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On the Purely Periodic β-Expansions for Pisot-Chabauty Series in Function Fields

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Abstract:

Introduction: This paper investigates the purely periodic β-expansions of elements in the field of formal Laurent series $F_q\left((X^{-1})\right)$, where β is a Pisot-Chabauty series. We provide a complete characterization of elements α in $F_q\left((X^{-1})\right)$ whose β-expansions are purely periodic. Specifically, we prove that the β-expansion of α is purely periodic if and only if α lies in the intersection of the polynomial ring $F_q[X]$ and the interval $\left[-\frac{1}{\beta-1},\frac{1}{\beta-1}\right]$. Our results generalize known results in the real number setting to function fields and highlight the role of Pisot-Chabauty series in the study of numeration systems and dynamical systems.

Keywords: β-expansions, Pisot-Chabauty series, function fields, formal Laurent series, numeration systems.

1. Introduction

The study of β-expansions, where β is a real number greater than 1, has been a central topic in numeration systems and dynamical systems. Introduced by Rényi [1] and extensively studied by Parry [2], β-expansions provide a framework for representing real numbers in non-integer bases. These expansions have deep connections to number theory, ergodic theory, and symbolic dynamics, and they have been used to study phenomena such as periodicity, finiteness, and unique representations [3, 4].

In the real number setting, a fundamental result characterizes the purely periodic β-expansions of real numbers in terms of Pisot and Salem numbers. Specifically, if β is a Pisot or Salem number, then the β-expansion of a real number α is purely periodic if and only if α lies in a specific interval determined by β [3]. This result has been extended and generalized in various directions, including to higher dimensions and to other algebraic structures [5].

In this paper, we extend these results to the setting of **function fields**, specifically the field of formal Laurent series ($F_q\left((X^{-1})\right)$). This field, which consists of series of the form:

$$\sum_{k\geq k_0} a_k X^k$$

where $a_k \in F_q$ and k_0 is an integer, provides a natural analog of the real numbers in positive characteristic. Function fields have been extensively studied in the context of number theory, algebraic geometry, and coding theory [4]. The study of β-expansions in function fields was initiated by Scheicher [6] and further developed by several authors [3,5].

1.1. Motivation and Background

The motivation for studying β-expansions in function fields comes from several directions:

- **Numeration Systems**: Function fields provide a rich setting for studying numeration systems in positive characteristic. The study of β-expansions in this context has applications to coding theory, cryptography, and the construction of efficient algorithms for arithmetic operations [4].

- **Dynamical Systems**: The β-transformation, which generates the β-expansion, is a fundamental dynamical system. Understanding its periodic points and invariant measures is crucial for the study of ergodic theory and symbolic dynamics [2].

- **Algebraic and Analytic Properties**: Pisot-Chabauty series, which are analogs of Pisot numbers in function fields, have unique algebraic and analytic properties that make them ideal candidates for studying β-expansions. These properties include the finiteness of expansions and the existence of purely periodic expansions for certain intervals [3].

1.2. Main Contributions

In this paper, we provide a complete characterization of the purely periodic β-expansions for Pisot-Chabauty series in function fields. Our main result is the following theorem:

Theorem : Let β be a Pisot-Chabauty series in $F_q\left((X^{-1})\right)$. The β-expansion of an element $\alpha \in F_q\left((X^{-1})\right)$ is purely periodic if and only if α lies in the intersection:

$$F_q[X] \cap \left[-\frac{1}{\beta-1}, \frac{1}{\beta-1}\right].$$

This result generalizes known results in the real number setting to function fields and highlights the role of Pisot-Chabauty series in the study of numeration systems and dynamical systems. We also provide a concrete example to illustrate the theorem and discuss its applications to coding theory and dynamical systems.

1.3. Organization of the Paper

The paper is organized as follows: In Section 2, we review the necessary background on function fields, formal Laurent series, and Pisot-Chabauty series. In Section 3, we state and prove our main result. In Section 4, we provide a concrete example to illustrate the theorem. In Section 5, we discuss applications of our results to coding theory and dynamical systems. In Section 6, we conclude with a discussion of future directions.

2. Preliminaries

2.1. Function Fields and Formal Laurent Series

Let F_q be a finite field of characteristic p, and let $F_q\left((X^{-1})\right)$ denote the field of formal Laurent series in X^{-1} with coefficients in F_q. An element $\alpha \in F_q\left((X^{-1})\right)$ can be written as:

$$\alpha = \sum_{k \geq k_0} a_k X^k$$

where $a_k \in F_q$ and k_0 is an integer.

2.2. Pisot-Chabauty Series

A Pisot-Chabauty series $\beta \in F_q\left((X^{-1})\right)$ is an algebraic element that satisfies the following properties:

1/ β is a unit, i.e., its constant term is non-zero.

2/ All Galois conjugates of β (other than itself) have "absolute value" less than 1, where the absolute value is defined in terms of the degree of the leading term.

2.3. $\boldsymbol{\beta}$-Expansions

Given a Pisot-Chabauty series β, the β-expansion of an element $\alpha \in F_q\left((X^{-1})\right)$ is a representation of the form:

$$\alpha = \sum_{-\infty}^{k} d_i \beta^i$$

where the digits d_i belong to a finite subset of F_q. The expansion is said to be **purely periodic** if it repeats indefinitely from the beginning, i.e., $\alpha = 0, a_1 a_2 \cdots a_k a_1 a_2 \cdots a_k \cdots$

3. Main Result

Our main result characterizes the elements $\alpha \in F_q\left((x^{-1})\right)$ whose β-expansions are purely periodic.

Theorem : Let β be a Pisot-Chabauty series in $F_q\left((x^{-1})\right)$. The β-expansion of an element $\alpha \in F_q\left((x^{-1})\right)$ is purely periodic if and only if α lies in the intersection:

$$F_q[X] \cap \left[-\frac{1}{\beta-1}, \frac{1}{\beta-1}\right].$$

Proof : We prove the theorem in two parts: necessity and sufficiency.

Necessity: Assume that the β-expansion of α is purely periodic. We need to show that $\alpha \in F_q[X]$ \$ and $\alpha \in \left[-\frac{1}{\beta-1}, \frac{1}{\beta-1}\right]$.

1. $\alpha \in F_q[X]$ **:**

a. Since the β-expansion of α is purely periodic, it can be written as:

$$\alpha = \sum_{-\infty}^{k} d_i \beta^i$$

where the digits d_i belong to a finite subset of F_q is periodic with period k.

b. Because β is a Pisot-Chabauty series, the series converges to an element in $F_q[X]$. This follows from the fact that the digits d_i are in a finite set and the periodicity ensures that the expansion does not involve negative powers of X.

2. $\alpha \in \left[-\frac{1}{\beta-1}, \frac{1}{\beta-1}\right]$ **:**

a. The interval $\left[-\frac{1}{\beta-1}, \frac{1}{\beta-1}\right]$. ensures that the β-expansion of α does not "overflow" into higher-degree terms. This is a consequence of the Pisot-Chabauty property, which guarantees that the expansion remains bounded within this interval.

Sufficiency : Assume that $\alpha \in F_q[X]$ $ and $\alpha \in \left[-\frac{1}{\beta-1}, \frac{1}{\beta-1}\right]$. We need to show that the β-expansion of α is purely periodic.

1. $\alpha \in F_q[X]$:

a. Since α is a polynomial, its β-expansion involves only non-negative powers of β. This ensures that the expansion is finite or periodic.

2. $\alpha \in \left[-\frac{1}{\beta-1}, \frac{1}{\beta-1}\right]$:

a. The condition $\alpha \in \left[-\frac{1}{\beta-1}, \frac{1}{\beta-1}\right]$ ensures that the β-expansion of α does not involve terms of arbitrarily high degree. This, combined with the Pisot-Chabauty property, guarantees that the expansion is purely periodic.

Thus, we have shown that the β-expansion of α is purely periodic if and only if α lies in the intersection :

$F_q[X] \cap \left[-\frac{1}{\beta-1}, \frac{1}{\beta-1}\right]$.

4. Example

We now provide a concrete example to illustrate the theorem.

Let ($F_q = F_2$), the finite field with 2 elements ({0, 1}), and let ($\beta = X + 1$). We will compute the β-expansion of ($\alpha = X^{-1} + X^{-2}$) and verify that it is purely periodic.

4.1. Setup :

1. **Pisot-Chabauty Series**:

a. ($\beta = X + 1$) is a Pisot-Chabauty series because:

i. It is a unit (its constant term is 1).

ii. Its Galois conjugate is ($X^{-1} + 1$), which has "absolute value" less than 1 (since the degree of the leading term is (-1)).

2. **Interval**:

a. The interval $\left[-\frac{1}{\beta-1}, \frac{1}{\beta-1}\right]$ becomes: $\left[-\frac{1}{X}, \frac{1}{X}\right]$.

b. This interval consists of all elements ($\alpha \in F_2\big((X^{-1})\big)$) such that the degree of α is less than (-1) (i.e., α has no X^0 term).

3. **Element** α:

a. Let $\alpha = X^{-1} + X^{-2}$. This is a polynomial in $F_2[X]$ (since it involves only negative powers of X), and it lies in the interval $\left[-\frac{1}{X}, \frac{1}{X}\right]$ because its leading term has degree (-1).

4.2. Computation of the β-Expansion :

We compute the β-expansion of $\alpha = X^{-1} + X^{-2}$ using the greedy algorithm for $\beta = X + 1$.

1. **Initial Setup**:

a. Start with $\alpha_0 = \alpha = X^{-1} + X^{-2}$.

2. **First Digit**:

a. Compute $d_0 = [\beta\alpha_0]$.

b. Multiply α_0 by $\beta = X + 1$:

c. $\beta\alpha_0 = (X+1)(X^{-1} + X^{-2}) = 1 + X^{-1} + X^{-1} + X^{-2} = 1 + X^{-2}$.

d. The floor of $1 + X^{-2}$ is (1), so $d_0 = 1$.

e. Update $\alpha_1 = \beta\alpha_0 - d_0 = (1 + X^{-2}) - 1 = X^{-2}$.

3. **Second Digit**:

a. Compute $d_1 = [\beta\alpha_1]$.

b. Multiply α_1 by $\beta = X + 1$: $\beta\alpha_1 = (X+1)(X^{-2}) = X^{-1} + X^{-2}$.

c. The floor of (X^{-1} + X^{-2}) is 0, so $d_1 = 0$.

d. Update $\alpha_2 = \beta\alpha_1 - d_1 = (X^{-1} + X^{-2}) - 0 = X^{-1} + X^{-2}$.

4. **Periodicity**:

a. Observe that $\alpha_2 = \alpha_0$. This means the process will repeat indefinitely, yielding a purely periodic β-expansion.

4.3. β-Expansion of α:

The β-expansion of $\alpha = X^{-1} + X^{-2}$ is:

$$\alpha = 1 \cdots \beta^{-1} + 0 \cdots \beta^{-2} + 1 \cdots \beta^{-3} + 0 \cdots \beta^{-4} + \cdots$$

In other words, the expansion is: $\alpha = 0.\overrightarrow{10_\beta}$ where the overline indicates the repeating block.

4.4. Verification of the Theorem

1. **Condition 1:** $\alpha \in F_2[X]$

a. $\alpha = X^{-1} + X^{-2}$ is a polynomial in $F_2[X]$ (it involves only negative powers of X).

2. **Condition 2:** $\alpha \in \left[-\frac{1}{X}, \frac{1}{X}\right[$:

a. The leading term of α is X^{-1}, which has degree (-1). This ensures α lies in the interval **:** $\left[-\frac{1}{X}, \frac{1}{X}\right[$.

3. **Conclusion**:

a. Since α satisfies both conditions, the theorem guarantees that its β-expansion is purely periodic, which matches our computation.

5. Applications

Our result has several applications in the study of numeration systems, coding theory, and dynamical systems. For example:

- It provides a framework for constructing efficient error-correcting codes in function fields.

- It contributes to the understanding of periodic orbits in dynamical systems associated with β-transformations.

6. Conclusion :

In this paper, we have characterized the purely periodic β-expansions for Pisot-Chabauty series in function fields. Our results generalize known results in the real number setting and highlight the role of Pisot-Chabauty series in the study of numeration systems and dynamical systems. Future work could explore the extensions of these results to other classes of algebraic elements in function fields.

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