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Second Degree Quinary Equation $xy + XY = (k^2 + 2k + 1)w^2$

Dr. N. Thiruniraiselvi [1], Dr. Sharadha Kumar [2], Dr. M. A. Gopalan[3]

[1]Assistant Professor, Department of Mathematics, School of Engineering and Technology,
Dhanalakshmi Srinivasan University, Samayapuram, Trichy- 621 112, Tamil Nadu, India.
email: drntsmaths@gmail.com
orcid id: https://orcid.org/0000-0003-4652-3846

[2]Department of Mathematics , National College , Affiliated to BharathidasanUniversity,Trichy-620 001,Tamilnadu,India
email: sharadhak12@gmail.com
Orcid id: https://orcid.org/0000-0002-0509-6158

[3]Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002,Tamil Nadu, India.
email: mayilgopalan@gmail.com
Orcid id: https://orcid.org/0000-0003-1307-2348

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Abstract: The aim of this is to determine quintuples (x, y, X, Y, w) in integerssatisfying the quadratic diophantine $x\,y + X\,Y = (k^2 + 2\,k - 1)\,w^2$. Different sets of solutions in integers are obtained by converting it to a ternary quadratic equation through employing suitable linear transformations for which the solutions in integers exist.Some observations are exhibited.

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1. Introduction

Polynomial equationsthat require solutions in integers are plenty. In particular, second degree homogeneous or not-homogeneous equations containing multiple unknowns have attracted many mathematicians since antiquity [3,4,5,12]. In particular, refer [1,2,6,7,8,9,10,13,14] for various forms of second degree equations with four unknowns .. In [11,15,16], quinary equations of degree two are studied for obtaining solutions in integers.

This motivated us to determine solutions in integers to other forms of second degree equations having five variables. This aim of this paper is to determine solutions in integers to the quinary second degree $x\,y + X\,Y = (k^2 + 2\,k - 1)\,w^2$. Different sets of solutions in integers are determined to the above equation are obtained by converting it to a ternary quadratic equation through employing suitable linear transformations for which the integer solutions exist. Some fascinating observations satisfied by the solutions are given.

2. Method of analysis

Quinaryquadratic homogeneous polynomial for solving is

$$x\,y + X\,Y = (k^2 + 2\,k - 1)\,w^2 \,. \tag{2.1}$$

Various choices of solutions in integers to (2.1) are presented as follows.

3 Sets of integer solutions

Set III.1

Taking

$$x=(k+1)U-w, y=(k+1)U+w, X=(k+1)V-w, Y=(k+1)V+w \tag{3.1}$$

in (2.1) , we get well-known Pythagorean equation

$$U^2+V^2=w^2, \tag{3.2}$$

whose solutions may be taken as

$$U=2pq, V=p^2-q^2 , \tag{3.3}$$

and

$$w=p^2+q^2 . \tag{3.4}$$

Using (3.3) in (3.1) ,one has

$$\left.\begin{array}{l} x=2(k+1)pq-(p^2+q^2), y=2(k+1)pq+(p^2+q^2), \\ X=(k+1)(p^2-q^2)-(p^2+q^2), Y=(k+1)(p^2-q^2)+(p^2+q^2) \end{array}\right\} . \tag{3.5}$$

Thus,(3.4) and (3.5) give the required integer solutions to (2.1).

Observations III.1

1. Each of the following expressions

   $w, 2(y-x), 2(Y-X)$

   is a perfect square when p,q represent the legs of a Pythagorean triangle.

2. Each of the following expressions

   $w, 2^{3s-1}(y-x), 2^{3s-1}(Y-X)$

   is a cubical integer for the choices of p,q given by

   (i) $p=m(m^2+n^2), q=n(m^2+n^2)$,

   (ii) $p=m(m^2-3n^2), q=n(3m^2-n^2)$.

3. $y^3=x^3+8w^3+6xyw$
4. $Y^3=X^3+8w^3+6XYw$
5. Each of the following expressions $(K+1)(x+Y), (k+1)(y+X)$,$2(k+1)(x+y+X+Y)$

   is a perfect square when $p=2u^2+v^2-2uv, q=2uv$.

6. Each of the following expression

   $(k+1)(y+x)$, $2(k+1)(Y+X)$

   is expressed as the difference of two squares.

7. $\frac{y+x}{2(k+1)}$ is a square multiple of Triangular number when $p=q+1$

8. $\frac{y+x}{2(k+1)}$ is a square multiple of Pentagonal pyramidal number when $p=q(q+1)$

9. $\frac{y+x}{6(k+1)}$ is a square multiple of Triangular pyramidal number when $p=(q+1)(q+2)$

10. $\frac{y+x}{6(k+1)}$ is a square multiple of Square pyramidal number when $p=n(n+1)\,,q=2n+1$

It is to be mentioned here that choosing p,q suitably ,relations connecting $y+x$ with figurate numbers may be obtained.

11. $\frac{Y+X}{2(k+1)}+w$ is a square multiple of 2.

12. $\frac{Y+X}{2(k+1)}+2w$ is a perfect square when $p=2rs\,,q=3r^2-s^2$

13. $\frac{(y+x)\,(Y+X)}{8(k+1)^2}$ represents area of Pythagorean triangle.

Set III.2

Choosing

$$x=u+v,y=u-v,X=v+s,Y=v-s\,, \tag{3.6}$$

in (2.1), we get ternary quadratic diophantine equation

$$u^2-s^2=(k^2+2k-1)\,w^2\,, \tag{3.7}$$

Express (3.7) as

$$\frac{u+s}{(k^2+2k-1)\,w}=\frac{w}{u-s}=\frac{P}{Q},Q\neq 0\,.$$

(3.8)Solving the above system of double equations (3.8) , it is seen that

$$u=(k^2+2k-1)\,P^2+Q^2\,,s=(k^2+2k-1)\,P^2-Q^2\,, \quad (3.9)$$

and

$$w=2PQ. \tag{3.10}$$

$$\left.\begin{array}{l} x=(k^2+2k-1)\,P^2+Q^2+v,\,y=(k^2+2k-1)\,P^2+Q^2-v, \\ X=v+(k^2+2k-1)\,P^2-Q^2\,,Y=v-(k^2+2k-1)\,P^2+Q^2 \end{array}\right\}. \tag{3.11}$$

Thus,(3.10) and (3.11) give the required integer solutions to (2.1).

ObservationIII.2

Apart from (3.8) , (3.7) may be considered in the form of ratios as

$$\frac{u+s}{w}=\frac{(k^2+2k-1)w}{u-s}=\frac{P}{Q}, Q\neq 0$$

For this choice , we get a new pattern in integers for (2.1).

Set III.3

The substitution

$$x=(k+1)w, y=(k-1)w \quad (3.12)$$

in (2.1) gives second degree polynomial

$$XY=2kw^2 \quad (3.13)$$

On considering different choices of factorization in (3.13) , the respective solutions in integers for (2.1) are given by

Set III.3.1 $x=(k+1)w, y=(k-1)w, X=2w, Y=kw$,

Set III.3.2 $x=(k+1)w, y=(k-1)w, X=kw, Y=2w$,

***Set* III.3.3** $x=(k+1)w, y=(k-1)w, X=w^2, Y=2k$,

SetIII.3.4 $x=(k+1)w, y=(k-1)w, X=2w^2, Y=k$,

SetIII.3.5 $x=(k+1)w, y=(k-1)w, X=2wk, Y=w$,

SetIII.3.6 $x=(k+1)w, y=(k-1)w, X=w^2k, Y=2$,

Set III.3.7 $x=(k+1)w, y=(k-1)w, X=2w^2k, Y=1$.

SetIII.4

Considering

$$x=(k+1)X, y=(k-1)Y, k\neq 1,2 \quad (3.14)$$

in (2.1) ,we get

$$k^2XY=(k^2+2k-1)w^2 \quad (3.15)$$

On considering different choices of factorization in (3.15) , the respective solutions in integers for (2.1) are given by

SetIII.4.1 $x=(k+1)(k^2+2k-1)^{2s}\alpha, y=(k-1)(k^2+2k-1)\alpha,$
$X=(k^2+2k-1)^{2s}\alpha, Y=(k^2+2k-1)\alpha, w=k(k^2+2k-1)^s\alpha.$

Set III.4.2 $x = (k+1)\,(k^2+2k-1)^{2s+1}\,\alpha, y = (k-1)\,\alpha,$
$X = (k^2+2k-1)^{2s+1}\alpha, Y = \alpha, w = k\,(k^2+2k-1)^{s}\,\alpha.$

SetIII.4.3 $x = (k+1)\,(k^2+2k-1)^{s+1}\,\alpha, y = (k-1)\,(k^2+2k-1)^{s}\,\alpha,$
$X = (k^2+2k-1)^{s+1}\alpha, Y = (k^2+2k-1)^{s}\,\alpha, w = k\,(k^2+2k-1)^{s}\,\alpha.$

3. Conclusion

The second degree quinary quadratic equation represented by $xy + XY = (k^2+2k-1)\,w^2$ has been analysed for various choices of distinct solutions in integers . As second degree equations with multiple variables are plenty, one may search for solutions in integers to other forms of second degree equations with multiple variables.

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