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Sch-Continuous Function and T_{isch}-Spaces in Topological Spaces

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Abstract:

In this paper, some new concepts of generalized continuous and generalized of separation axioms T_i-space, $i = 0, 1, 2$ are studied. The notion of sch-continuous function and T_{isch} are given, and their basic characteristics are analyzed.

Keywords: sch-continuous, sch-open, sch-homeomorphism.

1. Introduction

One of the most important parts of research in general topology in the last few decades is the analysis of open sets, continuous functions, and generalized T_i-spaces, including their structural properties. In [4] Levine pioneered the research on generalized open sets with the publication of articles that featured $semi$-open sets, which was followed in [6] by α-open sets, and [5] preopen sets. Abbas [1] invented the concept of h-open sets. Other definitions of types of h-open set are αh-open, pre-h-open, $semi$-h-open set see ([2],[3] and [8]). This paper introduces the concept of $semi$-closure-h-continuous function (simple, sch-continuous function and T_{isch} and some of their properties are studied.

2. Sch-Continuous and Sch-Homeomorphism Functions.

Here we first define new classes of sch-continuous, sch-open, and sch-homeomorphism functions, and examine some of their properties.

Definition 2.1 [7]: For the set $B \subseteq X$ in (X, τ). It is a $semi$-$closure$-h-open set (in short sch- open set) if their h-open H such that $H \subseteq B \subseteq cl_h(H)$. τ_{sch} denoted to the family of all $semi$-$closure$-h-open set.

Definition 2.2: Let (X, τ) and (Y, ρ) be two topological space (T.S.) a function $f: X \to Y$ is said sch-continuous if $V \in \rho$ then $f^{-1}(V) \in \tau_{sch}$.

Example 2.3: Let $X = \{3,2,1\}$, $Y = \{1,2\}$, $\tau = \{\emptyset,\{1\}, X, \{3\}, \{2,1\}, \{1,3\}\}$, $\tau_h = \{\emptyset, X, \{1\}, \{3\}, \{1,2\}, \{1,3\}\}$, $\tau_{sch} = \{\emptyset, X, \{1\}, \{3\}, \{1,2\}, \{1,3\}\}$ and $\rho = \{\emptyset, Y, \{1\}\}$. The function $f: (X,\tau) \to (Y,\rho)$ define as: $f(1) = 1$, and $f(2) = f(3) = 2$, the function is sch-continuous.

Theorem 2.4: Any continuous function is sch-continuous.

Proof. Let a continuous function be $f: (X,\tau) \to (Y,\rho)$ and $V \in \rho$. Since, f is continuous, then $f^{-1}(V) \in \tau$. By (Proposition (3), [7]) every open set is sch-open, therefore $f^{-1}(V)$ is sch-open set in X. So, f is sch-continuous. ■

The following example show that a sch-continuous function may not be continuous.

Example 2.5: Let $X = \{3,2,1\}$, $Y = \{a,b,c\}$, $\tau = \{\emptyset,\{1\},X\}$, $\tau_{sch} = \{\emptyset, X, \{1\}, \{2,3\}\}$, and $\rho = \{\emptyset, Y, \{a,b\}\}$. The function $f: (X,\tau) \to (Y,\rho)$ define as: $f(1) = c$, $f(2) = a$ and $f(3) = b$, f is sch-continuous, but f isn't continuous.

Theorem 2.6: Every h-continuous function is sch-continuous.

Proof. Let a continuous function be $f: (X,\tau) \to (Y,\rho)$ and $V \in \rho$. Since, f is h-continuous, then $f^{-1}(V) \in \tau_h$. By (Proposition (4), [7]) every h-open set is sch-open set $f^{-1}(V)$ is sch-open set in X. So, f is sch-continuous in X. ■

Converse of the aforementioned theorem is false, as in example (2.6).

Example 2.7: Let $X = \{4,2,1,3\}$, $Y = \{3,2,1\}$, $\tau = \{\emptyset, X,\{2,1\}, \{2\}\}$, $\tau_h = \{\emptyset, X, \{1\}, \{2\}, \{1,2\}, \{1,3,4\}\}$, $\tau_{sch} = \{\emptyset, X, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{1,2,4\}, \{1,3,4\}, \{1,4\}, \{1,2,3\}\}$, and $\rho = \{\emptyset, Y\}$. The function $f: (X,\tau) \to (Y,\rho)$ define as: $f(1) = 1 = f(2), f(3) = 2, f(4) = 3$, f is sch-continuous, but f isn't h-continuous.

Theorem 2.8: Every sch-continuous function is sh-continuous.

Proof. Let a continuous function be $f: (X,\tau) \to (Y,\rho)$ and $V \in \rho$. Since, f is continuous, then $f^{-1}(V) \in \tau$. By (Proposition (5), [7]) every sch-open set is sh-open, then $f^{-1}(V)$ is sh-open set in X. So f is sh-continuous. ■

Converse of the aforementioned theorem is false, as in example (2.8).

Example 2.9: Let $X = \{4,2,1,3\}$, $Y = \{3,2,1\}$, $\tau = \{\emptyset, X, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$, $\tau_h = \{\emptyset, X, \{1\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \{2,3,4\}\}$ τ_{sh}=$\{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3\}, \{2,4\}, \{2,3,4\}, \{3,4\}\}$, τ_{sch}=$\{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3\}, \{2,4\}, \{2,3,4\}, \{3,4\}\}$, and ρ =$\{\emptyset, Y, \{1,3\}\}$. The function $f: (X,\tau) \to (Y,\rho)$ define as: $f(1) = f(2) = 1, f(3) = 2, f(4) = 3$, f is sh-continuous, but f isn't sch-continuous.

Example 2.10: Let $X = \{3,1,2\}$, $Y = \{c,b,a\}$ $\tau = \{\emptyset,\{2,1\}, X, \{2\}\}$, τ_h=$\{\emptyset, X, \{1\}, \{2\}, \{1,2\}, \{1,3\}\}$, τ_α =$\{\emptyset, X, \{2\}, \{1,2\}, \{2,3\}\}$, and τ_{sch}=$\{\emptyset, X, \{1\}, \{2\}, \{1,2\}, \{1,3\}\}$. The function $f: (X,\tau) \to (Y,\rho)$ define as: $f(1) = c, f(2) = a, f(3) = b$. When ρ =$\{\emptyset, Y, \{a,b\}\}$ then f is α-continuous but not sch-continuous. And when ρ=$\{\emptyset, Y, \{a\}\}$, Then f is sch-continuous but not α-continuous. It appears that there is no direct relationship between sch-continuity and α-continuity.

Example 2.11: Let $X = \{1,2,3\}$, $\tau = \{\emptyset, X, \{2\}, \{1,2\}\}$, τ_h={$\emptyset, X$, {1}, {2}, {1,2}, {1,3}}, τ_s={$\emptyset, X$, {2}, {1,2}, {2,3}} and τ_{sch}={$\emptyset, X$, {1}, {2}, {1,2}, {1,3}}. The function $f: (X, \tau) \to (Y, \rho)$ define as: $f(1) = c$, $f(2) = b$, $f(3) = a$. When $\rho = \{\emptyset, Y, \{c, b\}\}$ then f is semi-continuous but not sch-continuous. And when $\rho = \{\emptyset, Y, \{a\}\}$ then f is sch-continuous but not $semi$-continuous. Then we note that there is no relationship between sch-continuous and $semi$-continuous.

Theorem 2.12: If $f: (X, \tau) \to (Y, \rho)$ is sch-continuous and $g: (Y, \rho) \to (Z, \sigma)$ is continuous, then $g \circ f: (X, \tau) \to (Z, \sigma)$ is sch-continuous.

Proof: For $f: (X, \tau) \to (Y, \rho)$ define is sch- continuous and $g: (Y, \rho) \to (Z, \sigma)$ be continuous. Let $V \in \sigma$. Since, g is continuous, then $g^{-1}(V) \in \sigma$. Since, g is sch-continuous, so $f^{-1}((g^{-1}(V)) = (g \circ f)^{-1}(V)$ is sch-open set in X. We get, $g \circ f: (Y, \tau) \to (Z, \sigma)$ is sch- continuous. ■

Definition 2.13: Let $f: (X, \tau) \to (Y, \rho)$ be mapping if $f(v) \in \rho_{sch}$ for every $v \in \tau$, then f is called sch-open function.

Example 2.14: Let $f: (X, \tau) \to (Y, \rho)$ be an identity function, where $X = Y$ = {1, 2, 3}, and τ={$\emptyset, X$, {2,3}}, ρ={$\emptyset, Y$, {1}}, ρ_h={$\emptyset, Y$, {1}, {1,2}, {1,3}, {2,3}} and ρ_{sch}= {$\emptyset, Y$, {1}, {1,2}, {1,3}, {2,3}}. Clearly, the function f is sch-open.

Theorem 2.15: Any open function is also sch-open.

Proof. Let $f: (X, \tau) \to (Y, \rho)$ be an open function, and V be an open set in X. Since, f is open, $f(V)$ is open set in Y. By (Proposition (3), [7]) every open set is sch-open, which implies that $f(V)$ is sch-open in Y. Hence, f is sch-open. ■

The Converse of the aforementioned theorem is false, as shown in the example (2.16).

Example 2.16: In Example 2.14, The function $f: (X, \tau) \to (Y, \rho)$ is sch-open but not open.

Theorem 2.17: Every h-open function is sch-open.

Proof. Let $f: (X, \tau) \to (Y, \rho)$ be h-open function and V be any open set in X. Since, f is h-open, $f(V)$ is open set in Y. By (Theorem 2.1, [1]), every open set is h-open, and by (Proposition (4), [7]), every h-open set is sch-open. Thus, $f(V)$ is sch-open set in Y. Therefore, f is sch-open. ■

The converse of the theorem above is false, as demonstrated by the example (2.18).

Example 2.18: Let $f: (X, \tau) \to (Y, \rho)$ be an identity function, where X=Y={1, 2, 3}, τ={$\emptyset, X$, {1,3}}, ρ={$\emptyset, X$, {2}, {1,2}, {1}}, ρ_h={$\emptyset, X$, {2}, {1,2}, {1}} and ρ_{sch}={$\emptyset, X$, {2}, {2,3}, {1,2}, {1}, {1,3}}. Clearly, the function f is sch-open but not h-open.

Theorem 2.19: Every sch-open function is sh-open.

Proof. Let $f: (X, \tau) \to (Y, \rho)$ be open function and V be any open set in X. Since, f is open, then $f(V)$ is open set in Y. Since, every open set is sch-open by (Proposition (3), [7]) and every sch-open set is sh-open by (Proposition (5), [7]] then $f(v)$ is sch-open set in Y. Therefore, f is sch-open. ■

The Converse of the above theorem is false, as shown in the following example.

Example 2.20: Let $f:(X,\tau)\to(Y,\rho)$ be an identity function, where X=Y={1,2,3,4}, τ={$\emptyset$, X, {1,4}}, $\rho=\{\emptyset, X, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$, $\rho_h=\{\emptyset, X, \{1\},\{2\}, \{3\}, \{1,2\},\{1,3\},\{2,3\},\{1,2,3\},\{2,3,4\}\}$, ρ_{sh} ={$\emptyset$, X, {1}, {2}, {3}, {1,2}, {1,3}, {1,4}, {1,2,3}, {1,2,4}, {1,3,4}, {2,3}, {2,4}, {2,3,4}, {3,4}} and ρ_{sch}={$\emptyset$, X, {1}, {2}, {3}, {1,2}, {1,3}, {1,2,3}, {1,2,4}, {1,3,4}, {2,3}, {2,4}, {2,3,4}, {3,4}}. Clearly, the function f is sh-open but not sch-open

Theorem 2.21: Let $f:(X,\tau)\to(Y,\rho)$ is open and $g:(Y,\rho)\to(Z,\sigma)$ is sch-open, then $g\circ f:(X,\tau)\to(Z,\sigma)$ is sch-open.

Proof: Clear.

Definition 2.22: Let $f:(X,\tau)\to(Y,\rho)$ be a bijective function. f is a sch-homeomorphism if it is both sch-continuous and sch-open.

Theorem 2.23: If $f:(X,\tau)\to(Y,\rho)$ is homeomorphism, then f is sch-homeomorphism.

Proof: By Theorem (2.4), we get that every continuous mapping is sch-continuous. Furthermore, it can be deduced from Theorem (2.15) that every open mapping is also sch-open. Also, f is bijective then, f is sch-homeomorphism. ■

The converse of the above theorem is incorrect as in example (2.23)

Example 2.24: Let $f:(X,\tau)\to(Y,\rho)$ be the identity function, with X = {1,2,3} = Y, for τ ={$\emptyset$, X, {1,3}}, τ_{sch} = {$\emptyset$, Y, {1}, {2}, {3}, {1,2}, {1, 3}, {2,3}}, ρ = {$\emptyset$, Y, {2,3}} and ρ_{sch}={Ø, Y, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}. Clearly, f is sch-homeomorphism, but it isn't homeomorphism.

3. Some Properties of $sch-T_i$, $i=0,1,2.$

In the following, we prove some results on sch-T_i- space, $i=0,1,2$. We recall the following:

Definition 3.1: A T.S. (X,τ) is referred to as T_{0sch}-space, if for every $a,b\in X$, $a\neq b$, there exists sch-open set G, such that $G\ni a$, $G\not\ni b$.

Example 3.2: Let X={3,2,1}, $\tau=\{\emptyset, X, \{2\}, \{3\}, \{1,3\}, \{2,3\}\}$, $\tau_{sch}=\{\emptyset, X, \{2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. Then (X,τ) is T_{0sch}-space.

Definition 3.3: A T.S. (X,τ) is referred to as T_{1sch}--space if for every $a,b\in X$, $a\neq b$ there exists two sch-open sets G and H such that $G\ni a$, $G\not\ni b$, and $H\ni b$, $H\not\ni a$.

Example 3.4: Let X={1,2,3}, $\tau=\{\emptyset, X, \{2\}, \{1,2\}, \{2,3\}\}$, $\tau_{sch}=\{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$. Then (X,τ) is T_{2sch}-space.

Definition 3.5: A T.S. (X,τ) is referred to as T_{2sch}-space if for every $a,b\in X$, $a\neq b$ there exists two disjoint sch-open sets G and H such that $G\ni a$, and $H\ni b$.

Example 3.6: Let X={3,2,1}, $\tau=\{\emptyset, X, \{2\}, \{1,2\}, \{2,3\}\}$, $\tau_{sch}=\{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$. Then (X,τ) is T_{2sch}-space.

Theorem 3.7: Every T_0-space is T_{0sch}-space.

Proof: Let (X,τ) is T_0-space and $x,y\in X$, $x\neq y$. Then there exists open set $A\subseteq X$. Such that $x\in A, y\notin A$. By (Proposition (3), [7]) every open set is sch-open. Hence X is T_{0sch}-space. ■

Theorem 3.8: Every T_1-space is T_{1sch}-space.

Proof: Let (X,τ) is T_1-space and $x,y \in X$, $x \neq y$. Then there exists two open sets A,B in X such that $x \in A, y \notin A$ and $x \notin B, y \in B$. By (Proposition (3), [7]) every open set is sch-open. Therefore, X is T_{1sch}-space. ■

Theorem 3.9: Every T_2-space is T_{2sch}-space.

Proof: Let (X,τ) is T_2-space and $x,y \in X$, $x \neq y$. Then there exists two disjoint open sets A,B in X such that $x \in A$ and $y \in B$. By (Proposition (3), [7]) every open set is sch-open. Therefore, X is T_{2sch}-space. ■

Theorem 3.10:

1) Every T_{1sch}-space is T_{0sch}-space.

2) Every T_{2sch}-space is T_{1sch}-space.

Proof: (1) and (2) are Clear.

Theorem 3.11: A T.S. (X,τ) is T_{0sch}-space if and only if $cl_{sch}(\{x\}) \neq cl_{sch}(\{y\})$ for every pair of different points $x,y \in X$.

Proof: Let x,y be any two different points of T_{0sch}-space X. We show that $cl_{sch}(\{x\}) \neq cl_{sch}(\{y\})$. By hypothesis, assume that A is sch-open set such that $x \in A$ and $y \notin A$. Hence $y \in X - A$ and $X - A$ is sch-closed set. Therefore, $cl_{sch}(\{y\}) \subset X - A$. Hence $y \in cl_{sch}(\{y\})$, $x \notin X - A$. Hence $cl_{sch}(\{x\}) \neq cl_{sch}(\{y\})$.

Conversely: Assume that for all $x,y \in X$ with $x \neq y$, $cl_{sch}(\{x\}) \neq cl_{sch}(\{y\})$. Now, let $z \in X$ such that $z \in cl_{sch}(\{x\})$but $z \notin \{y\}^*$. If $x \in cl_{sch}(\{y\})$ then $\{x\} \subset cl_{sch}(\{y\})$ which implies that $cl_{sch}(\{x\}) \subset cl_{sch}(\{y\})$. Thus $x \in cl_{sch}(\{x\})$ and $z \notin cl_{sch}(\{y\})$. This is contradiction. Therefore, $x \in cl_{sch}(\{y\})$. Hence $X - \{y\}^*$ is sch-open set containing x but not y. So X is T_{0sch}-space. ■

Theorem 3.12:

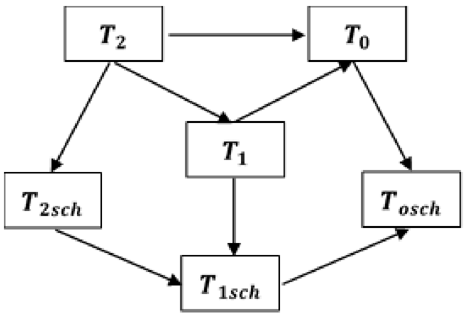
(1) Every T_{0h}-space is T_{0sch}-space.

(2) Every T_{1h}-space is T_{1sch}-space.

(3) Every T_{2h}-space is T_{2sch}-space.

Proof: Clear.

We have the following relations on T_{isch}-space, $i = 0,1,2$



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