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SOME PROPERTIES ON (q, h)-ANALOGUE OF EULER POLYNOMIALS

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Abstract. In this article, we define (q, h)-analogue of Euler polynomials and numbers and determine their properties. Based on these numbers and polynomials, we also confirm that the structure of the approximate root changes according to changes in q and h. We find differential equations that have (q, h)-analogue of Euler polynomials as solutions and also find differential equations that have other polynomials as coefficients, confirming the relationships among these.

1. Introduction

Recently, many mathematician [1, 7, 13] have defined and constructed generating maps for novel families of special polynomials, such as q-Bernoulli, q-Euler, q-Genocchi, etc., utilizing tangent polynomials and degenerate tangent polynomials. These types of studies have provided fundamental properties and diverse applications for these polynomials. For instance, not only several explicit and implicit summation formulas, recurrence formulas, symmetric properties and many correlations with the well-known polynomials in the literature have been derived intensely, but also we have derived some beautiful correlations between some special polynomials. Additionally, the aforementioned polynomials allow the derivation of utility properties in a quite basic procedure and assist in defining novel families of special polynomials. By motivating the above, here, we introduce (q, h)-analogue of Frobenius-tangent polynomials and analyze some properties by providing several relations and applications. We first attain diverse relations and formulas covering addition formulas, recurrence rules, implicit summation formulas and relations with the earlier polynomials in the literature.

The subject of q-calculus started appearing in the nineteenth century due to its applications in various fields of mathematics, physics and engineering. The definitions and notations of q-calculus reviewed here are taken from (see [6, 7]):

The q-analogue of the shifted factorial $(a)_n$ is given by

$$(a;q)_0 = 1, (a;q)_n = \prod_{m=0}^{n-1}(1-q^m a), n \in \mathbb{N}.$$

The q-analogue of a complex number a and of the factorial function are given by

$$[a]_q = \frac{1-q^a}{1-q}, q \in \mathbb{C} - \{1\}; a \in \mathbb{C},$$

$$[n]_q! = \prod_{m=1}^{n}[m]_q = [1]_q[2]_q \cdots [n]_q = \frac{(q;q)_n}{(1-q)^n}, q \neq 1; n \in \mathbb{N},$$

$$[0]_q! = 1, q \in \mathbb{C}; 0 < q < 1.$$

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The Gauss q-binomial coefficient $\left(\begin{array}{c} n \\ k \end{array}\right)_q$ is given by

$$\left(\begin{array}{c} n \\ k \end{array}\right)_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, k = 0, 1, \cdots, n.$$

The q-analogue of the function $(x+y)_q^n$ is given by

$$(x+y)_q^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right)_q q^{k(k-1)/2} x^{n-k} y^k, n \in \mathbb{N}_0. \tag{1}$$

The q-analogue of exponential functions are given by

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{((1-q)x;q)_\infty}, 0 <\mid q \mid< 1; \mid x \mid<\mid 1-q \mid^{-1}, \tag{2}$$

The q-Bernoulli $B_{n,q}(x)$, the q-Euler $\mathbb{E}_{n,q}(x)$ and q-Genocchi polynomials $\mathbb{G}_{n,q}(x)$ are defined by (see [11, 12]):

$$\frac{t}{e_q(t)-1} e_q(xt) = \sum_{n=0}^{\infty} \mathbb{B}_{n,q}(x) \frac{t^n}{[n]_q!} \ (\mid t \mid) < 2\pi, \tag{7}$$

$$\frac{2}{e_q(t)+1} e_q(xt) = \sum_{n=0}^{\infty} \mathbb{E}_{n,q}(x) \frac{t^n}{[n]_q!} \ (\mid t \mid) < \pi, \tag{8}$$

$$\frac{2t}{e_q(t)+1} e_q(xt) = \sum_{n=0}^{\infty} \mathbb{G}_{n,q}(x) \frac{t^n}{[n]_q!} \ (\mid t \mid< \pi), \tag{9}$$

respectively.

Clearly, we have

$$\mathbb{B}_{n,q} = \mathbb{B}_{n,q}(0), \mathbb{E}_{n,q} = \mathbb{E}_{n,q}(0), \mathbb{G}_{n,q} = \mathbb{G}_{n,q}(0). \tag{10}$$

Let $n \in \mathbb{Z}_0$ and $u \in \mathbb{Z}$, the q-Frobenius-tangent polynomials as follows (see [])

$$\frac{1-u}{e_q(2t)-u} e_q(tx) = \sum_{n=0}^{\infty} T_{n,q}(x;u) \frac{t^n}{[n]_q!}, \ \textit{where } e_q(2t) \neq u. \qquad ()$$

When $x = 0$, $T_{n,q}(u) = T_{n,q}(0;u)$ are called the q-Frobenius-tangent numbers.

The generalized quantum exponential function is defined by

$$\exp_{q,h}(\alpha x) = \sum_{i=0}^{\infty} \frac{\alpha^i (x-0)_{q,h}^i}{[i]!}, \qquad ()$$

where α is arbitrary nonzero constant.
Clearly $\exp_{q,h}(0) = 1$. As $h \to 0$ with $\alpha = 1$, the generalized quantum exponential function () becomes the so called q-exponential function $e_q(x)$ (see []). Likewise, as $q \to 1$ with $\alpha = 1$, the generalized quantum exponential function $\exp_{q,h}(\alpha x)$ reduces to the so-called h-exponential function $\exp_{1,h}(x) = (1+h)^{\frac{x}{h}}$.

The q-tangent numbers and polynomials are defined as

$$\frac{2}{e_q(2t)+1} e_q(tx) = \sum_{n=0}^{\infty} T_{n,q}(x) \frac{t^n}{[n]_q!}$$

For $q \to 1$, we note that q-tangent numbers and polynomials become tangent numbers and polynomials become tangent numbers and polynomials, respectively.

The degenerate tangent polynomials are defined by

$$\frac{2}{(1+\lambda t)^{\frac{2}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}T_{n,\lambda}(x)\frac{t^n}{n!}$$

When $x=0$, $T_{n,\lambda}=T_{n,\lambda}(0)$ are called the degenerate tangent numbers.

Recently, Kang [] introduced the degenerate exponential function define by

$$\exp_{q,h}(x:t)=\sum_{n=0}^{\infty}(x)_{q,h}^{n}\frac{t^n}{[n]_q!}. \qquad ()$$

Definition 1.1. Let $\mid q \mid< 1$ and h be non-negative integer. Then, we can define the degenerate q-tangent polynomials $T_{n,q}(x:h)$ as

$$\frac{2}{q_{q,h}(2:t)}e_{q,h}(x:t)=\sum_{n=0}^{\infty}T_{n,q}(x:h)\frac{t^n}{[n]_q!}, \qquad ()$$

so that

$$T_{n,q}(x:h)=\sum_{k=0}^{n}\binom{n}{k}_q(x)_{q,h}^{n-k}T_{k,q}(h).$$

When $x=0$, $T_{n,q}(h)=T_{n,q}(0:h)$ are called the degenerate q-tangent numbers.

The main purpose of this paper, we introduce (q,h)-analogue of Frobenius-tangent numbers and polynomials. Based on these polynomials, we construct some differential equation of these polynomials. Also, we derive some differential equations between q-Euler, q-Bernoulli and q-Genocchi polynomials.

2. (q,h)-ANALOGUE OF EULER POLYNOMIALS

In this section, we define (q,h)-analogue Euler polynomials using generalized quantum exponential functions. Using the (q,h)-derivatives, we obtain several differential equations related to (q,h)-analogue of Euler polynomials. Furthermore, we find relations among q-Euler polynomials, degenerate Euler polynomials and (q,h)-Euler polynomials.

Let $\mid q \mid< 1$, and h be non-negative integer. The, we can define the (q,h)-analogue of Euler polynomials $\mathbb{E}_{n,q}(x:h)$

$$\frac{2}{e_{q,h}(1:t)+1}e_{q,h}(x:t)=\sum_{n=0}^{\infty}\mathbb{E}_{n,q}(x:h)\frac{t^n}{[n]_q!}. \qquad (2.1)$$

For $x=0$ in (2.1), we note that

$$\sum_{n=0}^{\infty}\mathbb{E}_{n,q}(0:h)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}\mathbb{E}_{n,q}(h)\frac{t^n}{[n]_q!}=\frac{2}{e_{q,h}(1:t)+1}, \qquad (2.2)$$

where $\mathbb{E}_{n,q}(h)$ are called the (q,h)-Euler numbers. From (2.1), we can see certain relations between the Euler, degenerate Euler and q-Euler polynomials. Setting $h\to 0$ in (2.1) and (2.2), we can derive the q-Euler numbers and polynomials as follows:

$$\sum_{n=0}^{\infty}\mathbb{T}_{n,q}(u)\frac{t^n}{[n]_q!}=\frac{1-u}{e_q(1-u:t)-u}, \quad \sum_{n=0}^{\infty}\mathbb{T}_{n,q}(x;u)\frac{t^n}{[n]_q!}=\frac{1-u}{e_q(1-u:t)-u}e_q^{xt}.$$

As $h \to 0$ and $q \to 1$ in (2.1) and (2.3), we can derive the Frobenius-tangent numbers and polynomials as follows:

$$\sum_{n=0}^{\infty} \mathbb{T}_n(u)\frac{t^n}{n!} = \frac{1-u}{e^{(1-u)t}-u}, \ \sum_{n=0}^{\infty} \mathbb{T}_n(x;u)\frac{t^n}{n!} = \frac{1-u}{e^{(1-u)t}-u}e^{xt}.$$

When $q \to 1$ in (2.1) and (2.3), we can derive the degenerate Frobenius-tangent numbers and polynomials as follows:

$$\sum_{n=0}^{\infty} \mathbb{T}_{n,h}(u)\frac{t^n}{n!} = \frac{1-u}{(1+ht)^{\frac{1-u}{h}}-u}, \ \sum_{n=0}^{\infty} \mathbb{T}_{n,h}(x;u)\frac{t^n}{n!} = \frac{1-u}{(1+ht)^{\frac{1-u}{h}}-u}(1+ht)^{\frac{x}{h}}.$$

Here is a list of some (q,h)-Frobenius-tangent numbers:

Several (q,h)-Euler polynomials are as follows:

Figures

Theorem 2.1. For $\mid q \mid < 1$ and $h \in \mathbb{N}$, we have

$$D^{(1)}_{q,h,x}\mathbb{E}_{n,q}(x:h) = [n]_q\mathbb{E}_{n-1,q}(x:h). \tag{2.4}$$

Proof. Using (2.1), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{E}_{n,q}(x:h)\frac{t^n}{[n]_q!} &= \sum_{n=0}^{\infty} \mathbb{E}_{n,q}(h)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}(x)^n_{q,h}\frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_q \mathbb{E}_{k,q}(h)(x)^{n-k}_{q,h}\right)\frac{t^n}{[n]_q!}. \end{aligned} \tag{2.5}$$

From this equation, we have

$$\mathbb{E}_{n,q}(x:h) = \sum_{k=0}^{n}\binom{n}{k}_q \mathbb{E}_{k,q}(h)(x)^{n-k}_{q,h}. \tag{2.6}$$

Using the (q,h)-derivative in (2.6), we have

$$D^{(1)}_{q,h,x}\mathbb{E}_{n,q}(x:h) = \sum_{k=0}^{n}\binom{n}{k}_q [n-k]_q\mathbb{E}_{k,q}(h)(x)^{n-k}_{q,h} = [n]_q\mathbb{E}_{n-1,q}(x:h).$$

This completes the proof. □

Corollary 2.1. Let k be a non-negative integer. Then

$$\mathbb{E}_{n-k,q}(x:h) = \frac{[n-k]_q!}{[n]_q!}D^{(k)}_{q,h,x}\mathbb{E}_{n,q}(x:h).$$

Corollary 2.2. Setting $k \to 1$ in Theorem 2.1, we have

$$D^{(1)}_{h,x}\mathbb{E}_n(x:h) = n\mathbb{E}_{n-1}(x:h), \ \mathbb{E}_{n-k}(x:h) = \frac{(n-k)!}{n!}D^{(k)}_{h,x}\mathbb{E}_n(x:h),$$

where D_h is the h-derivative and $\mathbb{E}_n(x:h)$ are the degenerate Euler polynomials.

Corollary 2.3. Letting $h \to 0$ in Theorem 2.1, we have

$$D^{(1)}_{q,x}\mathbb{E}_{n,q}(x) = [n]_q\mathbb{E}_{n-1,q}(x), \ \mathbb{E}_{n-k,q}(x) = \frac{[n-k]_q!}{[n]_q!}D^{(k)}_{q,x}\mathbb{E}_{n,q}(x),$$

where D_q is the q-derivative and $\mathbb{E}_{n,q}(x)$ are q-Euler polynomials.

Theorem 2.2. The solutions of differential equation of the (q,h)-Euler polynomials as follows

$$\frac{(2)_{q,h}^{n}}{[n]_q!}D_{q,h,x}^{(n)}\mathbb{E}_{n,q}(x:h)+\frac{(2)_{q,h}^{n-1}}{[n-1]_q!}D_{q,h,x}^{(n-1)}\mathbb{E}_{n,q}(x:h)+\frac{(2)_{q,h}^{n-2}}{[n-2]_q!}D_{q,h,x}^{(n-2)}\mathbb{E}_{n,q}(x:h)$$
$$+\cdots+\frac{(2)_{q,h}^{2}}{[2]_q!}D_{q,h,x}^{(2)}\mathbb{E}_{n,q}(x:h)+2D_{q,h,x}^{(1)}(x;u)\mathbb{E}_{n}(x:h)-2(x)_{q,h}^{n}=0.$$

Proof. By using (2.1), we see that

$$\begin{aligned}2e_{q,h}(x:t)&=\sum_{n=0}^{\infty}\mathbb{E}_{n,q}(x:h)\frac{t^n}{[n]_q!}(e_{q,h}(1:t)+1)\\&=\sum_{n=0}^{\infty}\mathbb{E}_{n,q}(x:h)\frac{t^n}{[n]_q!}\left(\sum_{k=0}^{\infty}(2)_{q,h}^{k}\frac{t^k}{[k]_q!}+1\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_q(2)_{q,h}^{k}\mathbb{E}_{n-k,q}(x:h)+\mathbb{E}_{n,q}(x:h)\right)\frac{t^n}{[n]_q!}.\end{aligned}\tag{2.7}$$

On the other hand, we have

$$2e_{q,h}(x:t)=2\sum_{n=0}^{\infty}(x)_{q,h}^{n}\frac{t^n}{[n]_q!}.\tag{2.8}$$

Therefore, by (2.7) and (2.8), we get

$$\sum_{k=0}^{n}\binom{n}{k}_q(2)_{q,h}^{k}\mathbb{E}_{n-k,q}(x:h)=2(x)_{q,h}^{n}.\tag{2.9}$$

Taking the $k-th$ derivative of above equation, we obtain

$$\sum_{k=0}^{n}\frac{(2)_{q,h}^{k}}{[k]_q!}D_{q,h,x}^{(k)}\mathbb{E}_{n,q}(x:h)+\mathbb{E}_{n,q}(x:h)-2(x)_{q,h}^{n}=0.$$

Therefore, we obtain the desired result. ☐

Corollary 2.4. Letting $q\to 1$ in Theorem 2.2, we have

$$\frac{(2)_{1,h}^{n}}{n!}D_{h,x}^{(n)}\mathbb{E}_{n}(x:h)+\frac{(2)_{1,h}^{n-1}}{(n-1)!}D_{h,x}^{(n-1)}\mathbb{E}_{n}(x:h)+\frac{(2)_{1,h}^{n-2}}{(n-2)!}D_{h,x}^{(n-2)}\mathbb{E}_{n}(x:h)$$
$$+\cdots+\frac{(2)_{1,h}^{2}}{(2)!}D_{h,x}^{(2)}\mathbb{E}_{n}(x:h)+2D_{h,x}^{(1)}E_{n}(x:h)-2(x)_{1,h}^{n}=0.$$

Corollary 2.5. Letting $h\to 0$ in Theorem 2.2, we have

$$\frac{(1-u)}{[n]_q!}D_{q,x}^{(n)}\mathbb{T}_{n,q}(x;u)+\frac{(1-u)}{[n-1]_q!}D_{h,x}^{(n-1)}\mathbb{T}_{n,q}(x;u)+\frac{(1-u)}{[n-2]_q!}D_{q,x}^{(n-2)}\mathbb{T}_{n,q}(x;u)$$
$$+\cdots+\frac{(1-u)}{[2]_q!}D_{q,x}^{(2)}\mathbb{T}_{n,q}(x;u)+(1-u)D_{q,x}^{(1)}(x;u)-u\mathbb{T}_{n,q}(x;u)-(1-u)x^n=0.$$

Theorem 2.3. For $\omega\geq 0$, we have

$$\frac{\mathbb{E}_{n,q}(x:h)+\mathbb{E}_{n,q}(h)}{[n]_q!}D_{q,h,x}^{(n)}\mathbb{E}_{n,q}(x:h)+\frac{\mathbb{E}_{n-1,q}(x:h)+\mathbb{E}_{n-1,q}(h)}{[n-1]_q!}D_{q,h,x}^{(n-1)}\mathbb{E}_{n,q}(x:h)+$$
$$\cdots+\frac{\mathbb{E}_{2,q}(x:h)+\mathbb{E}_{2,q}(h)}{[2]_q!}D_{q,h,x}^{(2)}\mathbb{E}_{n,q}(x:h)+(\mathbb{E}_{1,q}(x:h)+\mathbb{E}_{1,q}(h))D_{q,h,x}^{(1)}\mathbb{E}_{n,q}(x:h)$$
$$+(\mathbb{E}_{0,q}(x:h)+\mathbb{E}_{0,q}(h)-2)\mathbb{E}_{n,q}(x:h)=0.$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty}\mathbb{E}_{n,q}(x:h)\frac{t^n}{[n]_q!}=\frac{2}{e_{q,h}(1:t)+1}e_{q,h}(x:t)$$

$$=\frac{1}{2}\left(\frac{2}{e_{q,h}(1:t)+1}e_{q,h}(x:t)+\frac{2}{e_{q,h}(1:t)+1}\right)\frac{2}{e_{q,h}(1:t)+1}e_{q,h}(x:t).$$

Using the generating function of (q,h)-Euler polynomials, we have

$$2\sum_{n=0}^{\infty}\mathbb{E}_{n,q}(x:h)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_q\left(\mathbb{E}_{k,q}(x:h)+\mathbb{E}_{k,q}(h)\right)\mathbb{E}_{n-k,q}(x:h)\right)\frac{t^n}{[n]_q!}.$$

Comparing the coefficients of both sides, we have

$$\sum_{k=0}^{n}\binom{n}{k}_q\left(\mathbb{E}_{k,q}(x:h)+\mathbb{E}_{k,q}(h)\right)\mathbb{E}_{n-k,q}(x:h)-2\mathbb{E}_{n,q}(x:h)=0. \tag{2.10}$$

Replacing $\mathbb{E}_{n-k,q}(x:h)$ with $D^{(k)}_{q,h,x}\mathbb{T}_{n,q}(x:h)$ in equation (2.10), we have

$$\sum_{k=0}^{n}\frac{\left(\mathbb{E}_{k,q}(x:h)+\mathbb{E}_{k,q}(h)\right)}{[k]_q!}D^{(k)}_{q,h,x}\mathbb{E}_{n,q}(x:h)-2\mathbb{E}_{n,q}(x:h)=0.$$

The complete proof of the theorem. □

Corollary 2.6. On setting $h\to 0$ in Theorem 2.3, we have

$$\frac{\mathbb{E}_{n,q}(x)+\mathbb{E}_{n,q}}{[n]_q!}D^{(n)}_{q,x}\mathbb{E}_{n,q}(x)+\frac{\mathbb{E}_{n-1,q}(x)+\mathbb{E}_{n-1,q}}{[n-1]_q!}D^{(n-1)}_{q,x}\mathbb{E}_{n,q}(x)+$$

$$\cdots+\frac{\mathbb{E}_{2,q}(x)+\mathbb{E}_{2,q}}{[2]_q!}D^{(2)}_{q,x}\mathbb{E}_{n,q}(x)+\left(\mathbb{E}_{1,q}(x)+\mathbb{E}_{1,q}\right)D^{(1)}_{q,x}\mathbb{E}_{n,q}(x)$$

$$+\left(\mathbb{E}_{0,q}(x)+\mathbb{E}_{0,q}-2\right)\mathbb{E}_{n,q}(x)=0.$$

Corollary 2.7. On setting $q\to 1$ in Theorem 2.3, we have

$$\frac{\mathbb{E}_{n}(x:h)+\mathbb{E}_{n}(h)}{n!}D^{(n)}_{h,x}\mathbb{E}_{n}(x:h)+\frac{\mathbb{E}_{n-1}(x:h)+\mathbb{E}_{n-1}(h)}{(n-1)!}D^{(n-1)}_{h,x}\mathbb{E}_{n}(x:h)+$$

$$\cdots+\frac{\mathbb{E}_{2}(x:h)+\mathbb{E}_{2}(h)}{2!}D^{(2)}_{h,x}\mathbb{E}_{n}(x:h)+(\mathbb{E}_{1}(x:h)+\mathbb{E}_{1}(h))D^{(1)}_{h,x}\mathbb{E}_{n}(x:h)$$

$$+(\mathbb{E}_{0}(x:h)+\mathbb{E}_{0}(h)-2)\mathbb{E}_{n}(x:h)=0.$$

Theorem 2.4. For $\omega\geq 0$, we have

$$\frac{q^n\left(\mathbb{E}_{n,q}(2:q^{-1}h)+\mathbb{E}_{n,q}(q^{-1}h)\right)}{[n]_q!}D^{(n)}_{q,h,x}\mathbb{E}_{n,q}(qx:h)$$

$$+\frac{q^{n-1}\left(\mathbb{E}_{n-1,q}(2:q^{-1}h)+\mathbb{E}_{n-1,q}(q^{-1}h)\right)}{[n-1]_q!}D^{(n-1)}_{q,h,x}\mathbb{E}_{n-1,q}(qx:h)$$

$$+\cdots+\frac{q^2\left(\mathbb{E}_{2,q}(2:q^{-1}h)+\mathbb{E}_{2,q}(q^{-1}h)\right)}{[2]_q!}D^{(2)}_{q,h,x}\mathbb{E}_{n,q}(qx:h)$$

$$+q\left(\mathbb{E}_{1,q}(2:q^{-1}h)+\mathbb{E}_{1,q}(q^{-1}h)\right)D^{(1)}_{q,h,x}\mathbb{E}_{n,q}(qx:h)$$

$$+\left(\mathbb{E}_{0,q}(2:q^{-1}h)+\mathbb{E}_{0,q}(q^{-1}h)-2\right)\mathbb{E}_{n,q}(x:h)=0.$$

Proof. By using () and (2.1), we have

$$\sum_{n=0}^{\infty}\mathbb{E}_{n,q}(x:h)\frac{t^n}{[n]_q!}=\frac{2}{e_{q,h}(1:t)+1}e_{q,h}(qx:t)$$

$$=\frac{1}{2}\left(\frac{2}{e_{q,q^{-1}h}(1:qt)+1}e_{q,q^{-1}h}(1:qt)+\frac{2}{e_{q,q^{-1}h}(1:t)+1}\right)\times\frac{2}{e_{q,h}(1:t)+1}e_{q,h}(qx:t).$$

$$2\sum_{n=0}^{\infty}\mathbb{E}_{n,q}(x:h)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_q q^k\left(\mathbb{E}_{k,q}(2:q^{-1}h)+\mathbb{E}_{n,q}(q^{-1}h)\right)\mathbb{E}_{n-k,q}(qx:h)\right)\frac{t^n}{[n]_q!}. \tag{2.11}$$

From the above equation (2.11), we get

$$\sum_{k=0}^{n}\binom{n}{k}_q q^k\left(\mathbb{E}_{k,q}(2:q^{-1}h)+\mathbb{E}_{n,q}(q^{-1}h)\right)\mathbb{E}_{n-k,q}(qx:h)-2\mathbb{E}_{n,q}(x:h)=0. \tag{2.12}$$

Substituting qx for x in Corollary 2.3, we get

$$\mathbb{E}_{n-k,q}(qx:h)=\frac{[n-k]_q!}{[n]_q!}D_{q,h,x}^{(k)}\mathbb{E}_{n,q}(qx:h). \tag{2.13}$$

Applying equations (2.12) and (2.13), we obtain

$$\sum_{k=0}^{n}\frac{q^k\left(\mathbb{E}_{k,q}(2:q^{-1}h)+\mathbb{E}_{n,q}(q^{-1}h)\right)}{[k]_q!}D_{q,h,x}^{(k)}\mathbb{E}_{n,q}(qx:h)-2\mathbb{E}_{n,q}(x:h)=0.$$

Therefore, we acquire at the desired result. □

Corollary 2.7. Setting $h\to 0$ in Theorem 2.4, we have

$$\begin{aligned}&\frac{q^n\left(\mathbb{E}_{n,q}(2)+\mathbb{E}_{n,q}\right)}{[n]_q!}D_{q,x}^{(n)}\mathbb{E}_{n,q}(qx)\\&+\frac{q^{n-1}\left(\mathbb{E}_{n-1,q}(2)+\mathbb{E}_{n-1,q}\right)}{[n-1]_q!}D_{q,x}^{(n-1)}\mathbb{E}_{n-1,q}(qx)\\&+\cdots+\frac{q^2\left(\mathbb{E}_{2,q}(2)+\mathbb{E}_{2,q}\right)}{[2]_q!}D_{q,x}^{(2)}\mathbb{E}_{n,q}(qx)\\&+q\left(\mathbb{E}_{1,q}(2)+\mathbb{E}_{1,q}\right)D_{q,x}^{(1)}\mathbb{E}_{n,q}(qx)\\&+\left(\mathbb{E}_{0,q}(2)+\mathbb{E}_{0,q}-2\right)\mathbb{E}_{n,q}(x)=0.\end{aligned}$$

Theorem 2.5. For $\mid q\mid\geq 1$ with $a,b\neq 0$, we derive a basic symmetry relation for difference equation as

$$\begin{aligned}&\frac{b^n\mathbb{E}_{n,q}(ay:b^{-1}h)}{[n]_q!}D_{q,h}^{(n)}\mathbb{E}_{n,q}(bx:a^{-1}h)+\frac{b^{n-1}a\mathbb{E}_{n-1,q}(ay:b^{-1}h)}{[n-1]_q!}D_{q,h}^{(n-1)}\mathbb{E}_{n,q}(bx:a^{-1}h)+\\&\cdots+\frac{b^2a^{n-2}\mathbb{E}_{2,q}(ay:b^{-1}h)}{[2]_q!}D_{q,h}^{(2)}\mathbb{E}_{n,q}(bx:a^{-1}h)+ba^{n-1}\mathbb{E}_{1,q}(ay:b^{-1}h)D_{q,h}^{(1)}\mathbb{E}_{n,q}(bx:a^{-1}h)\\&+a^n\mathbb{E}_{0,q}(ay:b^{-1}h)\mathbb{E}_{n,q}(bx:a^{-1}h)\end{aligned}$$

$$\begin{aligned}&=\frac{a^n\mathbb{E}_{n,q}(by:a^{-1}h)}{[n]_q!}D_{q,h}^{(n)}\mathbb{E}_{n,q}(ax:b^{-1}h)+\frac{a^{n-1}b\mathbb{E}_{n-1,q}(by:a^{-1}h)}{[n-1]_q!}D_{q,h}^{(n-1)}\mathbb{E}_{n,q}(ax:b^{-1}h)+\\&\cdots+\frac{a^2b^{n-2}\mathbb{E}_{2,q}(by:a^{-1}h)}{[2]_q!}D_{q,h}^{(2)}\mathbb{E}_{n,q}(ax:b^{-1}h)+ab^{n-1}\mathbb{E}_{1,q}(by:a^{-1}h)D_{q,h}^{(1)}\mathbb{E}_{n,q}(ax:b^{-1}h)\\&+b^n\mathbb{E}_{0,q}(by:a^{-1}h)\mathbb{E}_{n,q}(ax:b^{-1}h).\qquad()\end{aligned}$$

Proof. From (2.1), we find a relation

$$e_{q,h}(abx:t)=\sum_{n=0}^{\infty}a^{n}(bx)(bx-a^{-1}h)(bx-[2]_q a^{-1}h)\cdots(bx-[m-1]_q a^{-1}h)\frac{t^m}{[m]_q!}$$

$$=e_{q,a^{-1}h}(bx:at).$$

If $e_{q,h}(abx:t)=e_{q,a^{-1}h}(bx:at)$, we can take

$$A(t)=\frac{2^2 e_{q,h}(abx:t)e_{q,h}(aby:t)}{(e_{q,a^{-1}h}(1:at)+1)(e_{q,b^{-1}h}(1:bt)+1)}$$

$$=\frac{2}{e_{q,a^{-1}h}(1:at)+1}e_{q,h}(abx:t)\frac{2}{e_{q,b^{-1}h}(1:bt)+1}e_{q,h}(aby:t)$$

$$=\frac{2}{e_{q,a^{-1}h}(1:at)+1}e_{q,a^{-1}h}(bx:at)\frac{2}{e_{q,b^{-1}h}(1:bt)+1}e_{q,b^{-1}h}(ay:bt)$$

$$=\sum_{n=0}^{\infty}a^n\mathbb{E}_{n,q}(bx:a^{-1}h)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}b^k\mathbb{E}_{k,q}(ay:b^{-1}h)\frac{t^k}{[k]_q!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_q a^{n-k}b^k\mathbb{E}_{n-k,q}(bx:a^{-1}h)\mathbb{E}_{k,q}(ay:b^{-1}h)\right)\frac{t^n}{[n]_q!}. \qquad ()$$

Similarly, we obtain

$$A(t)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_q b^{n-k}a^k\mathbb{E}_{n-k,q}(ax:b^{-1}h)\mathbb{E}_{k,q}(by:a^{-1}h)\right)\frac{t^n}{[n]_q!}. \qquad ()$$

Comparing the coefficients of both sides in () and (), we obtain

$$\sum_{k=0}^{n}\binom{n}{k}_q a^{n-k}b^k\mathbb{E}_{n-k,q}(bx:a^{-1}h)\mathbb{E}_{k,q}(ay:b^{-1}h)$$

$$=\sum_{k=0}^{n}\binom{n}{k}_q b^{n-k}a^k\mathbb{E}_{n-k,q}(ax:b^{-1}h)\mathbb{B}_{k,q}(by:a^{-1}h). \qquad ()$$

From Corollary 2.3, we have

$$\mathbb{E}_{n-k,q}(bx,a^{-1}h)=\frac{[n-k]_q!}{[n]_q!}D^{(k)}_{q,h,x}\mathbb{E}_{n,q}(bx,a^{-1}h),$$

$$\mathbb{E}_{n-k,q}(ax,b^{-1}h)=\frac{[n-k]_q!}{[n]_q!}D^{(k)}_{q,h,x}\mathbb{E}_{n,q}(ax,b^{-1}h). \qquad ()$$

Replacing () with equation (), we have

$$\sum_{k=0}^{n}\frac{a^{n-k}b^k\mathbb{E}_{k,q}(ay:b^{-1}h)}{[k]_q!}D^{(k)}_{q,h,x}\mathbb{E}_{k,q}(bx,a^{-1}h)$$

$$\sum_{k=0}^{n}\frac{a^k b^{n-k}\mathbb{E}_{k,q}(by:a^{-1}h)}{[k]_q!}D^{(k)}_{q,h,x}\mathbb{E}_{n,q}(ax,b^{-1}h). \qquad ()$$

From (2.11), we complete the proof of Theorem 2.6.

□

Corollary 2.8. On setting $q \to 1$ in Theorem 2.5, we have

$$\frac{b^n\mathbb{E}_n(ay:b^{-1}h)}{n!}D_h^{(n)}\mathbb{E}_n(bx:a^{-1}h)+\frac{b^{n-1}a\mathbb{E}_{n-1}(ay:b^{-1}h)}{(n-1)!}D_h^{(n-1)}\mathbb{E}_n(bx:a^{-1}h)+$$

$$\cdots+\frac{b^2a^{n-2}\mathbb{E}_2(ay:b^{-1}h)}{2!}D_h^{(2)}\mathbb{E}_n(bx:a^{-1}h)+ba^{n-1}\mathbb{E}_1(ay:b^{-1}h)D_h^{(1)}\mathbb{E}_n(bx:a^{-1}h)$$

$$+a^n\mathbb{E}_0(ay:b^{-1}h)\mathbb{E}_n(bx:a^{-1}h)$$

$$=\frac{a^n\mathbb{B}_n(by:a^{-1}h)}{n!}D_h^{(n)}\mathbb{B}_n(ax:b^{-1}h)+\frac{a^{n-1}b\mathbb{B}_{n-1}(by:a^{-1}h)}{(n-1)!}D_h^{(n-1)}\mathbb{B}_n(ax:b^{-1}h)+$$

$$\cdots+\frac{a^2b^{n-2}\mathbb{B}_2(by:a^{-1}h)}{2!}D_h^{(2)}\mathbb{B}_n(ax:b^{-1}h)+ab^{n-1}\mathbb{B}_1(by:a^{-1}h)D_h^{(1)}\mathbb{B}_n(ax:b^{-1}h)$$

$$+b^n\mathbb{B}_0(by:a^{-1}h)\mathbb{B}_n(ax:b^{-1}h). \qquad ()$$

Corollary 2.9. Setting $h \to 0$ in Theorem 2.5, we have

$$\frac{b^n\mathbb{E}_{n,q}(ay)}{[n]_q!}D_q^{(n)}\mathbb{E}_{n,q}(bx)+\frac{b^{n-1}a\mathbb{E}_{n-1,q}(ay)}{[n-1]_q!}D_q^{(n-1)}\mathbb{E}_{n,q}(bx)+$$

$$\cdots+\frac{b^2a^{n-2}\mathbb{E}_{2,q}(ay)}{[2]_q!}D_q^{(2)}\mathbb{E}_{n,q}(bx)+ba^{n-1}\mathbb{E}_{1,q}(ay)D_q^{(1)}\mathbb{E}_{n,q}(bx)$$

$$+a^n\mathbb{E}_{0,q}(ay)\mathbb{E}_{n,q}(bx)$$

$$=\frac{a^n\mathbb{E}_{n,q}(by)}{[n]_q!}D_q^{(n)}\mathbb{E}_{n,q}(ax)+\frac{a^{n-1}b\mathbb{E}_{n-1,q}(by)}{[n-1]_q!}D_q^{(n-1)}\mathbb{E}_{n,q}(ax)+$$

$$\cdots+\frac{a^2b^{n-2}\mathbb{E}_{2,q}(by)}{[2]_q!}D_q^{(2)}\mathbb{E}_{n,q}(ax)+ab^{n-1}\mathbb{E}_{1,q}(by)D_q^{(1)}\mathbb{E}_{n,q}(ax)$$

$$+b^n\mathbb{E}_{0,q}(by)\mathbb{E}_{n,q}(ax). \qquad ()$$

Theorem 2.6. For $\mid q \mid\geq 1$ with $a,b\neq 0$, we derive a basic symmetry relation for difference equation as

$$\frac{\beta^m\mathbb{E}_{n,q}(\beta^{-1}h)}{[n]_q!}D_{q,h}^{(n)}\mathbb{E}_{n,q}(\beta x:\alpha^{-1}h)+\frac{\beta^{m-1}\alpha\mathbb{E}_{n-1,q}(\beta^{-1}h)}{[n-1]_q!}D_{q,h}^{(n-1)}\mathbb{E}_{n,q}(\beta x:\alpha^{-1}h)$$

$$+\cdots+\frac{\beta^2\alpha^{m-2}\mathbb{E}_{2,q}(\beta^{-1}h)}{[n]_q!}D_{q,h}^{(2)}\mathbb{E}_{n,q}(\beta x:\alpha^{-1}h)+\beta\alpha^{n-1}\mathbb{E}_{1,q}(\beta^{-1}h)D_{q,h}^{(1)}\mathbb{E}_{n,q}(\beta x:\alpha^{-1}h)$$

$$+\alpha^n\mathbb{E}_{0,q}(\beta^{-1}h)\mathbb{E}_{n,q}(\beta x:\alpha^{-1}h)$$

$$=\frac{\alpha^m\mathbb{E}_{n,q}(\alpha^{-1}h)}{[n]_q!}D_{q,h}^{(n)}\mathbb{E}_{n,q}(\alpha x:\beta^{-1}h)+\frac{\alpha^{m-1}\beta\mathbb{E}_{n-1,q}(\alpha^{-1}h)}{[n-1]_q!}D_{q,h}^{(n-1)}\mathbb{E}_{n,q}(\alpha x:\beta^{-1}h)$$

$$+\cdots+\frac{\alpha^2\beta^{m-2}\mathbb{E}_{2,q}(\alpha^{-1}h)}{[n]_q!}D_{q,h}^{(2)}\mathbb{E}_{n,q}(\alpha x:\beta^{-1}h)+\alpha\beta^{n-1}\mathbb{E}_{1,q}(\alpha^{-1}h)D_{q,h}^{(1)}\mathbb{E}_{n,q}(\alpha x:\beta^{-1}h)$$

$$+\beta^n\mathbb{E}_{0,q}(\alpha^{-1}h)\mathbb{E}_{n,q}(\alpha x:\beta^{-1}h). \qquad ()$$

Proof. To obtain another symmetric difference equation which is related to the degenerate quantum Bernoulli polynomials, we have

$$B(t) = \frac{2^2 e_{q,h}(\alpha\beta x : t)}{(e_{q,\alpha^{-1}h}(1 : \alpha t) + 1)(e_{q,\beta^{-1}h}(1 : \beta t) + 1)}$$

From the same way as proving Theorem 2.6, we find

$$\begin{aligned}&\sum_{k=0}^{n} \binom{n}{k}_q \beta^k \alpha^{n-k} \mathbb{E}_{k,q}(\beta^{-1}h)\mathbb{E}_{n-k,q}(\beta x : \alpha^{-1}h)\\ &= \sum_{k=0}^{n} \binom{n}{k}_q \alpha^k \beta^{n-k} \mathbb{E}_{k,q}(\alpha^{-1}h)\mathbb{E}_{n-k,q}(\alpha x : \beta^{-1}h). \qquad ()\end{aligned}$$

Using Corollary 2.3 in (), we obtain

$$\begin{aligned}&\sum_{k=0}^{n} \frac{\beta^k \alpha^{n-k} \mathbb{E}_{k,q}(\beta^{-1}h)}{[k]_q!} D_{q,h}^{(k)} \mathbb{E}_{n,q}(\beta x : \alpha^{-1}h)\\ &= \sum_{k=0}^{n} \frac{\alpha^k \beta^{n-k} \mathbb{E}_{k,q}(\alpha^{-1}h)}{[k]_q!} D_{q,h}^{(k)} \mathbb{E}_{n,q}(\alpha x : \beta^{-1}h). \qquad ()\end{aligned}$$

From (), we complete the proof of Theorem 2.7.

□

Theorem 2.7. The degenerate quantum Bernoulli polynomials are solutions of the following higher-order differential equation combined with the degenerate quantum tangent numbers and polynomials

$$\begin{aligned}&\frac{(T_{n,q}(2 : h) + T_{n,q}(h))}{[n]_q!} D_{q,h,x}^{(n)} \mathbb{E}_{n,q}(x : h) + \frac{(T_{n-1,q}(2 : h) + T_{n-1,q}(h))}{[n-1]_q!} D_{q,h,x}^{(n-1)} \mathbb{E}_{n,q}(x : h)\\ &+ \cdots + \frac{(T_{2,q}(2 : h) + T_{2,q}(h))}{[2]_q!} D_{q,h,x}^{(2)} \mathbb{E}_{n,q}(x : h) + (T_{1,q}(2 : h) + T_{1,q}(h))\, D_{q,h,x}^{(1)} \mathbb{E}_{n,q}(x : h)\\ &+ (T_{0,q}(2 : h) + T_{0,q}(h) - 2)\, \mathbb{E}_{n,q}(x : h) = 0.\end{aligned}$$

Proof. Using (2.1), we note that By using () and (), we have

$$\begin{aligned}&\sum_{n=0}^{\infty} \mathbb{E}_{n,q}(x : h) \frac{t^n}{[n]_q!} = \frac{2}{e_{q,h}(1 : t) + 1} e_{q,h}(x : t)\\ &= \frac{1}{2}\left(\frac{2}{e_{q,h}(2 : t) + 1} e_{q,h}(1 : t) + \frac{2}{e_{q,h}(2 : t) + 1} \right) \frac{2}{e_{q,h}(1 : t) + 1} e_{q,h}(x : t)\\ &= \frac{1}{2} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k}_q (T_{k,q}(1 : h) + T_{k,q}(h))\, \mathbb{E}_{n-k,q}(x : h) \right) \frac{t^n}{[n]_q!}. \qquad ()\end{aligned}$$

Comparing the coefficients on both sides of equation (), we have

$$2\mathbb{E}_{n,q}(x : h) = \sum_{k=0}^{n} \binom{n}{k}_q (T_{k,q}(1 : h) + T_{k,q}(h))\, \mathbb{E}_{n-k,q}(x : h). \qquad ()$$

Using the relationship of the (q, h)-Euler polynomials to the k-times (q, h)-derivative in (), we get

$$\sum_{k=0}^{n} \frac{(T_{k,q}(1 : h) + T_{k,q}(h))}{[k]_q!} D_{q,h,x}^{(k)} \mathbb{E}_{n,q}(x : h) - 2\mathbb{E}_{n,q}(x : h) = 0.$$

The above equation completes the proof. □

Corollary 2.10. Setting $h \to 0$ in Theorem 2.7, we have

$$\begin{aligned}&\frac{T_{n,q}+T_{n,q}(1)}{[n]_q!}D_{q,x}^{(n)}\mathbb{E}_{n,q}(x)+\frac{T_{n-1,q}+T_{n-1,q}(1)}{[n-1]_q!}D_{q,x}^{(n-1)}\mathbb{E}_{n,q}(x)\\&+\cdots+\frac{T_{2,q}+T_{2,q}(1)}{[2]_q!}D_{q,x}^{(2)}\mathbb{E}_{n,q}(x)+T_{1,q}+T_{1,q}(1)D_{q,x}^{(1)}\mathbb{E}_{n,q}(x)\\&+\left(T_{0,q}+T_{0,q}(1)-2\right)\mathbb{E}_{n,q}(x)=0.\end{aligned}$$

Corollary 2.11. Setting $q \to 1$ in Theorem 2.7, we have

$$\begin{aligned}&\frac{(T_n(2:h)+T_n(h))}{n!}D_{h,x}^{(n)}\mathbb{E}_n(x:h)+\frac{(T_{n-1}(2:h)+T_{n-1}(h))}{(n-1)!}D_{h,x}^{(n-1)}\mathbb{E}_n(x:h)+\cdots+\frac{(T_2(2:h)+E_2(h))}{2!}D_{h,x}^{(2)}\mathbb{E}_n(x:h)\\&+\left(T_1(2:h)+T_1(h)\right)D_{h,x}^{(1)}\mathbb{E}_n(x:h)+\left(T_0(2:h)+T_0(h)-2\right)\mathbb{E}_n(x:h)=0,\end{aligned}$$

where D_h is the h-derivative and $\mathbb{E}_n(x:h)$ are called the degenerate Euler polynomials.

3. Conclusion

We constructed (q,h)-analogue of Euler polynomials and numbers and found several differential equations with these polynomials as solutions. We also found differential equations combining with q-Bernoulli and q-Genocchi polynomials. We established some properties of (q,h)-analogue of Euler polynomials and numbers. The results from this paper have highlighted interesting topics for constructing Euler polynomials with bivariate quantum numbers and properties.

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