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Using Simulation to Estimate Parameters for a Novel Extension Rayleigh Distribution, Properties and Failure Rate Data Application

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Abstract:

Simulation is necessary to indicate the superiority of methods for discriminating a particular distribution. This search was employed to illustrate the possibility of estimating the distribution parameters of a novel Generalized exponentiated Rayleigh distribution (NGERay), in several different ways Maximum Likelihood, Least Square, Weighted Least Square, Maximum Product Spacing, Anderson- Darling and Right Anderson-Darling methods. It has also been studied the statistical characteristics of the proposed distribution, including Moment, probability-weighted moments, Incomplete moments, and Quantile function. For examining the flexibility of the proposed distribution, it was compared with several previously studied distributions that have proven their flexibility in modeling failure data for various phenomena. The new distribution showed success over the rest of the aforementioned distributions, based on many criteria of conformity and suitability. The research produced encouraging results, and the researchers recommended using it to study evidence of failure, similar to the data analyzed.

Keywords: Rayleigh distribution, Moment, Probability Weighted Moments, Incomplete moments, Estimation methods.

1. Introduction

The primary objective of statistics is to identify effective statistical models that represent real-world phenomena using fixed probability distributions. Uncertain and risky natural life occurrences are represented using probability distributions. The purpose of the parameters of the extra figure is to induce deflection and alter the weight distribution on the tail. In addition, several models have been constructed by expanding and studying useful lifetime distributions in response to different parameters.

Recently, researchers have developed many extensions of the Rayleigh distribution to enhance its usefulness in medical science, physical analysis, and survival analysis. Several researchers have explored different generalizations of the Rayleigh distribution. Ahmad et al. [1] introduced the Weibull-Rayleigh distribution, Bhat and Ahmad [2] proposed an extension of the exponentiated Rayleigh distribution, Kilai et al. [3] proposed a versatile modification of the Rayleigh distribution, and Ahmad et al. [4], Saima et al. [5], Bhat et al. [6], Ahmad et al. [7], Habib et al. [8], Khaleel et al. [9] also contributed to this field. As may be shown from references [10-15] an extensive amount of research has used different estimation methods.

In 2025, Murtadha et al. [16] introduced a novel family of generalized distributions known as the Odd Generalized Exponential -G family. The cumulative distribution function (CDF) and probability density function (PDF) are:

$$G(x,\lambda,\theta)=\left[1-e^{-\lambda\left(\frac{(F(x))^2}{1-F(x)}\right)}\right]^{\theta} \quad , \ x>0,\theta,\lambda>0 \tag{1}$$

$$g(x,\lambda,\theta)=\theta\lambda F(x)f(x)(2-F(x))(1-F(x))^{-2}\left[1-e^{-\lambda\left(\frac{(F(x))^2}{1-F(x)}\right)}\right]^{\theta-1} e^{-\lambda\left(\frac{(F(x))^2}{1-F(x)}\right)} \tag{2}$$

The main motivation behind the study is the challenge of not modeling data for some real-world phenomena, which was an incentive for us to break this problem and present a new model. The scientific contribution to this study is to extend distribution based on that can have an important role in the data modeling phenomena that may arise due to rapid development in the real world.

This study included six parts, the first of which included the introduction, the second part included the method of generating the family, the third of which included the process of expansion, the fourth included the statistical characteristics of the distribution, the fifth of which included Simulation and the sixth of which included application.

2. Methods

2.1 Novel Generalized Exponential Rayleigh Distribution

Rayleigh Distribution has the CDF and PDF declared respectively as:[17]

$$F(x)=1-e^{-ax^2}, x>0, a>0 \tag{3}$$

$$f(x)=2axe^{-ax^2}, x\geq 0, a>0 \tag{4}$$

With a is the parameter of the scale,

Then by way of compensation (3) in (1) get the CDF of Odd Generalized Exponential Rayleigh Distribution (NGERay) as follows

$$G(x,\lambda,\theta,a)=\left[1-e^{-\lambda\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)}\right]^{\theta}, x>0,\theta,\lambda,a>0 \tag{5}$$

And by way of compensation (3) and (4) in (2) to be obtained on the pdf Odd Generalized Exponential Rayleigh Distribution (NGERay) as follows

$$g(x,\lambda,\theta,a) = 2\lambda\theta a x e^{ax^2}\left(1-e^{-ax^2}\right)\left(1+e^{-ax^2}\right)\left[1-e^{-\lambda\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)}\right]^{\theta-1} e^{-\lambda\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)} \qquad (6)$$

The survival function for the NGERay distribution can be obtained as follows:

$$S(x;\alpha,\gamma,\delta) = 1-\left[1-e^{-\gamma\left(\frac{\left(1-e^{-\delta x^2}\right)^2}{e^{-\delta x^2}}\right)}\right]^{\alpha} \quad x \geq 0, \alpha, \gamma, \delta > 0 \qquad (7)$$

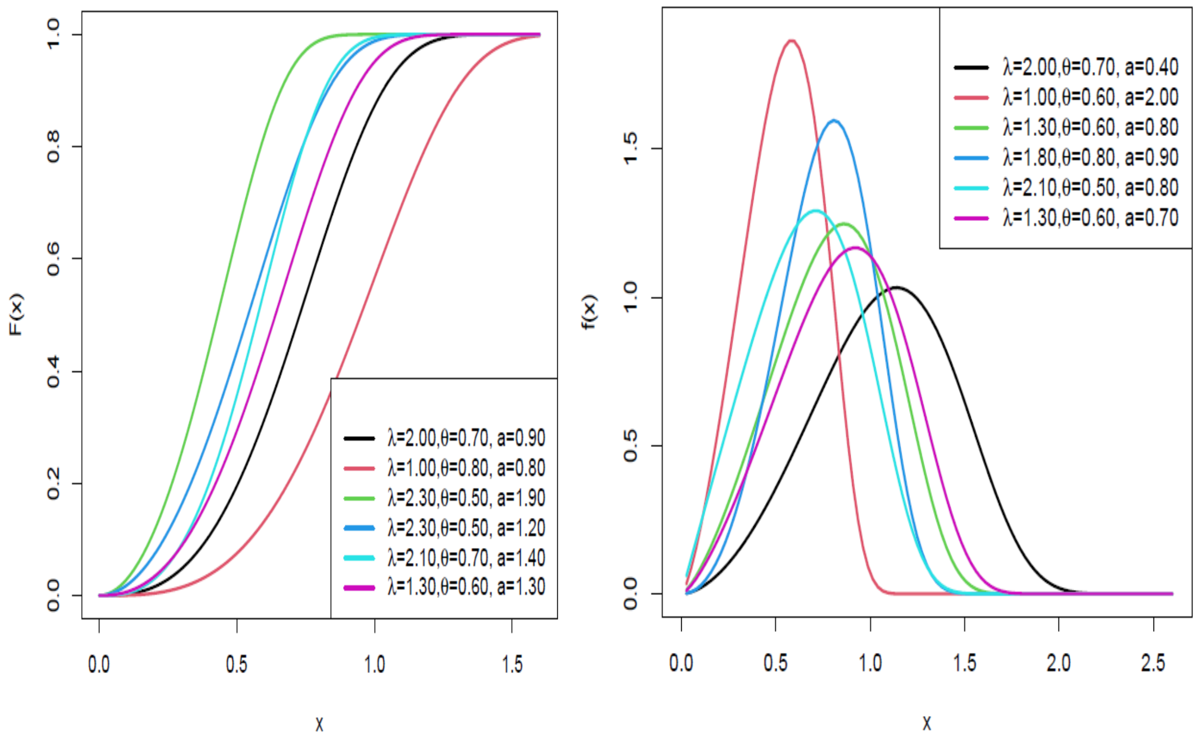

Figure 1. The CDFs and PDFs for the NGERay distribution

2.2 Expansion of NGERay

This section presents the expansion CDF, power CDF, pdf, and power pdf of the (NGERAY)

2.2.1 Expansion CDF of (NGERay)

The CDF expansion of the NGERay distribution can be obtained as follows:

$$G(x,\lambda,\theta,a)=\left[1-e^{-\lambda\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)}\right]^{\theta}=\sum_{c=0}^{\infty}(-1)^c\binom{\theta}{c}e^{-\lambda c\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)}$$

Now Using the generalized binomial theorem as

$$e^{-\lambda c\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)}=\sum_{v=0}^{\infty}\frac{(-1)^v}{v!}(\lambda c)^v\left(1-e^{-ax^2}\right)^{2v}e^{+vax^2}$$

The following is an application of the generalized binomial theorem:

$$\left(1-e^{-ax^2}\right)^{2v}=\sum_{b=0}^{\infty}(-1)^b\binom{2v}{b}e^{-\lambda b\left(ax^2\right)}$$

Then

$$G(x,\lambda,\theta,a)=e^{-ax^2(\lambda b-v)} \tag{8}$$

Where $\Phi=\sum_{c=0}^{\infty}\sum_{v=0}^{\infty}\sum_{b=0}^{\infty}\frac{(-1)^{c+v+b}}{v!}\binom{\theta}{c}(\lambda c)^v\binom{2v}{b}$

2.2.2 Expansion $(CDF)^{\omega}$ of (NGERay)

The following is how to get the power CDF expansion of the NGERay distribution:

$$G^{\omega}(x,\lambda,\theta,a)=\left[1-e^{-\lambda\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)}\right]^{\theta\omega}$$

Now Using the generalized binomial theorem as

$$\left[1-e^{-\lambda\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)}\right]^{\theta\omega}=\sum_{c=0}^{\infty}(-1)^c\binom{\theta\omega}{c}e^{-\lambda c\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)}$$

The following is an application of the generalized binomial theorem:

$$e^{-\lambda c\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)}=\sum_{v=0}^{\infty}\frac{(-1)^v}{v!}(\lambda c)^v\left(1-e^{-ax^2}\right)^{2v}e^{+vax^2}$$

And $\left(1-e^{-ax^2}\right)^{2v}=\sum_{b=0}^{\infty}(-1)^b\binom{2v}{b}e^{-\lambda b\left(ax^2\right)}$

Then

$$G^{\omega}(x,\lambda,\theta,a) = T e^{-ax^2(\lambda b - v)} \quad (9)$$

$$where T = \sum_{c=0}^{\infty}\sum_{v=0}^{\infty}\sum_{b=0}^{\infty}\frac{(-1)^{c+v+b}}{v!}\binom{\theta\omega}{c}(\lambda c)^{v}\binom{2v}{b}$$

2.2.3 Expansion pdf of (NGERay)

You may obtain the PDF expansion of the NGERay distribution by following these steps:

$$g(x,\lambda,\theta,a) = 2\lambda\theta a x e^{ax^2}\left(1-e^{-ax^2}\right)\left(1+e^{-ax^2}\right)\left[1-e^{-\lambda\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)}\right]^{\theta-1} e^{-\lambda\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)}$$

Now Using the generalized binomial theorem as

$$\left[1-e^{-\lambda\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)}\right]^{\theta-1} = \sum_{c=0}^{\infty}(-1)^{c}\binom{\theta-1}{c}e^{-\lambda c\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)}$$

The following is an application of the generalized binomial theorem:

$$g(x,\lambda,\theta,a) = 2\lambda\theta a x e^{ax^2}\left(1-e^{-ax^2}\right)\left(1+e^{-ax^2}\right)\sum_{c=0}^{\infty}(-1)^{c}\binom{\theta-1}{c}e^{-\lambda\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)(c+1)}$$

By using the generalized binomial theorem as

$$e^{-\lambda\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)(c+1)} = \sum_{v=0}^{\infty}\frac{(-1)^{v}}{v!}\lambda^{v}(c+1)^{v}e^{vax^2}\left(1-e^{-ax^2}\right)^{2v}$$

Thus

$$g(x,\lambda,\theta,a) = \sum_{c=0}^{\infty}\sum_{v=0}^{\infty}\frac{(-1)^{c+v}}{v!}2a\theta\lambda^{v+1}(c+1)^{v}\binom{\theta-1}{c}e^{ax^2(v+1)}\left(1-e^{-ax^2}\right)^{2v+1}\left(1+e^{-ax^2}\right)$$

And using the generalized binomial theorem as

$$\left(1-e^{-ax^2}\right)^{2v+1} = \sum_{b=0}^{\infty}(-1)^{b}\binom{2v+1}{b}e^{-abx^2}$$

$$g(x,\lambda,\theta,a) = \sum_{c=0}^{\infty}\sum_{v=0}^{\infty}\sum_{b=0}^{\infty}\frac{(-1)^{c+v+b}}{v!}2a\theta\lambda^{v+1}(c+1)^{v}\binom{\theta-1}{c}\binom{2v+1}{b}\, x\, e^{-a(b-v-1)x^2}\left(1+e^{-ax^2}\right)$$

$$g(x,\lambda,\theta,a) = \Upsilon\, x\, e^{-a(b-v-1)x^2} + \Upsilon\, x\, e^{-a(b-v)x^2} \qquad (10)$$

Where $\Upsilon = \sum_{c=0}^{\infty}\sum_{v=0}^{\infty}\sum_{b=0}^{\infty}\frac{(-1)^{c+v+b}}{v!}2a\theta\lambda^{v+1}(c+1)^v\binom{\theta-1}{c}\binom{2v+1}{b}$

2.2.4 Expansion $(pdf)^{\omega}$ of (NGERay)

You may obtain the power PDF expansion of the NGERay distribution by following these steps:

$$g^{\omega}(x,\lambda,\theta,a) = (2\lambda\theta a x)^{\omega} e^{a\omega x^2}\left(1-e^{-ax^2}\right)^{\omega}\left(1+e^{-ax^2}\right)^{\omega}\left[1-e^{-\lambda\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)}\right]^{\theta\omega-\omega} e^{-\lambda\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)^{\omega}}$$

Using the generalized binomial theorem as

$$\left[1-e^{-\lambda\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)}\right]^{\theta\omega-\omega} = \sum_{c=0}^{\infty}(-1)^c\binom{\theta\omega-\omega}{c}e^{-\lambda c\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)}$$

Now we substitute the equivalent of (3) and (4) into (13) we got

$$g^{\omega}(x,\lambda,\theta,a) = (2\lambda\theta a x)^{\omega} e^{a\omega x^2}\left(1-e^{-ax^2}\right)^{\omega}\left(1+e^{-ax^2}\right)^{\omega}\sum_{c=0}^{\infty}(-1)^c\binom{\theta\omega-\omega}{c}e^{-\lambda\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)(c+\omega)}$$

Using the generalized binomial theorem as

$$e^{-\lambda\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)(c+\omega)} = \sum_{v=0}^{\infty}\frac{(-1)^v}{v!}\lambda^v(c+\omega)^v e^{vax^2}\left(1-e^{-ax^2}\right)^{2v}$$

$$g^{\omega}(x,\lambda,\theta,a) = \sum_{c=0}^{\infty}\sum_{v=0}^{\infty}\frac{(-1)^{c+v}}{v!}\binom{\theta\omega-\omega}{c}(2a\theta)^{\omega}\lambda^{v+\omega}(c+\omega)^v\binom{\theta-1}{c}x^{\omega}e^{ax^2(v+\omega)}\left(1-e^{-ax^2}\right)^{2v+\omega}\left(1+e^{-ax^2}\right)^{\omega}$$

By using the generalized binomial theorem as

$\left(1-e^{-ax^2}\right)^{2v+\omega}=\sum_{b=0}^{\infty}(-1)^b\binom{2v+\omega}{b}e^{-abx^2}$ and $\left(1+e^{-ax^2}\right)^{\omega}=\sum_{n=0}^{\infty}\binom{\omega}{n}e^{-\omega a x^2}$

hence that

$$g^{\omega}(x,\lambda,\theta,a)=\Psi x^{\omega}e^{-ax^2(b-v)} \quad (11)$$

Where $\Psi=\sum_{c=0}^{\infty}\sum_{v=0}^{\infty}\sum_{b=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{c+v+b}}{v!}\binom{\theta\omega-\omega}{c}\binom{2v+\omega}{b}(2a\theta)^{\omega}\lambda^{v+\omega}(c+\omega)^{v}\binom{\theta-1}{c}\binom{\omega}{n}$

3. Mathematical properties of NGERay

In this section, we will explore several important statistical properties of the NGERAY distribution The characteristics we will cover include, moments, incomplete moments, Probability Weighted Moments (PWMs), Renyi Entropy, Quantile function and entropy

3.1 Moment

The moments of the NGERay distribution are obtained by the following relationship [18-19]

$$M_r=E(X^r)=\int_0^{\infty}x^r g(x,\lambda,\theta,a)dx \quad (12)$$

Now makeup Eq (9) in Eq (11) we obtain that

$$M_r=\int_0^{\infty}x^r\left(\Upsilon\, x\, e^{-a(b-v-1)x^2}+\Upsilon\, x\, e^{-a(b-v)x^2}\right)dx$$

$$M_r=\int_0^{\infty}\Upsilon x^{r+1}e^{-a(b-v-1)x^2}dx+\int_0^{\infty}\Upsilon x^{r+1}e^{-a(b-v)x^2}\,dx$$

We take the part $\int_0^{\infty}\Upsilon x^{r+1}e^{-a(b-v-1)x^2}dx$

Put $y=a(b-v-1)x^2\Rightarrow x^2=\frac{y}{a(b-v-1)}\ \Rightarrow\ x=\frac{y^{\frac{1}{2}}}{\left(a(b-v-1)\right)^{\frac{1}{2}}}\ \Rightarrow\frac{dx}{dy}=\frac{y^{\frac{-1}{2}}}{2\left(a(b-v-1)\right)^{\frac{1}{2}}}\Rightarrow$
$dx=\frac{dy}{2y^{\frac{1}{2}}(a(b-v-1))^{\frac{1}{2}}}$

$$\Rightarrow\Upsilon\int_0^{\infty}\left(x=\frac{y^{\frac{1}{2}}}{\left(a(b-v-1)\right)^{\frac{1}{2}}}\right)^{r+1}e^{y}\frac{dy}{2y^{\frac{1}{2}}(a(b-v-1))^{\frac{1}{2}}}=\frac{\Upsilon}{2(a(b-v-1))^{\frac{r}{2}+1}}\int_0^{\infty}y^{\frac{r}{2}}e^{y}dy$$

And by the same way obtaining that $\int_0^{\infty}\Upsilon x^{r+1}e^{-a(b-v)x^2}\,dx$, thus

$$M_r=\frac{\Upsilon}{2(a(b-v-1))^{\frac{r}{2}+1}}\Gamma\left(\frac{r}{2}+1\right)+\frac{\Upsilon}{2(a(b-v))^{\frac{r}{2}+1}}\Gamma\left(\frac{r}{2}+1\right) \quad (13)$$

The mean and variance of the NGERAY distribution can be obtained from the equation as follows

$$M_1=\frac{\Upsilon}{2(a(b-v-1))^{\frac{3}{2}}}\Gamma\left(\frac{3}{2}\right)+\frac{\Upsilon}{2(a(b-v))^{\frac{3}{2}}}\Gamma\left(\frac{3}{2}\right) \quad (14)$$

$$M_2 = \frac{\Upsilon}{2(a(b-v-1))^2} + \frac{\Upsilon}{2(a(v-b))^2} \quad (15)$$

$$M_3 = \frac{\Upsilon}{2(a(b-v-1))^{\frac{5}{2}}}\Gamma\left(\frac{5}{2}\right) - \frac{\Upsilon}{2(a(b-v))^{\frac{5}{2}}}\Gamma\left(\frac{5}{2}\right) \quad (16)$$

$$M_4 = \frac{2\Upsilon}{2(a(b-v-1))^3} + \frac{2\Upsilon}{2(a(b-v))^3} \quad (17)$$

The variance can be found using the following form

$$Var = M_2 - (M_1)^2$$

$$Var = \left(\frac{\Upsilon}{2(a(b-v-1))^2} + \frac{\Upsilon}{2(a(b-v))^2}\right) - \left(\frac{\Upsilon}{2(a(b-v-1))^{\frac{3}{2}}}\Gamma\left(\frac{3}{2}\right) + \frac{\Upsilon}{2(a(b-v))^{\frac{3}{2}}}\Gamma\left(\frac{3}{2}\right)\right)^2$$

(18)

3.2 Probability Weighted Moments (PWMs)

Theorem: the probability-weighted moments of NGERay distribution as follows:

$$\rho_{q,\omega} = \frac{T}{2(a(\lambda b-v))^{\frac{q}{2}+1}}\Gamma\left(\frac{q}{2}+1\right)$$

proof: Since PWMs for a random variable X are defined as

$$\rho_{q,\omega} = E\left(X^q(G^\omega(x))\right) = \int_{-\infty}^{\infty} x^q G^\omega(x) g(x) dx$$

Now recoup Eq (10) in Eq above we get that

$$\rho_{q,\omega} = \int_0^{\infty} x^q \left(Te^{-ax^2(\lambda b-v)}\right) dx$$

$$\rho_{q,\omega} = T\int_0^{\infty} x^{\ q} e^{-ax^2(\lambda b-v)}\, dx$$

Put $y = ax^2(\lambda b - v) \Rightarrow x^2 = \frac{y}{a(\lambda b-v)} \Rightarrow x = \frac{y^{\frac{1}{2}}}{(a(\lambda b-v))^{\frac{1}{2}}} \Rightarrow \frac{dx}{dy} = \frac{y^{\frac{-1}{2}}}{2(a(\lambda b-v))^{\frac{1}{2}}} \Rightarrow dx = \frac{dy}{2y^{\frac{1}{2}}(a(\lambda b-v))^{\frac{1}{2}}}$

$$\Rightarrow T\int_0^{\infty}\left(\frac{y^{\frac{1}{2}}}{(a(\lambda b-v))^{\frac{1}{2}}}\right)^q e^{-y}\frac{dy}{2y^{\frac{1}{2}}(a(\lambda b-v))^{\frac{1}{2}}} = \frac{T}{2(a(\lambda b-v))^{\frac{q}{2}+1}}\int_0^{\infty} y^{\frac{q-1}{2}} e^{-y} dy$$

thus

$$\rho_{q,\omega} = \frac{T}{2(a(\lambda b-v))^{\frac{q}{2}+1}}\Gamma\left(\frac{q+1}{2}\right) \quad (19)$$

3.3 Incomplete moments

Theorem: The incomplete moments of NGERay distribution by formula [20-21]

$$M_r(y) = \frac{\Upsilon}{2(a(b-v-1))^{\frac{r}{2}+1}}\Gamma\left(\frac{r}{2}+1, -ay^2(b-v-1)\right) + \frac{\Upsilon}{2(a(b-v))^{\frac{r}{2}+1}}\Gamma\left(\frac{r}{2}+1, -ay^2(b-v)\right)$$

Proof: incomplete moments of the random variable X are given by the formula

$$M_r(y) = \int_0^y x^r\, f(x) dx \qquad (20)$$

Now we recoup Eq (9) in Eq (20) as

$$M_r(y) = \int_0^y x^r \left(\Upsilon\, x\, e^{-a(b-v-1)x^2} + \Upsilon\, x\, e^{-a(b-v)x^2}\right) dx$$

$$M_r(y) = \Upsilon \int_0^y x^{r+1} e^{-a(b-v-1)x^2}\, dx + \Upsilon \int_0^y x^{r+1} e^{a(b-v)x^2}\, dx$$

Now take the part that $\Upsilon \int_0^y x^{r+1} e^{-a(b-v-1)x^2}\, dx$

Let $t = a(b-v-1)x^2 \Rightarrow x = \frac{t^{\frac{1}{2}}}{(a(b-v-1))^{\frac{1}{2}}}$

If $x = 0 \Rightarrow t = 0$ and if $x = y \Rightarrow t = ay^2(b-v-1) \Rightarrow dx = \frac{dt}{2t^{\frac{1}{2}}(a(b-v-1))^{\frac{1}{2}}}$

$$\Rightarrow \Upsilon \int_0^{ay^2(b-v-1)} \left(\frac{t^{\frac{1}{2}}}{(a(b-v-1))^{\frac{1}{2}}}\right)^{r+1} e^t \left(\frac{dt}{2t^{\frac{1}{2}}(a(b-v-1))^{\frac{1}{2}}}\right) = \frac{\Upsilon}{2(a(b-v-1))^{\frac{r}{2}+1}} \int_0^{ay^2(b-v-1)} t^{\frac{r}{2}} e^t dt$$

And by the same way obtaining that $\Upsilon \int_0^y x^{r+1} e^{a(b-v)x^2}\, dx$, hence

$$M_r(y) = \frac{\Upsilon}{2(a(b-v-1))^{\frac{r}{2}+1}}\Gamma\left(\frac{r}{2}+1, -ay^2(b-v-1)\right) + \frac{\Upsilon}{2(a(b-v))^{\frac{r}{2}+1}}\Gamma\left(\frac{r}{2}+1, -ay^2(b-v)\right)$$
(21)

3.4 Renyi Entropy

Theorem: The Renyi entropy of the NGERay distribution as follows [22]

$$I_R(\omega) = \frac{\Psi}{2(a(b-v))^{\frac{\omega+1}{2}}}\Gamma\left(\frac{\omega+1}{2}\right)$$

Proof: since Rennie entropy of the random variable X is defined by

$$I_R(\omega) = \frac{1}{1-\omega}\ell og \int_0^\infty g(x, \lambda, \theta, a)^\omega\, dx$$
(22)

Now we make up Eq (10) in Eq (21) as

$$I_R(\omega) = \frac{1}{1-\omega}\ell og \int_0^\infty \Psi x^\omega e^{-ax^2(b-v)}\, dx$$

Put $y = ax^2(b-v) \Rightarrow x^2 = \frac{y}{a(b-v)} \Rightarrow x = \frac{y^{\frac{1}{2}}}{(a(b-v))^{\frac{1}{2}}} \Rightarrow \frac{dx}{dy} = \frac{y^{\frac{-1}{2}}}{2(a(b-v))^{\frac{1}{2}}} \Rightarrow dx = \frac{dy}{2y^{\frac{1}{2}}(a(b-v))^{\frac{1}{2}}}$

$$I_R(\omega) = \Psi \int_0^{\infty} \left(\frac{y^{\frac{1}{2}}}{(a(b-v))^{\frac{1}{2}}} \right)^{\omega} e^{-y} \frac{dy}{2y^{\frac{1}{2}}(a(b-v))^{\frac{1}{2}}} \Rightarrow I_R(\omega) = \frac{\Psi}{2(a(b-v))^{\frac{\omega+1}{2}}} \int_0^{\infty} y^{\frac{\omega-1}{2}} e^{-y} dy$$

$$I_R(\omega) = \frac{-\Psi}{2(a(b-v))^{\frac{\omega+1}{2}}} \Gamma\left(\frac{\omega+1}{2}\right) \tag{23}$$

3.5 The Quantile function

The Quantile function of NGERay distribution can be retained by reversing CDF, which is defined in

$$G(x, \lambda, \theta, a) = \left[1 - e^{-\lambda \left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}} \right)} \right]^{\theta}$$

Now the CDF equal to the u

$$\left[1 - e^{-\lambda \left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}} \right)} \right]^{\theta} = u$$

After using some algebraic operations, we obtain that

$$x = \sqrt{\frac{-\ln\left(\frac{-\left(2+\frac{-1}{\lambda}\ln\left(1-u^{\frac{1}{\theta}}\right)\right) \mp \sqrt{\left(2+\frac{-1}{\lambda}\ln\left(1-u^{\frac{1}{\theta}}\right)\right)^2 - 4}}{2} \right)}{a}} \tag{24}$$

4. Methods of Estimation

This part, will explore some techniques to estimate the parameters of the NGERay distribution**,** such as the maximum Product Spacing approach, the maximum likelihood method, the Weighted Least Squares method, the Right Anderson-Darling Method, and the Least Squares method.

4.1 Least-Square Estimation Method

By minimizing, may get the LSE of the parameters of the NGERay distribution [23]:

$$V(\Theta) = \sum_{m=1}^{n} \left[G(x_m/\Theta) - \frac{m}{1+n} \right]^2$$

Where $G(x_m)$ as defined in Eq (5)

4.2 Weighted Least-Squares Estimation Method

To find the (WLSE) of the NGERay distribution, minimize the following equation

$$W(\Theta) = \sum_{m=1}^{n} \frac{(n+1)^n (n+2)}{m(n-m+1)} \left[G(x_m / \Theta) - \frac{m}{1+n}\right]^2$$

Where $G(x)$ as defined in Eq (5)

4.3 Maximum Product Spacing Method

Minimizing the following equation will give you the maximum probability of the NGERay distribution (MPSE).

$$M_i(x) = G(x_i) - G(x_{i-1}): i = 1, \dots, n+1$$

Where $G(x)$ as defined in Eq (5)

4.4 Anderson- Darling Method

$$A(\varphi) = -n - \frac{1}{n} \sum_{m=1}^{n} (2m-1)\, ln[G(x_m)] + ln[S(x_m)]$$

Where $G(x_m)$ and $S(x_m)$ as defined in Eq (5) and (7)

4.5 Right Anderson-Darling Method

To obtain (RADE) for NGERay distribution parameters:

$$R(\alpha, \gamma, \delta) = \frac{n}{2} - 2 \sum_{m=1}^{n} G(x_m) - \frac{1}{n} \sum_{m=1}^{n} (2m-1)\, ln[S(x_{n+1-m:n})]$$

Where $G(x_m)$ and $S(x_m)$ as defined in Eq (5) and (7)

4.6 Maximum Likelihood Method

Suppose $x_1, x_2, \dots, x_n$ is a random sample of n size of the NGERay distribution. Then the probability function of the corresponding logarithm is given by [24-26]

$L(\lambda,\theta,a \backslash x) = \prod_{i=1}^{n} g(x_i,\lambda,\theta,a)$ and let $I = \ln L(\gamma \backslash x)$ where $\gamma = \theta, \lambda, a$

$$I = (2\lambda\theta a)^n \prod_{i=1}^{n} xi\, e^{\sum_{i=1}^{n} ax^2} \prod_{i=1}^{n}\left(1 - e^{-ax_i^2}\right) \prod_{i=1}^{n}\left(1 + e^{-ax_i^2}\right) \prod_{i=1}^{n} \left[1 - e^{-\lambda\left(\frac{\left(1-e^{-axi^2}\right)^2}{e^{-axi^2}}\right)}\right]^{\theta-1} e^{-\lambda \sum_{i=1}^{n}\left(\frac{\left(1-e^{-ax^2}\right)^2}{e^{-ax^2}}\right)}$$

Then

$$\frac{dI}{d\lambda}=\frac{n}{\lambda}+(\theta-1)\sum_{i=1}^{n}\frac{\left(\frac{\left(1-e^{-ax_i^2}\right)^2}{e^{-ax_i^2}}\right)}{1-e^{-\lambda\left(\frac{\left(1-e^{-ax_i^2}\right)^2}{e^{-ax_i^2}}\right)}}-\sum_{i=1}^{n}\left(\frac{\left(1-e^{-ax_i^2}\right)^2}{e^{-ax_i^2}}\right) \quad (25)$$

$$\frac{dI}{d\theta}=\frac{n}{\theta}+ln\sum_{i=1}^{n}\left[1-e^{-\lambda\left(\frac{\left(1-e^{-axi^2}\right)^2}{e^{-axi^2}}\right)}\right] \quad (26)$$

$$\frac{dI}{da}=\frac{n}{a}+\sum_{i=1}^{n}x_i^2+\sum_{i=1}^{n}\frac{x_i^2e^{-ax_i^2}}{1-e^{-ax_i^2}}+\sum_{i=1}^{n}\frac{-x_i^2e^{-ax_i^2}}{1-e^{-ax_i^2}}$$

$$+(\theta-1)\sum_{i=1}^{n}\frac{\left[-2\lambda x_i\left(1-e^{-ax_i^2}\right)-\lambda x_i^2e^{ax_i^2}\left(1-e^{-ax_i^2}\right)^2\right]e^{-\lambda\left(\frac{\left(1-e^{-ax_i^2}\right)^2}{e^{-ax_i^2}}\right)}}{1-e^{-\lambda\left(\frac{\left(1-e^{-ax_i^2}\right)^2}{e^{-ax_i^2}}\right)}}$$

$$-\lambda\sum_{i=1}^{n}2x_i^2(1-e^{-ax_i^2})+\frac{x_i^2\left(1-e^{-ax_i^2}\right)^2}{e^{-ax_i^2}} \quad (27)$$

It becomes challenging to solve equations (25), (26), and (27) manually once they are equal to zero. For this investigation, mathematical software is required. R has been employed.

5. Results

5.1 Simulation study

Simulation is a major tool for demonstrating the superiority of methods for estimating parameters of a statistical distribution, as it relies primarily on the quantum function, which is simulated using the Monte Carlo method. Using a Monte Carlo experiment, we examine the asymptotic behavior of MLEs, LSE, WLSE, MPSE, ADE, and RTADE methods for NGERay distribution parameters. The research looks at four different combinations of parameters. Over the course of 1000 repetitions, we examine four different sample sizes: 30, 60,90, 120, and 200. To prove the accuracy of the simulation of the parameters of the proposed distribution, we relied on three error criteria, which included (Mean, Root Mean Square Error (RMSE), and Bais), to indicate that the consistency condition was met, with the aim of determining the best methods used to estimate the parameters presented in the theoretical aspect. Table (1) refers to the simulation result for ($\lambda=0.8,\theta=0.7,a=0.3$) and Table (2) refers to the simulation result for ($\lambda=0.7,\theta=0.2,a=0.05$) Table (3) refers to the simulation result for ($\lambda=1.5,\theta=0.3,a=0.07$)and Table (4) refers to the simulation result for ($\lambda=2,\theta=0.2,a=0.09$)for the different sample sizes 30,60,90,120 and 200.

Table 1. Mean, RMSE, and Bias with the MLE, LSE, WLSE, MPSE, ADE, and RTADE for the NGERay model with $\lambda = 0.8, \theta = 0.75, a = 0.3$

n		Est. Par.	MLE	LSE	WLSE	MPSE	ADE	RTADE
30	Mean	$\hat{\lambda}$	0.8213269	0.7469153	0.7617419	0.9172168	0.7939249	0.7953831
		$\hat{\theta}$	1.4782713	2.310696	2.778762	1.4541204	2.806463	2.743779
		$\hat{a}$	0.0400096	0.0385931	0.0369428	0.0431282	0.0360123	0.0365262
	RMSE	$\hat{\lambda}$	0.2637455	0.2571766	0.2483263	0.3372940	0.2515078	0.2898153
		$\hat{\theta}$	3.0669044	3.317313	4.086646	3.0165310	4.302484	4.114784
		$\hat{a}$	0.0290989	0.0311239	0.0305907	0.0320856	0.0278671	0.0279029
	Bias	$\hat{\lambda}$	0.0213269	0.0530846	0.0382580	0.1172168	0.0060750	0.0046168
		$\hat{\theta}$	0.9282713	1.560696	2.028762	0.7041204	2.056463	1.993779
		$\hat{a}$	0.0100096	0.0085931	0.0069428	0.0131282	0.0060123	0.0065262
60	Mean	$\hat{\lambda}$	0.8103316	0.7643233	0.7833485	0.8590910	0.7975604	0.7981643
		$\hat{\theta}$	1.5573801	1.994327	2.494632	1.4209436	2.453584	2.435926
		$\hat{a}$	0.0340768	0.0352392	0.0326618	0.0371695	0.0320342	0.0322864
	RMSE	$\hat{\lambda}$	0.1747118	0.1921438	0.1798097	0.2023276	0.2023276	0.2038983
		$\hat{\theta}$	2.8867611	2.621552	3.542232	2.6835678	3.534133	3.491509
		$\hat{a}$	0.0198377	0.0236804	0.0214360	0.0206627	0.0197815	0.0202041
	Bias	$\hat{\lambda}$	0.0103316	0.0356766	0.0166514	0.0590910	0.0054395	0.0038356
		$\hat{\theta}$	0.8673801	1.244327	1.744632	0.6709436	1.703584	1.685926
		$\hat{a}$	0.0040768	0.0052392	0.0026618	0.0071695	0.0020342	0.0022864
90	Mean	$\hat{\lambda}$	0.8047179	0.7693046	0.7855675	0.8358856	0.7951142	0.7987661
		$\hat{\theta}$	1.7188169	1.805628	2.157754	1.3222768	2.140925	2.045214
		$\hat{a}$	0.0330673	0.0342962	0.0320706	0.0357866	0.0316422	0.0316541
	RMSE	$\hat{\lambda}$	0.1376280	0.1500899	0.1378070	0.1499003	0.1356535	0.1659914
		$\hat{\theta}$	2.8050957	2.375211	3.070392	2.4635494	3.102579	2.871856
		$\hat{a}$	0.0170412	0.0202553	0.0182001	0.0173958	0.0169514	0.0172342
	Bias	$\hat{\lambda}$	0.0047179	0.0306953	0.0144324	0.0358856	0.0048857	0.0032338
		$\hat{\theta}$	0.8188169	1.055628	1.407754	0.5722768	1.390925	1.295214
		$\hat{a}$	0.0030673	0.0042962	0.0020706	0.0057866	0.0016422	0.0016541
120	Mean	$\hat{\lambda}$	0.8039400	0.7738403	0.7898501	0.8263026	0.7959157	0.7979073
		$\hat{\theta}$	1.5443385	1.6788426	1.973455	1.3125233	1.916206	1.840074
		$\hat{a}$	0.0322414	0.0334397	0.0312562	0.0348294	0.0311077	0.0310814
	RMSE	$\hat{\lambda}$	0.1211005	0.1334994	0.1226071	0.1287282	0.1211244	0.1418116
		$\hat{\theta}$	2.7735992	2.1385116	2.760020	2.4180303	2.675734	2.493919
		$\hat{a}$	0.0147099	0.0181787	0.0159621	0.0151295	0.0151852	0.0153316
	Bias	$\hat{\lambda}$	0.0039400	0.0261596	0.0101498	0.0263026	0.0040842	0.0020926
		$\hat{\theta}$	0.7943385	0.9288426	1.223455	0.5625233	1.166206	1.090074
		$\hat{a}$	0.0022414	0.0034397	0.0012562	0.0048294	0.0011077	0.0010814
200	Mean	$\hat{\lambda}$	0.8011666	0.7824976	0.7929133	0.8140750	0.7965321	0.7984078
		$\hat{\theta}$	1.3896819	1.4581426	1.5917842	1.2089318	1.6015661	1.5582578
		$\hat{a}$	0.0311428	0.0320070	0.0306401	0.0332964	0.0304376	0.0303903
	RMSE	$\hat{\lambda}$	0.0953132	0.1083881	0.0975981	0.0993361	0.0970032	0.1130010
		$\hat{\theta}$	2.4260602	1.7586836	2.1354229	2.1385765	2.1599662	2.0012281
		$\hat{a}$	0.0118656	0.0149074	0.0129135	0.0121394	0.0125077	0.0125234
	Bias	$\hat{\lambda}$	0.0011666	0.0175023	0.0070866	0.0140750	0.0034678	0.0015921
		$\hat{\theta}$	0.6396819	0.7081426	0.8417842	0.4589318	0.8515661	0.8082578
		$\hat{a}$	0.0011428	0.0020070	0.0006401	0.0032964	0.0004376	0.0003903

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Table 2. Mean, RMSE, and Bias with the MLE, LSE, WLSE, MPSE, ADE, and RTADE for the NGERay model with $\lambda = 0.7$, $\theta = 0.2$, $a = 0.05$

n		Est. Par.	MLE	LSE	WLSE	MPSE	ADE	RTADE
30	Mean	$\hat{\lambda}$	0.7426800	0.6727114	0.6889552	0.8304122	0.7215017	0.7367249
		$\hat{\theta}$	0.3333140	0.6558753	0.5865837	0.3487676	0.5493210	0.6399046
		$\hat{a}$	0.0573875	0.0571248	0.0546906	0.0599196	0.0532026	0.0541417
	RMSE	$\hat{\lambda}$	0.2694604	0.2605185	0.2511023	0.3430756	0.2551884	0.3101106
		$\hat{\theta}$	0.5845328	1.0932777	1.0401864	0.5969298	0.9676623	1.0730083
		$\hat{a}$	0.0314544	0.0381542	0.0359309	0.0326754	0.0314748	0.0331429
	Bias	$\hat{\lambda}$	0.0426800	0.0272885	0.0110447	0.1304122	0.0215017	0.0367249
		$\hat{\theta}$	0.1333140	0.4558753	0.3865837	0.1487676	0.3493210	0.4399046
		$\hat{a}$	0.0073875	0.0071248	0.0046906	0.0099196	0.0032026	0.0041417
60	Mean	$\hat{\lambda}$	0.7225887	0.6882968	0.7045686	0.7659417	0.7169935	0.7283787
		$\hat{\theta}$	0.3411443	0.5850426	0.5578378	0.3228890	0.5110014	0.5049660
		$\hat{a}$	0.0528779	0.0525932	0.0504583	0.0550194	0.0071212	0.0505804
	RMSE	$\hat{\lambda}$	0.1734798	0.1944800	0.1840272	0.1999329	0.1798136	0.2201885
		$\hat{\theta}$	0.5189545	0.9271493	0.9107300	0.4902852	0.8292290	0.8052349
		$\hat{a}$	0.0205903	0.0279466	0.0245288	0.0208592	0.0225155	0.0237565
	Bias	$\hat{\lambda}$	0.0225887	0.0117032	0.0045686	0.0659417	0.0169935	0.0283787
		$\hat{\theta}$	0.1211443	0.3850426	0.3578378	0.1228890	0.3110014	0.3049660
		$\hat{a}$	0.0028779	0.0025932	0.0004583	0.0050194	0.0021109	0.0005804
90	Mean	$\hat{\lambda}$	0.7121242	0.6894891	0.7009519	0.7399755	0.7087938	0.7205541
		$\hat{\theta}$	0.3007137	0.4942591	0.4480735	0.2908084	0.4330120	0.4563296
		$\hat{a}$	0.0526343	0.0519477	0.0506634	0.0544543	0.0503981	0.0504692
	RMSE	$\hat{\lambda}$	0.1331398	0.1528382	0.1398518	0.1447416	0.1376423	0.1761769
		$\hat{\theta}$	0.3887937	0.7760244	0.7295492	0.3933275	0.6955207	0.7102560
		$\hat{a}$	0.0168745	0.0231369	0.0198007	0.0169649	0.0186765	0.0197437
	Bias	$\hat{\lambda}$	0.0121242	0.0105108	0.0009519	0.0399755	0.0087938	0.0205541
		$\hat{\theta}$	0.1007137	0.2942591	0.2480735	0.0908084	0.2330120	0.2563296
		$\hat{a}$	0.0026343	0.0019477	0.0006634	0.0044543	0.0003981	0.0004692
120	Mean	$\hat{\lambda}$	0.7090564	0.6921895	0.7016869	0.7294894	0.7062822	0.7143061
		$\hat{\theta}$	0.2775966	0.4447830	0.3773488	0.2645899	0.3755394	0.3968107
		$\hat{a}$	0.0520364	0.0512885	0.0503671	0.0537581	0.0503387	0.0503650
	RMSE	$\hat{\lambda}$	0.1157065	0.1363561	0.1229466	0.1230462	0.1205630	0.1493396
		$\hat{\theta}$	0.3305651	0.6767009	0.5703662	0.3290557	0.5708367	0.5842815
		$\hat{a}$	0.0142731	0.0203486	0.0169875	0.0145281	0.0161501	0.0170227
	Bias	$\hat{\lambda}$	0.0090564	0.0078104	0.0006869	0.0294894	0.0062822	0.0143061
		$\hat{\theta}$	0.0775966	0.2447830	0.1773488	0.0645899	0.1755394	0.1968107
		$\hat{a}$	0.0020364	0.0012885	0.0003671	0.0037581	0.0003387	0.0003650
200	Mean	$\hat{\lambda}$	0.7042817	0.6951187	0.6999968	0.7162770	0.7030800	0.7085631
		$\hat{\theta}$	0.2522246	0.3523951	0.3126523	0.2364838	0.3101128	0.3202034
		$\hat{a}$	0.0512306	0.0507114	0.0503396	0.0526812	0.0501870	0.0501513
	RMSE	$\hat{\lambda}$	0.0900652	0.1093789	0.0965546	0.0936998	0.0957639	0.1167738
		$\hat{\theta}$	0.2403465	0.4848814	0.4344236	0.2157724	0.4146609	0.4167183
		$\hat{a}$	0.0109834	0.0158707	0.0130237	0.0112471	0.0127018	0.0132379
	Bias	$\hat{\lambda}$	0.0042817	0.0048812	0.0000302	0.0162770	0.0030800	0.0085631
		$\hat{\theta}$	0.0522246	0.1523951	0.1126523	0.0364838	0.1101128	0.1202034
		$\hat{a}$	0.0012306	0.0007114	0.0003396	0.0026812	0.0001870	0.0001513

Table 3. Mean, RMSE, and Bias with the MLE, LSE, WLSE, MPSE, ADE, and RTADE for the NGERay model with $\lambda = 1.5$, $\theta = 0.3$, $a = 0.07$

n		Est. Par.	MLE	LSE	WLSE	MPSE	ADE	RTADE
30	Mean	$\hat{\lambda}$	1.6697117	1.4573493	1.4918641	1.9203431	1.5726168	1.6257527
		$\hat{\theta}$	0.5516680	0.9126946	0.9742819	0.589125	0.9402122	0.9769641
		$\hat{a}$	0.0793161	0.0787335	0.0772627	0.0812608	0.0765022	0.0766825
	RMSE	$\hat{\lambda}$	0.8196643	0.6845582	0.6828290	1.0967594	0.7152082	0.8567740
		$\hat{\theta}$	0.9620364	1.3915184	1.5504554	1.067287	1.5165338	1.5115508
		$\hat{a}$	0.0473394	0.0486744	0.0485299	0.0530515	0.0456175	0.0459408
	Bias	$\hat{\lambda}$	0.1697117	0.0426506	0.0281358	0.4203431	0.0726168	0.1257527
		$\hat{\theta}$	0.2516680	0.6126946	0.6742819	0.289125	0.6402122	0.6769641
		$\hat{a}$	0.0093161	0.0087335	0.0072627	0.0112608	0.0065022	0.0066825
60	Mean	$\hat{\lambda}$	1.5954099	1.4826903	1.5258835	1.7161985	1.5580835	1.5961585
		$\hat{\theta}$	0.5762625	0.781070	0.818336	0.5621368	0.8076484	0.8177102
		$\hat{a}$	0.0728619	0.0747446	0.0721325	0.0755796	0.0715315	0.0723561
	RMSE	$\hat{\lambda}$	0.5197748	0.5326310	0.5193004	0.6151470	0.5156064	0.6320209
		$\hat{\theta}$	0.8979017	1.126817	1.208954	0.9081604	1.2261072	1.1994752
		$\hat{a}$	0.0311126	0.0376976	0.0348893	0.0314605	0.0331591	0.0351674
	Bias	$\hat{\lambda}$	0.0954099	0.0373096	0.0258835	0.2161985	0.0580835	0.0961585
		$\hat{\theta}$	0.2262625	0.4810770	0.518336	0.2621368	0.5076484	0.5177102
		$\hat{a}$	0.0028619	0.0047446	0.0021325	0.0055796	0.0015315	0.0023561
90	Mean	$\hat{\lambda}$	1.5539091	1.4761639	1.5096883	1.6302740	1.5296563	1.5650408
		$\hat{\theta}$	0.4947846	0.6513150	0.6517102	0.4813457	0.6514501	0.6900195
		$\hat{a}$	0.4947846	0.0741881	0.0719872	0.0749553	0.0716168	0.0719337
	RMSE	$\hat{\lambda}$	0.3991355	0.4376538	0.4103777	0.4420132	0.4038326	0.5139972
		$\hat{\theta}$	0.6996945	0.8912608	0.9374762	0.7096316	0.9469533	0.9817117
		$\hat{a}$	0.0255944	0.0323864	0.0290128	0.0255574	0.0275993	0.0298347
	Bias	$\hat{\lambda}$	0.0539091	0.0238360	0.0196883	0.1302740	0.0296563	0.0650408
		$\hat{\theta}$	0.1947846	0.3513150	0.3517102	0.1813457	0.3514501	0.3900195
		$\hat{a}$	0.0027886	0.0041881	0.0019872	0.0049553	0.0013168	0.0019337
120	Mean	$\hat{\lambda}$	1.5412908	1.4835057	1.5125524	1.5974920	1.5227413	1.5474253
		$\hat{\theta}$	0.4533507	0.6017696	0.5853758	0.4392345	0.5728223	0.5934267
		$\hat{a}$	0.0721184	0.0731454	0.0711631	0.0741331	0.0711916	0.0715238
	RMSE	$\hat{\lambda}$	0.3457701	0.4002648	0.3628626	0.3726418	0.3539629	0.4464546
		$\hat{\theta}$	0.6097939	0.8040310	0.8372224	0.6178235	0.8207047	0.7853713
		$\hat{a}$	0.0218282	0.0289577	0.0249728	0.0219751	0.0239324	0.0264570
	Bias	$\hat{\lambda}$	0.0412908	0.0164942	0.0125524	0.0974920	0.0227413	0.0474253
		$\hat{\theta}$	0.1533507	0.3017696	0.2853758	0.1392345	0.2728223	0.2934267
		$\hat{a}$	0.0021184	0.0031454	0.0011631	0.0041331	0.0011916	0.0015238
200	Mean	$\hat{\lambda}$	1.5224136	1.4894279	1.5060437	1.5561289	1.5136918	1.5324357
		$\hat{\theta}$	0.3980772	0.4911702	0.4641766	0.3842513	0.4636151	0.4978058
		$\hat{a}$	0.0712141	0.0719038	0.0707175	0.0728675	0.0705755	0.0706244
	RMSE	$\hat{\lambda}$	0.2652074	0.3280542	0.2876244	0.2796352	0.2850042	0.3588845
		$\hat{\theta}$	0.4098888	0.5793187	0.5566380	0.4299347	0.5604245	0.6081748
		$\hat{a}$	0.0168649	0.0234732	0.0194589	0.0170827	0.0190331	0.0208653
	Bias	$\hat{\lambda}$	0.0224136	0.0105720	0.0060437	0.0561289	0.0136918	0.0324357
		$\hat{\theta}$	0.0980772	0.1911702	0.1641766	0.0842513	0.1636151	0.1978058
		$\hat{a}$	0.0012141	0.0019038	0.0007175	0.0028675	0.0005755	0.0006244

Table 4. Mean, RMSE, and Bias with the MLE, LSE, WLSE, MPSE, ADE, and RTADE for the NGERay model with $\lambda = 2, \theta = 0.2, a = 0.09$

n		Est. Par.	MLE	LSE	WLSE	MPSE	ADE	RTADE
30	Mean	$\hat{\lambda}$	2.3955808	2.0385526	2.0773283	2.8062145	2.2088396	2.3692388
		$\hat{\theta}$	0.4018997	0.5834636	0.6381358	0.4456675	0.5956757	0.6437182
		$\hat{a}$	0.1005527	0.0987506	0.0988786	0.1050421	0.0978367	0.0974610
	RMSE	$\hat{\lambda}$	1.5175606	1.0944738	1.1240041	2.0715611	1.2317887	1.5433478
		$\hat{\theta}$	0.6835378	0.8888925	1.0274313	0.7690723	0.9605748	1.0138276
		$\hat{a}$	0.0583643	0.0511834	0.0535697	0.0613341	0.0514706	0.0527685
	Bias	$\hat{\lambda}$	0.3955808	0.0385526	0.0773283	0.8062145	0.2088396	0.3692388
		$\hat{\theta}$	0.2018997	0.3834636	0.4381358	0.2456675	0.3956757	0.4437182
		$\hat{a}$	0.0105527	0.0087506	0.0088786	0.0150421	0.0078367	0.0074610
60	Mean	$\hat{\lambda}$	2.2162797	2.0514290	2.1039398	2.4089282	2.1499589	2.2598673
		$\hat{\theta}$	0.3840222	0.4826782	0.4912452	0.3958171	0.4882192	0.5377602
		$\hat{a}$	0.0930657	0.0951798	0.0932370	0.0955696	0.0927308	0.0928168
	RMSE	$\hat{\lambda}$	0.9250806	0.8848601	0.8850965	1.1149635	0.8915285	1.1388071
		$\hat{\theta}$	0.5623801	0.6787157	0.7106223	0.6162232	0.7176931	0.8033994
		$\hat{a}$	0.0368318	0.0423107	0.0406930	0.0378539	0.0390612	0.0413246
	Bias	$\hat{\lambda}$	0.2162797	0.0214290	0.0639398	0.4089282	0.1499589	0.2598673
		$\hat{\theta}$	0.1840222	0.2826782	0.2912452	0.1958171	0.2882192	0.3377602
		$\hat{a}$	0.0030657	0.0051798	0.0032370	0.0055696	0.0029308	0.0038168
90	Mean	$\hat{\lambda}$	2.1248010	2.0183456	2.0580438	2.2444015	2.0811107	2.1732984
		$\hat{\theta}$	0.3280679	0.4079071	0.4047249	0.3211615	0.3956699	0.4471604
		$\hat{a}$	0.0930489	0.0951329	0.0930506	0.0948664	0.0928186	0.0929378
	RMSE	$\hat{\lambda}$	0.6969831	0.7477368	0.7164708	0.7808405	0.7020556	0.9171785
		$\hat{\theta}$	0.4422141	0.5450840	0.5692541	0.4305082	0.5622492	0.6401337
		$\hat{a}$	0.0298436	0.0375083	0.0339407	0.0299043	0.0325671	0.0361016
	Bias	$\hat{\lambda}$	0.1248010	0.0183456	0.0580438	0.2444015	0.0811107	0.1732984
		$\hat{\theta}$	0.1280679	0.2079071	0.2047249	0.1211615	0.1956699	0.2471604
		$\hat{a}$	0.0030489	0.0051329	0.0030506	0.0048664	0.0028186	0.0029378
120	Mean	$\hat{\lambda}$	2.0955166	2.0202996	2.0525339	2.1846024	2.0631027	2.1322409
		$\hat{\theta}$	0.3037883	0.3717067	0.3667916	0.3010690	0.3530667	0.4006748
		$\hat{a}$	0.0922695	0.0940776	0.0920200	0.0939027	0.0921391	0.0923796
	RMSE	$\hat{\lambda}$	0.5995915	0.6901550	0.6332248	0.6533523	0.6159171	0.8062775
		$\hat{\theta}$	0.3997044	0.4747558	0.5219085	0.4158419	0.4901975	0.5568475
		$\hat{a}$	0.0252271	0.0334149	0.0292639	0.0254786	0.0280815	0.0320934
	Bias	$\hat{\lambda}$	0.0955166	0.0152996	0.0525339	0.1846024	0.0631027	0.1322409
		$\hat{\theta}$	0.1037883	0.1717067	0.1667916	0.1010690	0.1530667	0.2006748
		$\hat{a}$	0.0022695	0.0040776	0.0020200	0.0039027	0.0021391	0.0023796
200	Mean	$\hat{\lambda}$	2.0530561	2.0133546	2.0304935	2.1078852	2.0389611	2.0854098
		$\hat{\theta}$	0.2671442	0.3133929	0.2944362	0.2618455	0.2910516	0.3231203
		$\hat{a}$	0.0912918	0.0926395	0.0912477	0.0926158	0.0911635	0.0912952
	RMSE	$\hat{\lambda}$	0.4463108	0.5669247	0.5013117	0.4770628	0.4903799	0.6413514
		$\hat{\theta}$	0.2991356	0.3644064	0.3434788	0.2933557	0.3300490	0.3872105
		$\hat{a}$	0.0193186	0.0273802	0.0226274	0.0196260	0.0221194	0.0252081
	Bias	$\hat{\lambda}$	0.0530561	0.0133546	0.0304935	0.1078852	0.0389611	0.0854098
		$\hat{\theta}$	0.0671442	0.1133929	0.0944362	0.0618455	0.0910516	0.1231203
		$\hat{a}$	0.0012918	0.0026395	0.0012477	0.0026158	0.0011635	0.0012952

5.2 Application

In the section, the new distribution is an analogy with real data to verify the superiority and suitability of the proposed distribution with the next distribution: Truncated Exponentiated Exponential Rayleigh (TEER), Beta Rayleigh distribution (BeR), Kumaraswamy Rayleigh distribution (KuR), Exponentiated Generalized Rayleigh distribution (EGR), Weibull Rayleigh distribution (WeR) [27], Gompertz Rayleigh distribution (GoR) [28], Rayleigh distribution (R).

- TEER

$$F(x) = \frac{1 - exp\left((-\theta\, (1 - \exp\,(-ax^2))^{\lambda})\right)}{1 - exp(-\theta\,)}$$

- BeR

$$F(x) = \; pbeta(1 - \exp\,(-ax^2), \lambda, \theta)$$

- KuR

$$F(x) = 1 \; - \; \left(1 \; - \; (1 - \exp(-ax^2))^{\theta}\right)^{\lambda}$$

- EGR

$$F(x) = \left(1 \; - \; \left(1 \; - (1 - \exp\,(-ax^2))\right)^{\theta}\right)^{\lambda}$$

- WeR

$$F(x) = 1 \; - \; exp\left(-\theta^{\,-\lambda}\left(-log(1 - (1 - \exp\,(-ax^2))\right)^{\lambda}\right)$$

- GoR

$$\text{F(x)=}\; 1 \; - \; exp\left(\left(\frac{\theta}{\lambda}\right) \left(1 - \left((1 - \exp\,(-ax^2))\right)^{-\lambda}\right) \right)$$

- R

$$F(x) = 1 - \exp\,(-\lambda x^2).$$

We used different statistical criteria, namely (log-likelihood), AIC (Akaike Information Criterion), AIC (corrected Akaike Information Criterion), BIC (Bayesian Information Criterion), and HQIC (Hannan-Quinn Information Criterion).

The first data set: data on the breaking stress of carbon fibers of 50 mm length [29]

0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90

The second dataset consists of 74 observations; specifically indicating measurement lengths of 20 mm. Analysis of this data was performed [30]

1.312, 1.314, 1.479 ,1.552,1.700 ,1.803, 1.861 ,1.865 ,1.944, 1.958 ,1.966, 1.997 ,2.006, 2.021 ,2.027, 2.055, 2.063 ,2.098, 2.140, 2.179 ,2.224 ,2.240, 2.253 ,2.270, 2.272, 2.274, 2.301, 2.301 ,2.359 ,2.382 ,2.382 ,2.426 ,2.434, 2.435, 2.478 ,2.490, 2.511, 2.514, 2.535 ,2.554, 2.566, 2.570, 2.586, 2.629 ,2.633, 2.642, 2.648, 2.684 ,2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848 ,2.880, 2.954, 3.012, 3.067 ,3.084, 3.090, 3.096, 3.128, 3.233, 3.433 ,3.585, 3.585

Table 5. Statistical description of data sets

	var	n	mean	sd	median	min	max	skew	kurtosis
Data 1	x	66	2.76	0.89	2.84	0.39	4.9	-0.13	0.13
Data 2	x	69	2.45	0.5	2.48	1.31	3.58	-0.03	-0.14

Table 6. Parameters estimate values of NGERay distribution with respect to first data set.

Distribution	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$
NGERay	10.1059	0.7651	0.0292
TEER	0.6896	2.1824	0.1593
BR	1.8231	1.0794	0.1592
KuR	1.8079	1.1616	0.1470
EGR	1.0065	1.8539	0.1696
WeR	1.1266	0.1690	0.0198
GoR	0.2050	0.3596	0.2857
R	0.1190	----------	------------

Table 7. The values of the -2 l, AIC, CAIC, BIC, HQIC

Distribution	-LL	AIC	CAIC	BIC	HQIC
NGERay	85.993	177.98	178.37	184.55	180.58
TEER	89.746	185.49	185.88	192.06	188.09
BR	89.464	184.95	185.33	191.52	187.54
KuR	89.265	184.56	184.95	191.13	187.16
EGR	89.433	184.88	185.26	191.45	187.47
WeR	93.882	193.78	194.16	200.34	196.37
GoR	88.032	182.06	182.45	188.63	184.66
R	98.208	198.41	198.47	200.60	199.28

Table 8. The values of the W, A, K-S, and p-value of the data set

Distribution	NGERay	TEER	BR	KuR	EGR	WeR	GoR	R
W	0.0859	0.2013	0.1654	0.1625	0.1675	0.1390	0.1136	0.1552
A	0.5186	1.0734	0.8833	0.8674	0.8940	0.7448	0.7587	0.8299
KS	0.0844	0.1385	0.1537	0.1392	0.1507	0.2053	0.1092	0.2265
p-value	0.7336	0.1587	0.0884	0.1546	0.0996	0.0076	0.4101	0.0022

According to the values shown in Table 7 and 8, the superiority of the distribution is obvious, as the extension and newly expanded distribution displays an accurate representation because it has the lowest values of the informational and statistical standards 2 L, AIC, CAIC, BIC, HQIC, K-S, A, W and the largest value of the p-value .

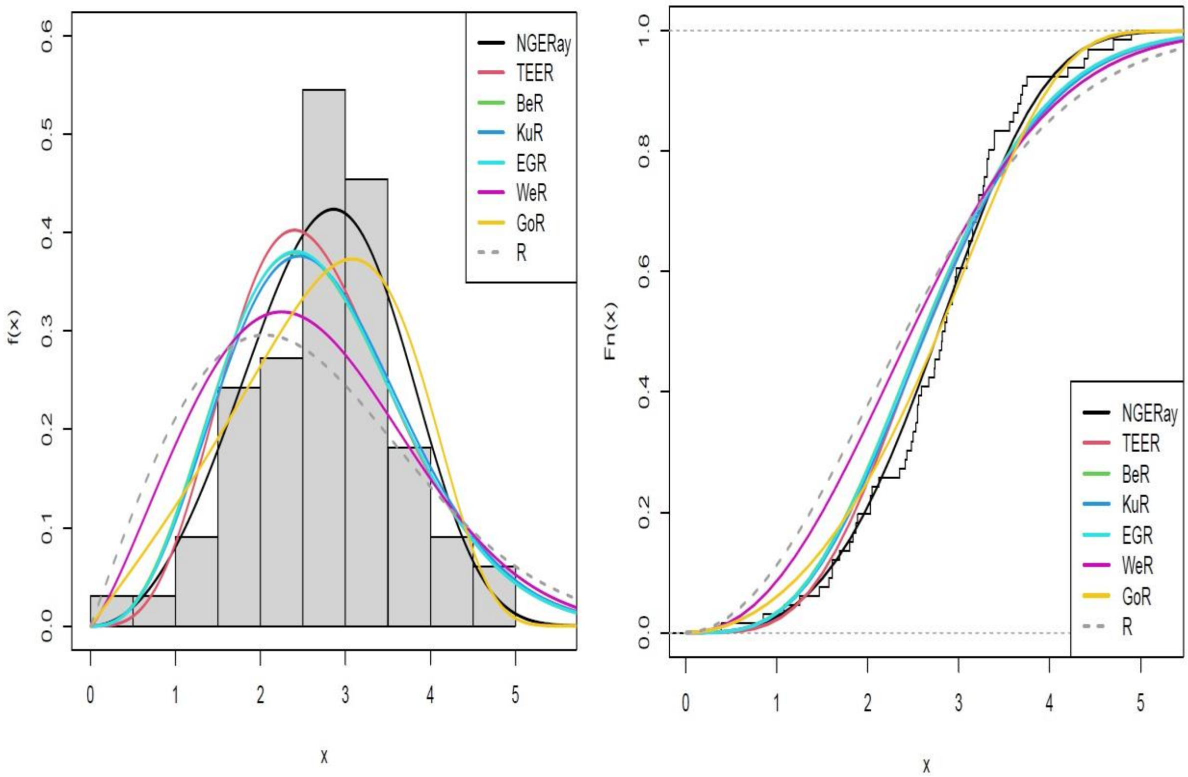

Figure 2. The CDFs and PDFs for the elected models for the data set 1.

Table 9. Parameters estimate values of NGERay with respect to the second data set.

Distribution	$\hat{\lambda}$	$\hat{\theta}$	$\hat{a}$
NGERay	8.2086	1.8448	0.0616
TEER	0.7594	3.1565	0.2451
BR	4.4189	1.2281	0.2949

KuR	4.6020	2.6049	0.2223
EGR	0.6103	5.6182	0.6103
WeR	2.0690	1.0063	0.1478
GoR	0.0708	0.6389	0.4935
R	0.1599	-------------	------------

Table 10. The values of the -2 l, AIC, CAIC, BIC, HQIC

Distribution	**-LL**	**AIC**	**CAIC**	**BIC**	**HQIC**
NGERay	48.892	103.78	104.15	110.48	106.44
TEER	60.326	126.70	127.07	133.40	129.36
BR	53.782	113.58	133.95	120.28	116.24
KuR	49.828	105.65	106.02	112.36	108.31
EGR	52.310	110.62	110.99	117.32	113.28
WeR	53.813	113.68	114.05	120.38	116.34
GoR	56.193	118.38	118.75	125.08	121.04
R	87.246	176.49	176.55	178.72	177.37

Table 11. The values of the W , A , K-S and p-value of the data set.

Distribution	**NGERay**	**TEER**	**BR**	**KuR**	**EGR**	**WeR**	**GoR**	**R**
W	0.0173	0.0457	0.0378	0.0239	0.0462	0.0188	0.1664	0.0268
A	0.1563	0.3347	0.2852	0.1929	0.3399	0.1678	1.1317	0.2141
KS	0.0429	0.1533	0.1116	0.0661	0.0907	0.1402	0.1101	0.3384
p-value	0.9995	0.0779	0.3554	0.9229	0.6211	0.1323	0.3723	2.7305

The values in Tables 10 and 11, show that the superiority of the distribution is obvious, as the extension and newly expanded distribution displays an accurate representation because it has the lowest values of the informational and statistical standards 2 L, AIC, CAIC, BIC, HQIC, K-S, A, W and the largest value of the p-value.

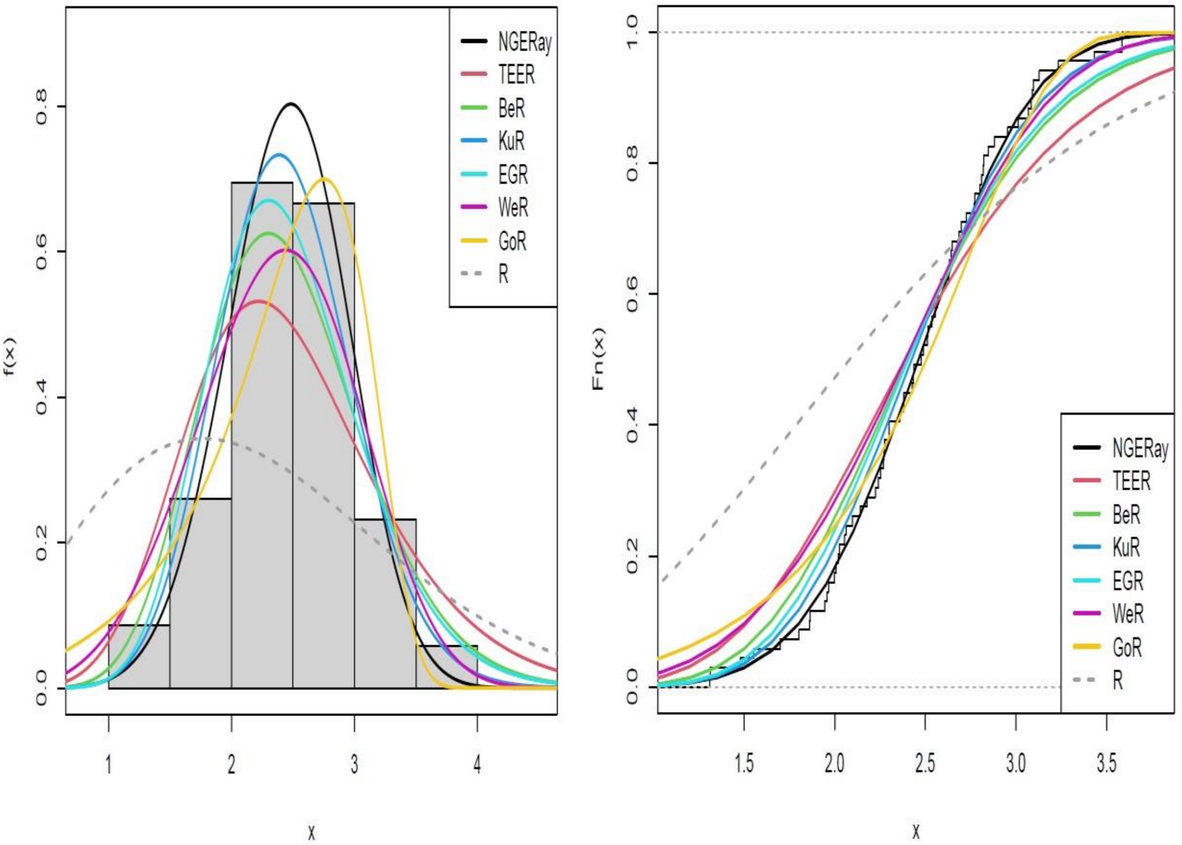

Figure 3. The CDFs and PDFs for the elected models for the second data set.

6. Discussion

The one-parameter Rayleigh distribution was expanded based on the new family (Odd Generalized Exponential- G family) to obtain a new three-parameter distribution that can handle failure data with good flexibility. When comparing it to a group of distributions that have proven to be flexible according to previous studies, simulation was used to demonstrate the best method out of six different methods for estimating the parameters of the proposed distribution, where the best method in Table 1 was (MLE), in Table 2 was (ADE), in Table 3 was (MLE) and method in Table 4 was (MLE) Through the graph of the probability density function and comparing it with the shape of the studied data, it became clear that the distribution can be handled in the proposal with semi-symmetric and right-skewed data.

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