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Efficient Adaptive Numerical Technique for Singularly Perturbed Reaction-Diffusion problem

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Abstract: In the proposed work, an efficient numerical strategy has been developed for solving a singularly perturbed reaction-diffusion problem (SPRDP). Shishkin mesh has been considered for tackling the boundary layers arising in the solution. Domain discretization has been carried out using Galerkin finite element method. To demonstrate the effectivity of proposed numerical scheme, numerical experiments have been carried out which shows that sharp boundary layers appear in the solution as singular perturbation parameter $\mu \to 0$.

Keywords: Reaction-diffusion problem, Galerkin finite element method, Shishkin mesh, Singular perturbation.

1. Introduction

A wide variety of physical phenomena can be simulated using differential equations that rely on parameters. The differential equations in which small parameter $\mu \in (0,1)$ is multiplied by the largest derivative term is said to be singularly perturbed. Here $'\mu'$ is known as the variation parameter [15]. Singular perturbation problem arises frequently in incompressible fluid flow, hydrodynamics, semi-conductor modeling, elasticity, etc.

These problems exhibit regions where solution changes rapidly resulting in sharp boundary layers. Conventional numerical techniques like finite difference method (FDM), Collocation method, etc. fails to give efficient numerical solution due to multiscale nature of such type of problems. Till date, various researchers have proposed numerical strategies for approximating reaction-diffusion problem for $\mu = 1$ on uniform grid. Adaptive mesh can be used for tackling boundary layers arising in the solution for $\mu \ll 1$.

Numerous numerical techniques have been developed by various researchers for approximating reaction-diffusion problem with single perturbation. Filé and Reddy [7] developed domain decomposition method for solving singular perturbation problem.

Kadalbajoo and Arora proposed B-spline collocation method for reaction diffusion problem. Wondimu et al. [22] gave fourth order compact finite difference method for solving singularly perturbed one-dimensional reaction diffusion problem.

From the literature review, it can be observed that very few μ-uniform convergent schemes are proposed for approximating SPRDP for the case $\mu \ll 1$. In the present work, Galerkin finite element method has been used for spatial discretization. Shishkin mesh has been used to treat boundary layers arising in the solution. The motive of the current work is to propose an efficient numerical strategy for treating one-dimensional SPRDP. Numerical experiments have been carried out. It has been proved that scheme is uniformly convergent with respect to singular perturbation parameter$'\mu'$. At the end, numerical validation of the code has been carried out.

The paper is organized as follows:

In Section 2, formulation of SPRDP is carried out. In Section 3, Shishkin mesh methodology is discussed in detail. Further spatial discretization is done using Galerkin finite element method. Numerical validation of proposed scheme is carried in Section 4. Conclusion is presented in Section 5.

2. Description of the problem

Consider the following singularly perturbed problem

$$-\mu \frac{d^2 s}{dx^2} + b(x)s(x) = f(x), x \in \phi = (0,1),$$

subjected to the conditions

$$s(0) = \alpha, s(1) = \gamma,$$

where α, γ are specific constants and μ is singularly perturbation parameter. Here $f(x)$, $b(x)$ are continuous functions and $b(x) \geq \beta > 0$, β is some appropriate constant.

3. Adaptive Numerical Scheme

3.1 Shishkin mesh

The considered SPRDP exhibit sharp boundary layers near $x = 0$ as $\mu \to 0$. Shishkin mesh methodology has considered for tackling boundary layers so that more mesh points are created in the region as $\mu \to 0$.

Let M be the total number of elements. Here M is a positive even integer. Spatial domain is divided into subparts $[0, \rho]$ and $[\rho, 1]$ where $'\rho'$ is transition parameter given as

$$\rho = \min \{\frac{1}{2}, C \sqrt{\mu} \log M\}$$

Here C is the constant. The two subintervals $[0, \rho]$ and $[\rho, 1]$ are divided into $\frac{M}{2}$ equal mesh parts with mesh spaces given by

$$x_j = \begin{Bmatrix} 2\rho \frac{j}{M}, j = 1,2, \dots \dots \frac{M}{2} \\ \rho + 2\left(j - \frac{M}{2}\right)\frac{(1-\rho)}{M}\ , j = \frac{M}{2} + 1, \dots . . M \end{Bmatrix}$$

$$h_j = \begin{Bmatrix} \frac{2\rho}{M}, j = 1,2, \dots \dots \dots . . \frac{M}{2} \\ \frac{2(1-\rho)}{M}, j = \frac{M}{2}, \dots \dots \dots . M \end{Bmatrix}$$

The piecewise uniform discretization of domain is given by

$$\overline{\phi_h} = \cup_e^N \phi_e^h.$$

3.2 Galerkin finite element method

Let H_0^1 represent the collection of all functions with order less than or 1 that vanish at the boundary and are square integrable over $\bar{\phi} = [0,1]$.The weak formulation of the SPRDP is given as

$$F(s,t) = G(t),$$

where $$F(s,t) = \int_0^1 [\mu \frac{ds(x)}{dx}\frac{dt(x)}{dx} + b(x)s(x)t(x)]dx,$$

and

$$G(t) = \int_0^1 f(x)t(x)dx.$$

Consequently, $F(s,t)$ is a bilinear function in s and t while $G(t)$ is a linear functional of t, $\forall\ t \in\ H_0^1$.The areas of the outer layer (0, ρ) and (ρ, 1) are distinguished inside the field ϕ = [0,1].

Consider a set of piecewise linear basis function $\phi_i(x) \in H_0^1$ of the form

$$\phi_j(x) = \begin{Bmatrix} \dfrac{x - x_j}{h_j} & if\ x_{j-1} \le x \le x_j \\ \dfrac{x_{j+1} - x}{h_{i+1}} & if\ x_j \le x \le x_{j+1} \\ 0, & otherwise \end{Bmatrix}$$

consider an element $\phi_r^e = [x_1^e, x_2^e]$ from the domain ϕ_ρ^M.Here e is the element number. An appropriate form of the problem for each element is provided as

$$\int_{x_1^e}^{x_2^e} [\mu \frac{ds^e(x)}{dx}\frac{dt(x)}{dx} + b(x)s(x)t(x)]dx = \int_{x_1^e}^{x_2^e} f(x)t(x)dx,$$

where the limitation of $u(x)$ on the element $\phi_r^e = [x_1^e, x_2^e]$.Using the linear basis function on each element to represent the numerical solution as

$$s_M^e\ (x) = \sum_{j=1}^{M} c_k^e\ \phi_k^e(x),$$

where ϕ_k^e is the fundamental functions, the values of c_k^e are unidentified which have to be determined.

By differentiating, it gives

$$\frac{ds_M^e(x)}{dx} = \sum_{k=1}^{N} c_k^e \frac{d\phi_k^e(x)}{dx}.$$

Substituting all the values, we get

$$\int_{x_1^e}^{x_2^e} \sum_{k=1}^{M_e} \mu \frac{d\phi_k^e(x)}{dx}\frac{dt(x)}{dx} + b(x)\phi_k^e(x)t(x))c_k^e]dx = \int_{x_1^e}^{x_2^e} f(x)t(x)dx$$

Applying Galerkin method

$$\int_{x_1^e}^{x_2^e} \sum_{k=1}^{M_e} \mu \frac{d\phi_k^e(x)}{dx}\frac{dt(x)}{dx} + b(x)\phi_k^e(x)t(x))c_k^e]dx = \int_{x_1^e}^{x_2^e} f(x)t(x)dx,$$

where $\phi_k^e(x)$ are the piecewise linear base function. We rewrite as

$$\sum_{k=1}^{N^e} l_{j,k}^e\, c_k^e = O_j^e \text{for j=1,2, …}$$

Here $l_{j,k}^e = \int_{x_1^e}^{x_2^e} [(\mu \frac{d\phi(x)}{dx}\frac{d\phi_j^e(x)}{dx} + b(x)\phi_k^e(x)\phi_j^e(x))]dx$

is called as the stiffness matrix and

$$O_i^e = \int_{x_1^e}^{x_2^e} f(x)\varphi_j^e(x)dx,$$

is the load vector.

Each element of the form ϕ_ρ^e has two nodes, because the basic functions are linear and each element of the form has two equations.

$$l_{11}^e c_1^e + l_{12}^e c_2^e = O_1^e,$$

$$l_{21}^e c_1^e + l_{22}^e c_2^e = O_2^e.$$

In this case, on a typical element, the end point node label is denoted by the subscripts 1 and 2. When the components are assembled, these subscripts are to be relabelled in order to correspond with the proper nodes in the final mesh i.e. *1, 2, 3..., M + 1*. Given that there are M pieces in the mesh, *M + 1* equation in *M + 1* degree of freedom describe *M+ 1* nodes and *M+ 1* equation assembled system of equations. The linear system of equations is given by

$$\begin{bmatrix} l_{11} & l_{12} & 0 \ldots\ldots.. & 0 \\ l_{21} & l_{22} & l_{23} \ldots\ldots & 0 \\ 0 & l_{32} & l_{33} \ldots\ldots & 0 \\ 0 & 0 & l_{43} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & l_{M+1,M+1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ \vdots \\ c_{N+1} \end{bmatrix} = \begin{bmatrix} O_1 \\ O_2 \\ O \\ \vdots \\ \vdots \\ O_{N+1} \end{bmatrix}$$

where

$$\begin{cases} l_{11} = l_{11}^{1}, l_{12} = l_{12}^{1}, \\ l_{21} = l_{21}^{1}, l_{22} = l_{21}^{1} + l_{11}^{2}, l_{23} = l_{12}^{2} \\ \vdots \\ l_{M,M+1} = l_{12}^{M}, l_{M+1,M} = l_{21}^{M}, l_{M+1,M+1} = l_{22}^{M}, \end{cases}$$

and

$$\begin{cases} O_1 = O_1^1 \\ O_2 = O_2^1 + O_1^2 \\ \vdots \\ O_{N+1} = O_2^N \end{cases}$$

Applying appropriate boundary conditions $s_M(0) = \alpha$ and $s_M(1) = \gamma$ then $M-1$ unknown values $c_2, c_{3,} \dots, c_N$ remain. So, equation becomes:

$$\begin{bmatrix} l_{11} & l_{12} & 0 \dots \dots .. & 0 \\ l_{21} & l_{22} & l_{23} \dots \dots & 0 \\ 0 & l_{32} & l_{33} \dots \dots & 0 \\ 0 & 0 & l_{43} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & l_{M+1,M+1} \end{bmatrix} \begin{bmatrix} c_2 \\ c_3 \\ c_4 \\ \vdots \\ \vdots \\ c_M \end{bmatrix} = \begin{bmatrix} O_2 - l_{21}\alpha \\ O_3 \\ O_4 \\ \vdots \\ \vdots \\ O_N - O_{M,M+1\gamma} \end{bmatrix}$$

And two auxiliary equations corresponding to two nodes 1 and $M+1$ are

$$l_{11}\alpha + lc_1 = O_1$$

$$l_{M+1,N}c_{M+1} + l_{M+1,M+1}\gamma = O_{M+1}.$$

We obtain the global system of linear equations after the assembly of element level matrices.

$$AX=B.$$

4. Numerical Results

Consider the singularly perturbed problem

$$-\mu\frac{d^2s(x)}{dx^2} + s(x) = x,$$

subject to boundary conditions $s(0) = 1, s(1) = 1 + e^{-\frac{1}{\sqrt{\mu}}}$. The exact solution is given by $u(x) = x + e^{-\frac{x}{\sqrt{\mu}}}$.

In Table 1, maximum absolute errors have been presented for different values of 'μ' with increase in number of elements. In Fig.1, Comparison of exact solution and numerical solution is presented for $M = 128$ and $\mu = 2^{-13}$. Further, in Fig. 2, numerical solution profile is shown for $M = 256,\ \mu = 2^{-14}$. In Fig.3., sharp boundary layers arising in the solution is shown for $M = 512,\ \mu = 2^{-16}$.

Table 1 shows maximum absolute error for considered SPRDP for small values of perturbation parameter.

Table 1

Our method						
μ ↓	$M = 16$	$M = 32$	$M = 64$	$M = 128$	$M = 256$	$M = 512$
2^{-0}	4.02e-02	1.99e-02	9.92e-03	4.94e-03	2.47e-03	1.23e-03
2^{-2}	4.66e-02	2.30e-02	1.14e-02	5.71e-03	2.85e-03	1.42e-03
2^{-4}	5.84e-02	2.91e-02	1.45e-02	7.24e-03	3.62e-03	1.81e-03
2^{-5}	6.14e-02	3.06e-02	1.53e-02	7.66e-03	3.82e-03	1.91e-03
2^{-6}	6.23e-02	3.11e-02	1.56e-02	7.79e-03	3.89e-03	1.94e-03
2^{-7}	6.37e-02	3.12e-02	1.56e-02	7.81e-03	3.90e-03	1.95e-03
2^{-8}	8.16e-02	3.54e-02	1.56e-02	7.85e-03	3.90e-03	1.95e-03
2^{-9}	9.43e-02	4.33e-02	1.97e-02	8.92e-03	3.98e-03	1.95e-03
2^{-10}	1.03e-01	4.89e-02	2.31e-02	1.08e-02	5.10e-03	2.38e-03
2^{-11}	1.09e-01	5.29e-02	2.55e-02	1.23e-02	5.89e-03	2.82e-03
2^{-12}	1.14e-01	5.57e-02	2.71e-02	1.32e-02	6.46e-03	3.14e-03
2^{-13}	1.17e-01	5.77e-02	2.83e-02	1.39e-02	6.85e-03	3.37e-03
2^{-14}	1.19e-01	5.91e-02	2.92e-02	1.44e-02	7.13e-03	3.52e-03
2^{-15}	1.21e-01	6.01e-02	2.98e-02	1.47e-02	7.33e-03	3.63e-03
2^{-16}	1.22e-01	6.08e-02	3.02e-02	1.50e-02	7.47e-03	3.71e-03

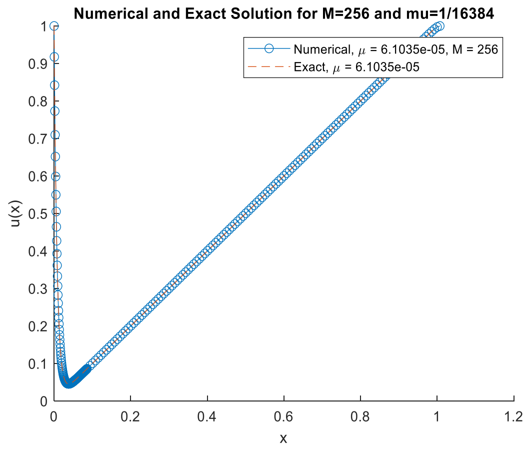

Fig.1

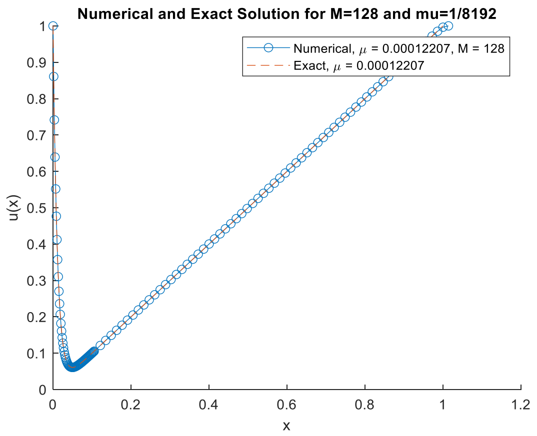

Fig.2

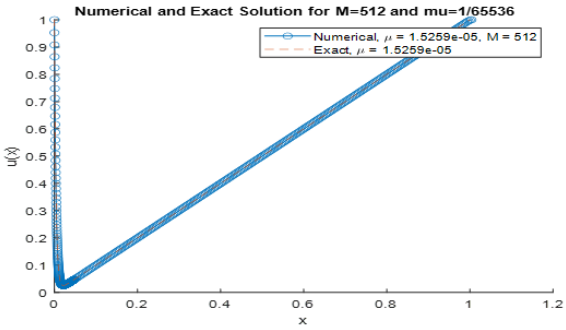

Fig.3

Fig.1: Comparison of Exact Sol and Numerical Sol for $M = 128, \mu = 2^{-13}$.

Fig.2: Comparison of Exact Sol and Numerical Sol for $M = 256,\ \mu = 2^{-14}$.

Fig.3: Comparison of Exact Sol and Numerical Sol for $M = 512,\ \mu = 2^{-16}$.

5. Conclusion

The reaction-diffusion problem is efficiently approximated using Galerkin finite element method. This method provides good approximation to the considered SPRDP. This method is shown to be flexible and efficient in handling complex geometries. It can be used to capture sharp gradients and localized behavior in the solution. Domain discretization has been carried out using Galerkin finite element method. Shishkin mesh refinement has been used for capturing boundary layers. Numerical tests have been carried out which reflect the effectivity of developed adaptive strategy in capturing sharp changes in the solution for small values of μ.

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