left(x + a, x + \frac{a}{2}\right) = x + \frac{3a}{4}(\text{mod } 1)$$

$$\Rightarrow x_5 = T_a\left(x + \frac{a}{2}, x + \frac{3a}{4}\right) = x + \frac{5a}{4}(\text{mod } 1)$$

For the general term we can write  $x_n = T(x_{n-2}, T_{n-1}) = x + \frac{pa}{2^{n-2}}(\text{mod } 1)$ , for some natural  $p$

So  $T_a$  has periodic point for every period  $n$  for  $a = \frac{2^{n-2}}{p} \in \mathbb{Q}$  then  $x_n = x$  and every point is periodic for suitable  $\in \mathbb{Q}$ .

Since  $\frac{pa}{2^{n-2}}$  is not a natural number, we will not receive periods and periodic points when  $a$  is an irrational number.

### Cellular Automata in General Systems

Let  $A = \{0,1,2, \dots, m-1\}$  are called symbols the  $A^{\mathbb{Z}}$  denotes the set of infinite two sided sequences with symbols from  $A$ . Any  $x \in A^{\mathbb{Z}}$  called the configuration then  $x = \dots x_{-3}x_{-2}x_{-1}.x_0x_1x_2 \dots \dots$ . Here all  $x_i \in A$  and  $x_{[i,j]} = x_i x_{i+1} \dots \dots x_j$  is a word of length  $j - i + 1$ . The set  $A^{\mathbb{Z}}$  with binary operation modulo  $m$  forms an abelian group with respect to addition modulo  $m$ .

We can define metric on these spaces which induces the same topology.

The Metric is

$$d(x, y) = \frac{1}{2^k}$$

Where  $k = \min \{|i|, x_i \neq y_i\}$ . (9)

Cellular automata in the dynamical system is a map  $F: A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$  with a local rule  $f: A^{2r+1} \rightarrow A$  as  $F(x)_i = f(x_{[i-m, i+a]})$  where  $r = \max \{-m, a\} \geq 0$  is called the radius for integers  $m, a$ . (10)

However, every element in the general system's orbit is dependent upon the previous two components.

Definition 3.8: the Cellular automata(CA) in general system is the function  $f: A^{\mathbb{Z}} \times A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$  and any configuration in the orbit of  $x \in A^{\mathbb{Z}}$  is

$$(x_n)_i = f((x_{n-2})_{[i+m, i+a]}, (x_{n-1})_{[i+m, i+a]}), \text{ for } n \geq 2 \text{ for considering}$$

$r = \max \{-m, a\}$  our general form of CA is

$$(x_n)_i = f((x_{n-2})_{[i-r,i+r]}, (x_{n-1})_{[i-r,i+r]}), \text{ for } n \geq 2 \quad (11)$$

Here  $(x_n)_i = i^{\text{th}}$  position in the sequence of  $x_n$ , which is  $n^{\text{th}}$  term in the orbit of  $x \in A^{\mathbb{Z}}$ .

And  $(x_{n-2})_{[i-r,i+r]}$  = word from the positions from  $i - r$  to  $i + r$  in the configuration of  $x_{n-2} \in A^{\mathbb{Z}}$ .

The absence of shift, which is present in dynamical systems and helpful for deriving numerous properties and solving problems, is the primary issue when discussing cellular automata in general systems.

Definition 3.9: The CA defined in Eq.(11) is a linear map then it is called the additive CA

$$\begin{aligned} \text{That is } (x_n)_i &= a_{i-r}(x_{n-2})_{i-r} \dots \dots \dots + a_i(x_{n-2})_{i+\dots} \dots \dots (x_{n-2})_{i+r} + b_{i-r}(x_{n-1})_{i-r} \\ &+ \dots \dots \dots + b_i(x_{n-1})_i + \dots b_{i+r}(x_{n-1})_{i+r} \text{ for } n \geq 2. \end{aligned} \quad (12)$$

Theorem 3.10: Every CA in General system in the Eq.(10) is continuous.

Proof: We have  $f: A^{\mathbb{Z}} \times A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$  defined as

$$(x_n)_i = f((x_{n-2})_{[i-r,i+r]}, (x_{n-1})_{[i-r,i+r]}), \text{ for } n \geq 2$$

For any two configurations  $x_0, y_0 \in A^{\mathbb{Z}}$  and  $d(x_0, y_0) < 2^{-n-r}$

$\Rightarrow (x_0)_{[-n-r,n+r]} = (y_0)_{[-n-r,n+r]}$  by the definition of metric from Eq.(9)

As  $-r \leq m \leq a \leq r$

$$\Rightarrow (x_0)_{[-n+m,n+a]} = (y_0)_{[-n+m,n+a]}$$

As  $x_1 = f(x_0, x_0)$  then  $\Rightarrow (x_1)_{[-n,n]} = (y_1)_{[-n,n]} \Rightarrow d(x_1, y_1) < 2^{-n}$

Here  $(x_1)_i = f((x_0)_{[i+m,i+a]}, (x_0)_{[i+m,i+a]})$  so the CA is continuous .

Lemma 3.11: The additive CA defined in Eq.(12) is continuous.

Proof: In the theorem 3.10 define  $(x_n)_i = a_{i-r}(x_{n-2})_{i-r} \dots \dots \dots + a_i(x_{n-2})_{i+\dots} \dots \dots (x_{n-2})_{i+r} + b_{i-r}(x_{n-1})_{i-r} + \dots \dots \dots + b_i(x_{n-1})_{i+\dots} b_{i+r}(x_{n-1})_{i+r}$  for  $n \geq 2$

Theorem 3.12: Product of two CA in general system is again a CA.

Proof: Let  $A = \{0,1,2 \dots m - 1\}(\text{mod } m)$  and  $B = \{0,1,2 \dots n - 1\}(\text{mod } n)$

Here  $f: A^{\mathbb{Z}} \times A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$  and  $g: B^{\mathbb{Z}} \times B^{\mathbb{Z}} \rightarrow B^{\mathbb{Z}}$  are CA in general systems defined as

$$\begin{aligned} (x_n)_i &= f((x_{n-2})_{[i-r,i+r]}, (x_{n-1})_{[i-r,i+r]}), \text{ for } n \geq 2 \quad \text{and} \quad (y_n)_i = \\ &f((y_{n-2})_{[i-r,i+r]}, (y_{n-1})_{[i-r,i+r]}), \text{ for } n \geq 2. \end{aligned}$$

Define  $C = A \times B$  then  $C^{\mathbb{Z}} = A^{\mathbb{Z}} \times B^{\mathbb{Z}}$  and  $h: C^{\mathbb{Z}} \times C^{\mathbb{Z}} \rightarrow C^{\mathbb{Z}}$  is the CA and orbit of  $z \in C^{\mathbb{Z}}$  defined as

$$O(z) = \{z_0, z_1, z_2 \dots\}$$

$$z_0 = z_1 = z$$

And

$$(z_n)_i = ((x_n)_i, (y_n)_i) \\ = (f((x_{n-2})_{[i-r, i+r]}, (x_{n-1})_{[i-r, i+r]}), f((y_{n-2})_{[i-r, i+r]}, (y_{n-1})_{[i-r, i+r]})) \text{ for } n \geq 2.$$

So  $h: C^{\mathbb{Z}} \times C^{\mathbb{Z}} \rightarrow C^{\mathbb{Z}}$  is a CA in general Systems.

Lemma 3.13: Product of finite number of CA is again a CA in general system.

Proof: Consider  $C^{\mathbb{Z}} = A_1^{\mathbb{Z}} \times A_2^{\mathbb{Z}} \dots \times A_{k-1}^{\mathbb{Z}} \times A_k^{\mathbb{Z}}$  then apply the theorem 3.12 accordingly.

Finding periods and periodic points for CA is a difficult task. Moothathu [7] discussed the periodicity for the additive CA with prime modulo and Kumar, 2023 [8] found the periods in simplest form of Additive CA.

Every CA in General System need not have the periodic points for example  $A = \{0,1\}(\text{mod}2)$  and  $f: A^{\mathbb{Z}} \times A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$  defined as  $(x_n)_i = (x_{n-2})_i(x_{n-1})_i + 1$  for  $n \geq 2$ .

But this case is not true in additive CA in general system.

Theorem 3.14: Every additive CA in General system has a periodic point.

Proof: We have additive CA in general system defined for any  $f: A^{\mathbb{Z}} \times A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$  defined as

$$(x_n)_i = a_{i-r}(x_{n-2})_{i-r} \dots \dots + a_i(x_{n-2})_{i+\dots} \dots \dots (x_{n-2})_{i+r} + b_{i-r}(x_{n-1})_{i-r} \\ + \dots \dots + b_i(x_{n-1})_{i+\dots} b_{i+r}(x_{n-1})_{i+r} \text{ for } n \geq 2.$$

Consider  $\bar{0} \in A^{\mathbb{Z}}$  is zero sequence in which all symbols are zeros.

Then  $f(\bar{0}, \bar{0}) = \bar{0}$  as no constant terms in the equation and every fixed point also the periodic point of period one.

Theorem 3.15: Inverse of a fixed point for an additive CA defined in Eq.(12) is also a fixed point.

Proof: Additive Inverse of any  $x = (x_i) \in A^{\mathbb{Z}}$  is  $-x = (m - x_i)_i$ . We have  $x$  is fixed point so  $f(x, x) = x \implies f(x_i, x_i) = x_i$  for all  $i \in \mathbb{Z}$ .

Consider  $f((-x)_i, (-x)_i) = a_{i-r}(m - x_i) + \dots a_i(m - x_i) + \dots a_{i+r}(m - x_i) +$

$$b_{i-r}(m - x_i) + \dots b_i(m - x_i) + \dots + b_{i+r}(m - x_i)(\text{mod } m)$$

$$= m(a_{i-r} + \dots a_i + \dots a_{i-r} + b_{i-r} + \dots b_i + \dots b_{i+r}) - \{a_{i-r}(x_i) + \dots a_i(x_i) + \dots a_{i+r}(x_i) + \\ b_{i-r}(x_i) + \dots b_i(x_i) + \dots + b_{i+r}(x_i)\}(\text{mod } m)$$

$$= 0 - x_i(\text{mod } m) \text{ as } x \text{ is a fixed point.}$$

$$= -x_i.$$

Theorem 3.16: Discuss the fixed and periodic points of Additive cellular Automata in general system with radius zero.

Proof: We defined the additive cellular automata in Eq.(12). If the radius is zero for it then the additive CA is in the form of

$$(x_n)_i = f((x_{n-2})_i, (x_{n-1})_i) = a(x_{n-2})_i + b(x_{n-1})_i \text{ for } n \geq 2$$

Where  $a, b \in \mathbb{Z}$  on the mapping  $f: A^{\mathbb{Z}} \times A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$  for  $A = \{0,1,2, \dots m-1\}(\text{mod } m)$ .

Here we discuss the  $P_n(f)$  rather than periodic points. Eq. (5)

Fixed points are those which satisfies  $f(x, x) = x$ . As in the orbit of any  $x \in A^{\mathbb{Z}}$ ,

$$x_0 = x_1 = x.$$

So the fixed point are  $x_i = ax_i + bx_i$  for  $i \in \mathbb{Z}$ .

$$\Rightarrow x_i = (a + b)x_i, (\text{mod } m).$$

It is true for  $a + b = 1 (\text{mod } m)$ .

Points of  $P_2(f)$ :

In the orbit of any  $x \in A^{\mathbb{Z}}$ ,  $O(x) = \{x_0, x_1, x_2, x_3 \dots\}$

By the definition of general system  $x_0 = x_1 = x$

$$\Rightarrow x_2 = f(x, x) \Rightarrow (x_2)_i = (a + b)x_i$$

$$\Rightarrow x_3 = f(x_1, x_2),$$

The points in  $P_2(f)$  are  $f(x_1, x_2) = x$

$$\Rightarrow ax_i + b(a + b)x_i = x_i, (\text{mod } m), \text{ for all } i \in \mathbb{Z}$$

$$\Rightarrow (a + ab + b^2)x_i = x_i (\text{mod } m)$$

We get  $P_2(f) = A^{\mathbb{Z}}$  for  $a + ab + b^2 = 1 (\text{mod } m)$

The points in  $P_3(f)$  are  $x_4 = f(x_2, x_3) = x$

$$\Rightarrow a(a + b)x_i + b(a + ab + b^2)x_i = x_i, \text{ for all } i \in \mathbb{Z}$$

$$\Rightarrow (a^2 + 2ab + b^2a + b^3)x_i = x_i$$

We get  $P_3(f) = A^{\mathbb{Z}}$  for  $a^2 + 2ab + b^2a + b^3 = 1 (\text{mod } m)$

We get  $P_4(f) = A^{\mathbb{Z}}$  for  $a^2 + 2a^2b + 3ab^2 + 2ab^3 + b^4 = 1 (\text{mod } m)$

We get  $P_5(f) = A^{\mathbb{Z}}$  for  $a^3 + 3a^2b + 3a^2b^2 + 4ab^3 + 2ab^4 + b^5 = 1(\text{mod } m)$

Similarly  $P_6(f) = A^{\mathbb{Z}}$  for  $a^3 + 3a^3b + 6a^2b^2 + 5a^2b^3 + 5ab^4 + 2ab^5 + b^6 = 1(\text{mod } m)$ .

We need to find the general  $P_n(f)$  for every natural  $n$ .

Example 4.10: For  $A = \{0,1,2,3,4,5\}(\text{mod } 6)$  and  $(x_n)_i = 4(x_{n-2})_i + 3(x_{n-1})_i$  for  $n \geq 2$

Solution : By the theorem 4.9,  $a + b = 4 + 3 = 7 \equiv 1(\text{mod } 6)$

Every point in  $A^{\mathbb{Z}}$  is a fixed point.

Example 3.17: For  $A = \{0,1,2,3,4\}(\text{mod } 5)$

$$(x_n)_i = 2(x_{n-2})_i + 2(x_{n-1})_i \text{ for } n \geq 2$$

Then  $P_2(f) = A^{\mathbb{Z}}$  as  $2 + 2.2 + 2^2 = 6 \equiv 1(\text{mod } 5)$

## 5. Discussion

The various periodic points in general systems and their relationships were covered in this paper. Additionally, we defined a few well-known dynamical systems in general systems, such as the circular map and the tent map, and we deduced their periodic and fixed points. We talked about the invariance in the strong basin of attraction and defined it. We discovered the desirable fixed points in general systems for a variety of maps. We have defined the CA in general systems without sacrificing generality and we have also derived the additive CA. At the first level, we demonstrated that such a CA exists by demonstrating that it is continuous in general systems. We looked at the fixedness of the inverse of a fixed point. Additionally, we determined the fixed and periodic points in the newly defined additive CA.

## References

- [1] Mihail, A. (2008). Recurrent iterated function systems. *Revue Roumaine de Mathematiques Pures et Appliquees*, 53(1), 43-54. [https://www.csm.ro/reviste/Revue\\_Mathematique/pdfs/2008/1/mihail.pdf](https://www.csm.ro/reviste/Revue_Mathematique/pdfs/2008/1/mihail.pdf)
- [2] Nia, M.F, & Bahabadi, A. Z. (2022). Chaos and shadowing in general systems. *Kragujevac Journal of Mathematics*, 46(3), 383-394. DOI 10.46793/KgJMat2203.383N
- [3] Kurka, P. (2003). *Topological and symbolic dynamics*. Paris, France: Société mathématique de France.
- [4] Nagar, A., Shah, R., & Sridharan, S. (Eds.). (2022). *Elements of Dynamical Systems* (Vol. 79). Springer Nature.
- [5] Ulam, S. (1952, August). Random processes and transformations. In *Proceedings of the international congress on mathematics* (Vol. 2, pp. 264-275). Citeseer. <https://citeseerx.ist.psu.edu/document?repid=rep1&type=pdf&doi=d670e8d3c7e6dbf5f0620ddd678b7173471ed036>.
- [6] Hedlund, G. A. (1969). Endomorphisms and automorphisms of the shift dynamical system. *Mathematical systems theory*, 3(4), 320-375. doi.org/10.1007/BF01691062.
- [7] Moothathu, T. S. (2006). Set of periods of additive cellular automata. *Theoretical computer science*, 352(1-3), 226-231. <https://doi.org/10.1016/j.tcs.2005.10.050>.
- [8] Kumar, L. P., & Kumar, V. S. (2023). Periods and periodic points of linear cellular automata. *The Scientific Temper*, 14(03), 1020-1023. <https://doi.org/10.58414/SCIENTIFICTEMPER.2023.14.3.71> [https://www.csm.ro/reviste/Revue\\_Mathematique/pdfs/2008/1/mihail.pdf](https://www.csm.ro/reviste/Revue_Mathematique/pdfs/2008/1/mihail.pdf).