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Symmetry Reductions, Qualitative Analysis and Conservation Law of Gardner's Equation

Mitu Nagpal[1],*Rajeev Kumar[2 †]
Department of Mathematics[1,2]
Maharishi Markandeshwar Deemed to be University
Mullana, Ambala-133001(Haryana), India

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Abstract

This study secures Lie symmetry analysis and traveling wave solutions to the well-known generalized form of Gardner's equation with three dispersion sources. Using the Lie symmetry approach, infinitesimal generators and symmetry reductions of the Gardner's equation with triplet dispersion sources have been discovered. The equation's reduced form is found and it is solved by the $\frac{G'}{G}$ approach, which yields several solution forms. In a reduced form, bifurcation theory has been used to analyze the stability and properties of critical points by converting the equation into an autonomous system. Phase portraits have been plotted at various critical points to depict the system's behavior and assess its stability. The conservation laws are determined using the multipliers approach. First, the multipliers are computed based on independent variables, dependent variables and derivatives of dependent variables up to a certain order. The fluxes of the conservation laws are then obtained for conserved vector.

Key words: Gardner's equation; Lie group analysis method; Qualitative analysis; Traveling wave solutions.

1 Introduction

1.1 Scope

In recent decades, Non linear partial differential equations (NLPDEs)[1]-[2] have emerged significant attention for their wide-ranging applications in various nonlinear fields, including physics, mathematics, biology, chemistry, signal processing, viscoelasticity, electronics and soft materials. Finding explicit and exact solutions to NLPDEs [3]-[5], which are essential to the applied mathematical sciences, is one of the most exciting and active research topics at the moment. Numerous studies have presented numerous efficient techniques for generating explicit solutions for NLPDEs, lie symmetry analysis[6]-[7], adomian-decomposition method[8], laplace transform[9], variational- iteration method[10], homotopy-perturbation

*gakharmitu@gmail.com[1]
†rajeevkumarbudhiraja@gmail.com[2]

method[11], invariant subspace method[12], etc.

Lie's approach is one of the most comprehensive and effective way to study the analytical solutions and symmetry characteristics of NLPDEs[13]-[19]. One-parameter Lie group of transformations[20] are the foundation of Lie group methodology, which was initially introduced by mathematician Sophus Lie. The Lie method is an effective approach for solving a wide range of equations. This method continues to be widely utilized by researchers to derive precise solutions for NLPDEs. This article aims to explore the Lie symmetry analysis of the Dual-Nonlinearity Gardner's equation[21], commonly referred to as the KdV-mKdV equation.

The Korteweg-de Vries (KdV) equation[22] is one of the many well-studied models developed for shallow water waves. Numerous studies have examined its integrability, numerical methods, and vector formulations for addressing multi-layered flows[23]-[25]. Numerous forms of nonlinearity, such as random power law nonlinearities, have been applied to the KdV equation. The KdV equation was later generalized to include dual non-linear effects, resulting in Gardners Equation. This equation has since been extensively studied to better understand shallow water wave phenomena[26]-[30]. In this way, the well-known models, previously examined with third order dispersion are extended. The BenjaminBonaMahoney equation describes spatio-temporal dispersion in shallow water waves. However, Joseph and Egri[31] first introduced the concept of dual temporal-spatial dispersion in 1977 and it has not been reviewed since. In this work, the revised model incorporates triplet dispersive effects, including triple spatial dispersion, dual-temporal spatial dispersion and spatiotemporal dispersion.

1.2 Governing Model

The following structure contains both the KdV equation's type and generalized form[22] :

$$q_t + F(q)q_x + cq_{xxx} + dq_{xxt} + eq_{xtt} = 0, \tag{1.1}$$

These independent variables denoted by x and t represent temporal and spatial factors respectively. Therefore, the wave profile is represented by the dependent variable q(x,t). Consequently, the wave profile is expressed by the dependent variable q(x,t). The wave's temporal evolution is the first term. The second term comes from nonlinear effects, comprising two nonlinear terms that result in Gardner's Equations and one effect for the KdV equation. The coefficients c, d and e from the triple spatial dispersion, spatio-temporal dispersion and dual temporal-spatial dispersion effects, respectively, are carried by the three dispersion terms.

This study focuses on Equation (1.1), which contains two nonlinear terms, $F(q) = aq + bq^2$ and $F(q) = aq^n + bq^{2n}$, where a and b are real-valued constants. The paper further explores Gardners equation and its power-law nonlinear generalization in separate subsections.

Gardner Equation (KdV-mKdV Equation):

Here $F(q) = aq + bq^2$, for any real-valued constants a and b. From (1.1), this leads to

$$q_t + (aq + bq^2)q_x + cq_{xxx} + dq_{xxt} + eq_{xtt} = 0. \tag{1.2}$$

Power - Law Nonlinearity:

Here $F(q) = aq^n + bq^{2n}$, for any real-valued constants a and b. Equation (1.1) leads to

$$q_t + (aq^n + bq^{2n})q_x + cq_{xxx} + dq_{xxt} + eq_{xtt} = 0. \tag{1.3}$$

1.3 Motivation

KdV-mKdV equation with three sources of dispersion studied by applying the semi inverse variational principle and further by applying the separation of variables method and integrating, a solitary wave solution is obtained. In this study, Lie symmetry analysis and symmetry reduction have obtained with the following motivation.

Sophus Lie's concept of using transformation groups[32] to investigate differential equation systems and partial differential equation systems proposed theory: the theory of symmetry groups, sometimes referred to as the classical Lie method of infinitesimal transformations. In addition to determining group-invariant solutions for this PDE and group-invariant PDEs invariant under this symmetry group, Bluman and Cole [13]also discovered the symmetry group of the heat equation provided by Lie.

This approach, which examines the characteristics of differential equations has been used by numerous scholars to look into the precise solutions of differential equations. For example, Miller used the separable variables approach to apply the symmetry group theory to the second order linear PDE solutions. P.J. Olver[40] also used the Lie symmetry to solve the ordinary differential equations. Also, B. Cantwell[6] successfully apply the Lie symmetries analysis over ordinary differential equations. Also, Anjan Biswas and Amupma Bansal[21] successfully apply the Lie symmetries analysis over fractional NLPDEs. This study is motivated to find the solution of NLPDEs using Lie symmetry analysis.

1.4 Overview

The structure of this paper is as follows: Section 2 is devoted to obtaining the Lie symmetry analysis and deriving exact solutions for KdV-mKdV equation and power-law nonlinearity equation with three dispersion sources.Section 3 outlines the algorithm of the $\frac{G'}{G}$ - expansion method, while Section 4 applies this method to obtain the traveling wave solution of the KdV-mKdV equation. In addition, a bifurcation analysis of the power-law nonlinearity equation is conducted. Section 5 includes conservation law for Gardner's equation. Finally, Section 6 summarizes the findings and presents the conclusion.

2 Lie Symmetry Analysis

2.1. Lie Symmetry for NLPDES

This section provides a detailed explanation of the terminology and primary notations that are needed to obtain the symmetries of NLPDEs. The Lie group of point transformations[33] is considered to determine the symmetries. Lie group of transformation with a single parameter $\epsilon < 1$ is assumed to be used as

$$\begin{aligned}
\bar{x^*} &= \bar{x} + \epsilon.\tilde{\bar{\xi}}(x,t,q) + \bar{O}(\epsilon^2),\\
\bar{t^*} &= \bar{t} + \epsilon.\tilde{\bar{\tau}}(x,t,q) + \bar{O}(\epsilon^2),\\
\bar{q^*} &= \bar{q} + \epsilon.\tilde{\bar{\eta}}(x,t,q) + \bar{O}(\epsilon^2). \qquad (2.1)
\end{aligned}$$

2.1.1. KdV-mKdV Equation

Using the Lie symmetry analysis outlined in Section 2.1, this subsection derives the Lie symmetry group and infinitesimal generators of the KdV-mKdV equation (1.2).

Theorem 2.1. The Lie point symmetry admitted by KdV-mKdV equation expressed as are

$$\tilde{E}_1 = \frac{\partial}{\partial x}, \tilde{E}_2 = \frac{\partial}{\partial t}, \tilde{E}_3 = \frac{\partial}{\partial q} and \tilde{E}_4 = q\frac{\partial}{\partial q} \qquad (2.2)$$

Proof: Suppose equation (1.2) is invariant under the transformation (2.1), the obtain relation having first order of ϵ coefficients :

$$\tilde{\bar{\eta}}_t + aq\tilde{\bar{\eta}}_x + a\tilde{\bar{\eta}}q_x + bq^2\tilde{\bar{\eta}}_x + 2bqq_x\tilde{\bar{\eta}} + c\tilde{\bar{\eta}}_{xxx} + d\tilde{\bar{\eta}}_{xxt} + e\tilde{\bar{\eta}}_{xtt} = 0, \qquad (2.3)$$

where the extended infinitesimals $\tilde{\bar{\eta}}_t, \tilde{\bar{\eta}}_x, \tilde{\bar{\eta}}_{xxx}, \tilde{\bar{\eta}}_{xxt}$, and $\tilde{\bar{\eta}}_{xtt}$ inserting into invariance condition equation (2.3) and equating the coefficients of all derivatives of q_t, q_x, q_{xxx}, q_{xxt}, and q_{xtt}. Now we get the following determined equations:

$$\begin{aligned}
&\tilde{\bar{\xi}}_q = 0,\\
&\tilde{\bar{\tau}}_x = 0 = \tilde{\bar{\tau}}_q,\\
&\tilde{\bar{\eta}}_{qq} = 0 = \tilde{\bar{\eta}}_{xq},\\
&\tilde{\bar{\tau}}_t + \tilde{\bar{\xi}}_x = 0,\\
&d\tilde{\bar{\xi}}_x + e\tilde{\bar{\xi}}_t = 0,\\
&c\tilde{\bar{\tau}}_t + 3c\tilde{\bar{\xi}}_x + d\tilde{\bar{\xi}}_t = 0,\\
&d\tilde{\bar{\eta}}_{tq} - e\tilde{\bar{\xi}}_{tt} - 3c\tilde{\bar{\xi}}_{xx} - 2d\tilde{\bar{\xi}}_{xt} = 0,\\
&2e\tilde{\bar{\eta}}_{tq} - e\tilde{\bar{\tau}}_{tt} - d\tilde{\bar{\xi}}_{xx} - 2e\tilde{\bar{\xi}}_{xt} = 0,\\
&\tilde{\bar{\eta}}_t + bu^2\tilde{\bar{\eta}}_x + c\tilde{\bar{\eta}}_{xxx} + d\tilde{\bar{\eta}}_{xxt} + e\tilde{\bar{\eta}}_{xtt} + au\tilde{\bar{\eta}}_x - b\tilde{\bar{\xi}}_x + b\tilde{\bar{\tau}}_t = 0,\\
&-\tilde{\bar{\xi}}_t + a\tilde{\bar{\eta}} + e\tilde{\bar{\eta}}_{ttq} - d\tilde{\bar{\xi}}_{xxt} - c\tilde{\bar{\xi}}_{xxx} + a\tilde{\bar{\tau}}_t q - aq\tilde{\bar{\xi}}_x - e\tilde{\bar{\xi}}_{xtt} - bq^2\tilde{\bar{\xi}}_x + 2bq\bar{\eta} + bq^2\tilde{\bar{\tau}}_t = 0. \qquad (2.4)
\end{aligned}$$

These determining equations can be solved to get

$$\begin{aligned}
\tilde{\bar{\eta}} &= q\tilde{h}_3 + \tilde{h}_4,\\
\tilde{\bar{\xi}} &= \tilde{h}_1,\\
\tilde{\bar{\tau}} &= \tilde{h}_2,
\end{aligned}$$

where the constants $\tilde{h}_i (i = 1, 2, 3, 4)$ are arbitrary. The vector field can be expressed in the form of equation (2.2).

The appropriate invariant solution for the symmetry $\tilde{E}_1 + \tilde{E}_2$ is as follows:

$$\tilde{\zeta} = x + Zt, \tag{2.5}$$

where Z is arbitrary constant.

2.1.2. Power Law Nonlinearity Equation

In this subsection we investigate of Lie point symmetry of Power Law Nonlinearity equation (1.3).

Theorem 2.2. The Lie point symmetry admitted by KdV-mKdV equation expressed as equation (1.3) are

$$\tilde{E}_1 = \frac{\partial}{\partial x}, \tilde{E}_2 = \frac{\partial}{\partial t}, \tilde{E}_3 = \frac{\partial}{\partial q} and \tilde{E}_4 = q\frac{\partial}{\partial q} \tag{2.6}$$

Proof: Suppose equation (1.3) is invariant under the transformation (2.1), the obtain relation having first order of ϵ coefficients :

$$\tilde{\tilde{\eta}}_t + aq^n\tilde{\tilde{\eta}}_x + nq^{(n-1)}a\tilde{\tilde{\eta}}q_x + bq^{2n}\tilde{\tilde{\eta}}_x + 2nbq^{2n-1}q_x\tilde{\tilde{\eta}} + c\tilde{\tilde{\eta}}_{xxx} + d\tilde{\tilde{\eta}}_{xxt} + e\tilde{\tilde{\eta}}_{xtt} = 0, \tag{2.7}$$

where $\tilde{\tilde{\eta}}_t, \tilde{\tilde{\eta}}_x, \tilde{\tilde{\eta}}_{xxx}, \tilde{\tilde{\eta}}_{xxt}$, and $\tilde{\tilde{\eta}}_{xtt}$ inserting into invariance condition equation (2.7) and equating the coefficients of all derivatives of q_t, q_x, q_{xxx}, q_{xxt}, and q_{xtt}. Now we get the following determined equations:

$$\begin{aligned}
\tilde{\tilde{\xi}}_q &= 0,\\
\tilde{\tilde{\tau}}_x &= 0 = \tilde{\tilde{\tau}}_q,\\
\tilde{\tilde{\eta}}_{qq} &= 0 = \tilde{\tilde{\eta}}_{xq},\\
\tilde{\tilde{\tau}}_t + \tilde{\tilde{\xi}}_x &= 0,\\
d\tilde{\tilde{\xi}}_x + e\tilde{\tilde{\xi}}_t &= 0,\\
d\tilde{\tilde{\xi}}_t + 3c\tilde{\tilde{\xi}}_x - c\tilde{\tilde{\tau}}_t &= 0,\\
2e\tilde{\tilde{\eta}}_{tq} - e\tilde{\tilde{\tau}}_{tt} - 2e\tilde{\tilde{\xi}}_{xt} - d\tilde{\tilde{\xi}}_{xx} &= 0,\\
d\tilde{\tilde{\eta}}_{tq} - e\tilde{\tilde{\xi}}_{tt} - 2d\tilde{\tilde{\xi}}_{xt} - 3c\tilde{\tilde{\xi}}_{xx} &= 0,\\
bq^{2n}\tilde{\tilde{\eta}}_x + c\tilde{\tilde{\eta}}_{xxx} + d\tilde{\tilde{\eta}}_{xxt} + e\tilde{\tilde{\eta}}_{xtt} + aq^n\tilde{\tilde{\eta}}_x + \tilde{\tilde{\eta}}_t &= 0,
\end{aligned}$$

$$bq^{2n}\tilde{\tilde{\tau}}_t - bq^{2n}\tilde{\tilde{\xi}}_x - c\tilde{\tilde{\xi}}_{xxx} + e\tilde{\tilde{\eta}}_{ttu} - aq^n\tilde{\tilde{\xi}}_x - \tilde{\tilde{\xi}}_t + aq^n\tilde{\tilde{\tau}}_t - d\tilde{\tilde{\xi}}_{xxt} - e\tilde{\tilde{\xi}}_{xtt} + anq^{n-1}\bar{\eta} + 2bnq^{2n-1}\tilde{\tilde{\eta}} = 0 \tag{2.8}$$

These determining equations can be solved to get

$$\tilde{\tilde{\eta}} = q\tilde{h}_3 + \tilde{h}_4,$$

$$\tilde{\tilde{\xi}} = \tilde{h}_1,$$

$$\tilde{\tilde{\tau}} = \tilde{h}_2,$$

where the constants $\tilde{h}_i$ and $i = 1, 2, 3, 4$ are arbitrary. The vector field can be expressed in the form of equation (2.6).

2.2 Symmetry Reductions

In the prior section , we derived the infinitesimal generators for the NLPDEs In this subsection, we conduct the similarity reduction of the KdV-mKdV equation presented in equation (1.2) and subsequently compute the exact solutions. The vector fields that follow span the corresponding symmetry group:

$$\tilde{E}_1 = \frac{\partial}{\partial x},\ \tilde{E}_2 = \frac{\partial}{\partial t},\ \tilde{E}_3 = \frac{\partial}{\partial q} \text{ and } \tilde{E}_4 = q\frac{\partial}{\partial q}$$

The appropriate invariant solution for the symmetry $\tilde{E}_1 + \tilde{E}_2$ is as follows:

$$\tilde{\zeta} = x + Zt, \tag{2.9}$$

where Z is an arbitrary constant.

Also we perform the similarity reduction of the power law nonlinearity equation given in equation (1.3). The vector fields that follow span the corresponding symmetry group:

$$\tilde{E}_1 = \frac{\partial}{\partial x},\ \tilde{E}_2 = \frac{\partial}{\partial t},\ \tilde{E}_3 = \frac{\partial}{\partial q} \text{ and } \tilde{E}_4 = q\frac{\partial}{\partial q}$$

The appropriate invariant solution for the symmetry $\tilde{E}_1 + \tilde{E}_2$ is as follows:

$$\tilde{\zeta} = x + Zt, \tag{2.10}$$

where Z is arbitrary constant.

3 $\left(\frac{G'}{G}\right)$- Expansion Method Algorithm

To calculate exact traveling solutions of NLPDEs, Wang et al. in Chinese mathematics introduced the $\left(\frac{G'}{G}\right)$-expansion method[34]-[36]. Due to its simplicity and ability to construct traveling wave solutions, this method is especially useful.
Let $\tilde{x}$ and $\tilde{t}$ be two independent variables in a NLPDE.

$$Q(q, q_{\tilde{t}}, q_{\tilde{x}}, q_{\tilde{t}\tilde{t}}, q_{\tilde{x}\tilde{t}}, q_{\tilde{x}\tilde{x}}, ...), \tag{3.1}$$

in which Q is a polynomial involving q($\tilde{x}$,$\tilde{t}$) and its partial derivatives, where Q is an unknown function.
The mechanism's main points are :

Stage 1
To begin with, apply the wave transformation $\tilde{\zeta} = \tilde{x} - V\tilde{t}$ so that

$$q(\tilde{x}, \tilde{t}) = \tilde{H}(\tilde{\zeta}) \tag{3.2}$$

An ordinary differential equation is obtained by reducing equation (3.2) to

$$S(\tilde{H}, -V\tilde{H}', \tilde{H}', V^2\tilde{H}'', -V\tilde{H}'', \tilde{H}'', \tilde{H}''', ...). \tag{3.3}$$

If necessary, integrate the ordinary differential equation (ODE) (3.3) and to keep things simple, suppose that the integration constants are zero.

Stage 2
Now, suppose that the ODE (3.3) has a solution that may be expressed as $\left(\frac{G'}{G}\right)$

$$\tilde{H}(\tilde{\zeta}) = e_p\left(\frac{G'}{G}\right)^p + e_{p-1}\left(\frac{G'}{G}\right)^{p-1} + ..., \tag{3.4}$$

with the arbitrary constants $e_p = 0, 1, 2, 3, ...$, also $e_p \neq 0$. Since $G(\tilde{\zeta})$ satisfies the form's second-order LDE

$$G'' + \tilde{\omega}G' + \tilde{\delta}G = 0. \tag{3.5}$$

Here $\tilde{\omega}$ and $\tilde{\delta}$ are arbitrary constant. The value of p can be determined easily by establishing a homogeneous-balance between the highest-order derivative and the nonlinear term, as appear in the equation (3.3).

Stage 3
Using Equation (3.5), substitute equation (3.4) into (3.3) to arrange all terms of $(\frac{G'}{G})$ in the same order. Then, equate each coefficient to zero to derive a system of algebraic equations for $e_p, e_{p-1}, ...e_0, V, \tilde{\omega}$ and $\tilde{\delta}$.

Stage 4
Traveling wave solutions of Equation (3.1) are obtained by utilizing the data from Stage 3 and substituting the general solutions of Equation (3.5) into Equation (3.4). This is because Equation (3.5) has well-known general solutions. Equation (3.5)'s general solutions are provided as

$$\left(\frac{G'}{G}\right) = \begin{cases} \frac{\sqrt{\tilde{\omega}^2-4\tilde{\delta}}}{2}\left(\frac{J_1 sinh\left(\frac{\sqrt{\tilde{\omega}^2-4\tilde{\delta}}}{2}\right)\tilde{\zeta}+J_2 cosh\left(\frac{\sqrt{\tilde{\omega}^2-4\tilde{\delta}}}{2}\right)\tilde{\zeta}}{J_1 cosh\left(\frac{\sqrt{\tilde{\omega}^2-4\tilde{\delta}}}{2}\right)\tilde{\zeta}+J_2 sinh\left(\frac{\sqrt{\tilde{\omega}^2-4\tilde{\delta}}}{2}\right)\tilde{\zeta}}\right) - \frac{\tilde{\omega}}{2}, & \tilde{\omega}^2 - 4\tilde{\delta} > 0 \\ \frac{\sqrt{4\tilde{\delta}-\tilde{\omega}^2}}{2}\left(\frac{-J_1 sin\left(\frac{\sqrt{4\tilde{\delta}-\tilde{\omega}^2}}{2}\right)\tilde{\zeta}+J_2 cos\left(\frac{\sqrt{4\tilde{\delta}-\tilde{\omega}^2}}{2}\right)\tilde{\zeta}}{J_1 cos\left(\frac{\sqrt{4\tilde{\delta}-\tilde{\omega}^2}}{2}\right)\tilde{\zeta}+J_2 sin\left(\frac{\sqrt{4\tilde{\delta}-\tilde{\omega}^2}}{2}\right)\tilde{\zeta}}\right) - \frac{\tilde{\omega}}{2}, & 4\tilde{\delta} - \tilde{\omega}^2 > 0 \\ \frac{J_2}{J_1+J_2\tilde{\zeta}} - \frac{\tilde{\omega}}{2} & ,\tilde{\omega}^2 - 4\tilde{\delta} = 0. \end{cases}$$

4 APPLICATIONS

4.1.1. Exact Solutions of KdV-mKdV Equations Using the $\left(\frac{G'}{G}\right)$-Expansion Approach

In this part, we utilize the $\left(\frac{G'}{G}\right)$-expansion approach to obtain the traveling wave solutions of nonlinear KdV-mKdV equations.

Theorem 4.1. Induced by the general infinitesimal generator $\tilde{E}_1 + \tilde{E}_2$ can be reduced to the following form:

$$-V\tilde{H}'\left(\tilde{\zeta}\right) + (a\tilde{H}\left(\tilde{\zeta}\right) + b\tilde{H}^2\left(\tilde{\zeta}\right))\tilde{H}'\left(\tilde{\zeta}\right) + (c - dV + eV^2)\tilde{H}'''\left(\tilde{\zeta}\right) = 0 \tag{4.1}$$

where wave transformation is

$$q(x,t) = \tilde{H}(\tilde{\zeta}), \tilde{\zeta} = x - Vt \tag{4.2}$$

Proof: Let us consider the first form of KdV-mKdV equation :

$$q_t + (aq + bq^2)q_x + cq_{xxx} + dq_{xxt} + eq_{xtt} = 0. \tag{4.3}$$

Applying the wave transformation and also take Z=-V which is non-zero arbitrary constant. $q(x,t) = \tilde{H}(\tilde{\zeta}), \tilde{\zeta} = x - Vt$. Now use the transformations (4.2) convert equation (4.3) into an ODE

$$-V\tilde{H}'\left(\tilde{\zeta}\right) + (a\tilde{H}\left(\tilde{\zeta}\right) + b\tilde{H}^2\left(\tilde{\zeta}\right))\tilde{H}'\left(\tilde{\zeta}\right) + (c - dV + eV^2)\tilde{H}'''\left(\tilde{\zeta}\right) = 0 \tag{4.4}$$

Integrating it with respect to $\tilde{\zeta}$. Also put the integrating constant equal to zero, we get

$$-V\tilde{H}\left(\tilde{\zeta}\right) + a\frac{\tilde{H}^2\left(\tilde{\zeta}\right)}{2} + b\frac{\tilde{H}^3\left(\tilde{\zeta}\right)}{3} + (c - dV + eV^2)\tilde{H}''\left(\tilde{\zeta}\right) = 0 \tag{4.5}$$

Assuming that a polynomial in $\left(\frac{G'}{G}\right)$ will formulate the solutions of equation (4.5),

$$\tilde{H}\left(\tilde{\zeta}\right) = e_m\left(\frac{G'}{G}\right)^m + e_{m-1}\left(\frac{G'}{G}\right)^{m-1} + ..., \tag{4.6}$$

with the arbitrary constants $e_m, m = 0, 1, 2, 3, ...,$ also $e_m \neq 0$. Since $G(\tilde{\zeta})$ satisfies the form's 2nd-order LDE

$$G'' + \tilde{\omega}G' + \tilde{\delta}G = 0. \tag{4.7}$$

Here $\tilde{\omega}$ and $\tilde{\delta}$ are arbitrary constant. Using Equation (4.5) homogeneous balancing between $\tilde{H}''$ and $\tilde{H}^3$, we arrive at m=1. So equation (4.6) can be represented as :

$$\tilde{H}\left(\tilde{\zeta}\right) = e_0 + e_1\left(\frac{G'}{G}\right), e_1 \neq 0 \tag{4.8}$$

$$\tilde{H}\left(\tilde{\zeta}\right)^2 = {e_0}^2 + {e_1}^2\left(\frac{G'}{G}\right)^2 + 2\,e_0e_1\left(\frac{G'}{G}\right) \tag{4.9}$$

$$\tilde{H}\left(\tilde{\zeta}\right)^3 = {e_0}^3 + 3\,{e_0}^2e_1\frac{G'}{G} + 3\,e_0{e_1}^2\left(\frac{G'}{G}\right)^2 + {e_1}^3\left(\frac{G'}{G}\right)^3, \tag{4.10}$$

$$\tilde{H}''\left(\tilde{\zeta}\right) = e_1\tilde{\omega}\,\tilde{\delta} + e_1\tilde{\omega}^2\left(\frac{G'}{G}\right) + 3\,e_1\tilde{\omega}\left(\frac{G'}{G}\right)^2 + 2\,e_1\left(\frac{G'}{G}\right)\tilde{\delta} + 2\,e_1\left(\frac{G'}{G}\right)^3. \tag{4.11}$$

Equation (4.5) gives the function in $(\frac{G'}{G})$ using equations (4.8)-(4.11). Then, The following collection of algebraic equations is produced by setting the coefficients of the identical power of $\left(\frac{G'}{G}\right)$ to zero:

$$\left(\frac{G'}{G}\right)^3 : 2ce_1 + \frac{1}{3}be_1^3 - 2dVe_1 + 2eV^2e_1, \tag{4.12}$$

$$\left(\frac{G'}{G}\right)^2 : 3eV^2e_1\tilde{\omega} + \frac{1}{2}ae_1^2 - 3dVe_1\tilde{\omega} + 3ce_1\tilde{\omega} + be_0e_1^2, \tag{4.13}$$

$$\left(\frac{G'}{G}\right)^1 : ae_0e_1 - Ve_1 + eV^2e_1\tilde{\omega}^2 + 2ce_1\tilde{\delta} - dVe_1\tilde{\omega}^2 - 2dVe_1\tilde{\delta} + ce_1\tilde{\omega}^2 + be_0^2e_1 + 2eV^2e_1\tilde{\delta} \tag{4.14}$$

$$\left(\frac{G'}{G}\right)^0 : -Ve_0 + \frac{1}{2}ae_0^2 + \frac{1}{3}be_0^3 + eV^2e_1\tilde{\omega}\tilde{\delta} + ce_1\tilde{\omega}\tilde{\delta} - dVe_1\tilde{\omega}\tilde{\delta} \tag{4.15}$$

After using Maple to solve the previously mentioned algebraic equations, we receive the traveling wave solutions :
V = V, $a = -\frac{3V}{-e_0^2+\tilde{\omega}e_1^2}$, b = b, $c = \frac{-Ve_1^2-2e_0^2dV^2+2b\tilde{\delta}e_1^2-2be_0^2+2dV^2\tilde{\delta}e_1^2}{2V(-e_0^2+\tilde{\delta}e_1^2)}$, d = d, $\tilde{\delta} = \tilde{\delta}$, $\tilde{\omega} = \frac{2e_0}{e_1}$, $e_0 = e_0$, $e_1 = e_1$.

Enter these values into equation (4.8) and the result is presented as follows:

$$\tilde{H}\left(\tilde{\zeta}\right) = e_0 + e_1\left(\frac{G'}{G}\right), \tag{4.16}$$

If $\tilde{\omega}^2 - 4\tilde{\delta} > 0$, then the exact solution of given equation :

$$p(x,t) = \tilde{H}\left(x - Vt\right) = e_0 + \frac{e_1\sqrt{\tilde{\omega}^2 - 4\,\tilde{\delta}}\left(J_1 \sinh\left(\frac{\sqrt{\tilde{\omega}^2-4\,\tilde{\delta}}(x-Vt)}{2}\right) + J_2\cosh\left(\frac{\sqrt{\tilde{\omega}^2-4\,\tilde{\delta}}(x-Vt)}{2}\right)\right)}{2\left(J_1\cosh\left(\frac{\sqrt{\tilde{\omega}^2-4\,\tilde{\delta}}(x-Vt)}{2}\right) + J_2\sinh\left(\frac{\sqrt{\tilde{\omega}^2-4\,\tilde{\delta}}(x-Vt)}{2}\right)\right)} - \frac{\tilde{\omega}}{2} \tag{4.17}$$

where J_1 and J_2 are arbitrary constants.

If $\tilde{\omega}^2 - 4\tilde{\delta} < 0$, then the exact solution of given equation:

$$p(x,t) = \tilde{H}\left(x - Vt\right) = e_0 + \frac{e_1\sqrt{-\tilde{\omega}^2 + 4\,\tilde{\delta}}\left(-J_1 \sin\left(\frac{\sqrt{-\tilde{\omega}^2+4\,\tilde{\delta}}(x-Vt)}{2}\right) + J_2\cos\left(\frac{\sqrt{-\tilde{\omega}^2+4\,\tilde{\delta}}(x-Vt)}{2}\right)\right)}{2\left(J_1\cos\left(\frac{\sqrt{-\tilde{\omega}^2+4\,\tilde{\delta}}(x-Vt)}{2}\right) + J_2\sin\left(\frac{\sqrt{-\tilde{\omega}^2+4\,\tilde{\delta}}(x-Vt)}{2}\right)\right)} - \frac{\tilde{\omega}}{2} \tag{4.18}$$

where J_1 and J_2 are arbitrary constants.

If $\tilde{\omega}^2 - 4\tilde{\delta} = 0$, then the exact solution of given equation :

$$p(x,t) = \tilde{H}\left(x - Vt\right) = e_0 + e_1\frac{J_2}{J_1 + J_2\tilde{\zeta}} - \frac{\tilde{\omega}}{2} \tag{4.19}$$

where J_1 and J_2 are arbitrary constants.

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Theorem 4.2. Induced by the general infinitesimal generator $\tilde{E}_1 + \tilde{E}_2$ can be reduced to the following form:

$$-V\tilde{H}(\tilde{\zeta}) + a\frac{\tilde{H}^{n+1}(\tilde{\zeta})}{n+1} + b\frac{\tilde{H}^{2n+1}(\tilde{\zeta})}{2n+1} + (c - dV + eV^2)\tilde{H}''(\tilde{\zeta}) = 0 \qquad (4.20)$$

where the traveling wave transformation is

$$p(x,t) = \tilde{H}(\tilde{\zeta}), \tilde{\zeta} = x - Vt \qquad (4.21)$$

Proof: Let us consider the non linearity form of KdV -mKdV equation :

$$p_t + ap^n p_x + bp^{2n} p_x + cp_{xxx} + dp_{xxt} + ep_{xtt} = 0. \qquad (4.22)$$

Applying the wave transformation and also take Z=-V which is non-zero arbitrary constant. So wave transformation is $p(x,t) = \tilde{H}(\tilde{\zeta}), \tilde{\zeta} = x - Vt$, Now use the transformations (4.37) convert equation (4.21) into an ODE

$$-V\tilde{H}(\tilde{\zeta}) + a\frac{\tilde{H}^{n+1}(\tilde{\zeta})}{n+1} + b\frac{\tilde{H}^{2n+1}(\tilde{\zeta})}{2n+1} + (c - dV + eV^2)\tilde{H}''(\tilde{\zeta}) = 0 \qquad (4.23)$$

Assuming that a polynomial in $\left(\frac{G'}{G}\right)$ will formulate the solutions of Equation (4.23),

$$\tilde{H}\left(\tilde{\zeta}\right) = e_m \left(\frac{G'}{G}\right)^m + e_{m-1}\left(\frac{G'}{G}\right)^{m-1} + ..., \qquad (4.24)$$

with the arbitrary constants $e_m, m = 0, 1, 2, 3, ...,$ also $e_m \neq 0$. Since $G(\tilde{\zeta})$ satisfies the form's 2nd-order LDE

$$G'' + \tilde{\omega}G' + \tilde{\delta}G = 0. \qquad (4.25)$$

Here $\tilde{\omega}$ and $\tilde{\delta}$ are arbitrary constant. Using Equation (4.40) homogeneous balancing between $\tilde{H}''$ and $\tilde{H}^{n+1}$, we arrive at $m = \frac{2}{n}$. So equation (4.41) can be represented as :

$$\tilde{H}\left(\tilde{\zeta}\right) = K\left(\frac{G'}{G}\right)^{\frac{2}{n}}, K \neq 0, n > 2, \qquad (4.26)$$

$$\tilde{H}^{n+1}\left(\tilde{\zeta}\right) = K^{n+1}\left(\frac{G'}{G}\right)^{\frac{2.(n+1)}{n}}, \qquad (4.27)$$

$$\tilde{H}^{2n+1}\left(\tilde{\zeta}\right) = K^{2n+1}\left(\frac{G'}{G}\right)^{\frac{2(2n+1)}{n}}, \qquad (4.28)$$

$$\begin{aligned}\tilde{H}''\left(\tilde{\zeta}\right) = \frac{4K\tilde{\delta}^2}{n^2}\left(\frac{G'}{G}\right)^{\frac{2(1-n)}{n}} + \frac{8K\tilde{\omega}\tilde{\delta}}{n^2}\left(\frac{G'}{G}\right)^{\frac{2-n}{n}} + \frac{8K\tilde{\delta}}{n^2}\left(\frac{G'}{G}\right)^{\frac{2}{n}} - \frac{2K\tilde{\delta}^2}{n}\left(\frac{G'}{G}\right)^{\frac{2(1-n)}{n}} \\ - \frac{2K\tilde{\delta}\tilde{\omega}}{n}\left(\frac{G'}{G}\right)^{\frac{2-n}{n}} + \frac{4K\tilde{\omega}^2}{n^2}\left(\frac{G'}{G}\right)^{\frac{2}{n}} + \frac{8K\tilde{\omega}}{n^2}\left(\frac{G'}{G}\right)^{\frac{n+2}{n}} + \frac{4K}{n^2}\left(\frac{G'}{G}\right)^{\frac{2(n+1)}{n}} \\ + \frac{2K\tilde{\omega}}{n}\left(\frac{G'}{G}\right)^{\frac{n+2}{n}} + \frac{2K}{n}\left(\frac{G'}{G}\right)^{\frac{2(n+1)}{n}},\end{aligned} \qquad (4.29)$$

Equation (4.24) gives the function in $\left(\frac{G'}{G}\right)$ using Equations (4.26)-(4.28). Subsequently, The following collection of algebraic equations is produced by equating the coefficients of the identical power of $\left(\frac{G'}{G}\right)$ to zero:

$$\left(\frac{G'}{G}\right)^{\frac{2(n+1)}{n}} : -\frac{2dVK}{n} - \frac{4dVK}{n^2} + \frac{2cK}{n} + \frac{4eV^2K}{n^2} + \frac{4cK}{n^2} + \frac{2cV^2K}{n} + \frac{ak^{(n+1)}}{n+1}, \quad (4.30)$$

$$\left(\frac{G'}{G}\right)^{\frac{n+2}{n}} : \frac{8cK\tilde{\omega}}{n^2} - \frac{2dVK\tilde{\omega}}{n} - \frac{8dVK\tilde{\omega}}{n^2} + \frac{2cK\tilde{\omega}}{n} + \frac{8eV^2K\tilde{\omega}}{n^2} + \frac{2eV^2K\tilde{\omega}}{n}, \quad (4.31)$$

$$\left(\frac{G'}{G}\right)^{\frac{2}{n}} : \frac{4cK\tilde{\omega}^2}{n^2} + \frac{8cK\tilde{\delta}}{n^2} + \frac{8eV^2K\tilde{\delta}}{n^2} - VK + \frac{4eV^2K\tilde{\omega}^2}{n^2} - \frac{8dVK\tilde{\delta}}{n^2} - \frac{4dVK\tilde{\omega}^2}{n^2}, \quad (4.32)$$

$$\left(\frac{G'}{G}\right)^{\frac{2-n}{n}} : -\frac{8dVK\tilde{\omega}\tilde{\delta}}{n^2} - \frac{2cK\tilde{\omega}\tilde{\delta}}{n} + \frac{8cK\tilde{\omega}\tilde{\delta}}{n^2} + \frac{2dVK\tilde{\omega}\tilde{\delta}}{n} + \frac{8eV^2K\tilde{\omega}\tilde{\delta}}{n^2} - \frac{2eV^2K\tilde{\omega}\tilde{\delta}}{n}, \quad (4.33)$$

$$\left(\frac{G'}{G}\right)^{\frac{2(1-n)}{n}} : -\frac{4dVK\tilde{\delta}^2}{n^2} - \frac{2cK\tilde{\delta}^2}{n} + \frac{4cK\tilde{\delta}^2}{n^2} + \frac{2dVK\tilde{\delta}^2}{n} + \frac{4eV^2K\tilde{\delta}^2}{n^2} - \frac{2eV^2K\tilde{\delta}^2}{n}, \quad (4.34)$$

$$\left(\frac{G'}{G}\right)^{\frac{2(2n+1)}{n}} : \frac{bk^{(2n+1)}}{2n+1} \quad (4.35)$$

After using Maple to solve the previously mentioned algebraic equation, we receive the traveling wave solutions :

K = K, V = V, $d = \frac{(4c\omega^2+8c\delta+8V^2\delta+4eV^2\omega^2-Vn^2)}{4V(24+\tilde{\omega}^2)}$, ,c=c, d = d and $\tilde{\delta} = \tilde{\delta}$

Enter these values into equation (4.26) and the result is presented as follows :

$$\tilde{H}\left(\tilde{\zeta}\right) = K\left(\frac{G'}{G}\right)^{\frac{2}{n}}, K \neq 0, n > 2, \quad (4.36)$$

If $\tilde{\delta} > 0$, then the exact result of given equation is :

$$p(x,t) = \tilde{H}\left(x - Vt\right) = K\left(\frac{-J_1 sin(\sqrt{\tilde{\delta}}(x-Vt)) + J_2 cos(\sqrt{\tilde{\delta}}(x-Vt))}{J_1 cos(\sqrt{\tilde{\delta}}(x-Vt)) + J_2 sin(\sqrt{\tilde{\delta}}(x-Vt))}\right)^{\frac{2}{n}} \quad (4.37)$$

If $\tilde{\delta} < 0$, then the exact result of specified equation :

$$p(x,t) = \tilde{H}\left(x - Vt\right) = K\left(\frac{J_1 sinh(\sqrt{-\tilde{\delta}}(x-Vt)) + J_2 cosh(\sqrt{-\tilde{\delta}}(x-Vt))}{J_1 cosh(\sqrt{-\tilde{\delta}}(x-Vt)) + J_2 sinh(\sqrt{-\tilde{\delta}}(x-Vt))}\right)^{\frac{2}{n}} \quad (4.38)$$

If $\tilde{\delta} = 0$, then the exact expression of given equation :

$$p(x,t) = \tilde{H}\left(x - Vt\right) = K\left(\frac{J_2}{J_1 + J_2\tilde{(}x - Vt)}\right)^{\frac{2}{n}} \quad (4.39)$$

where J_1 and J_2 are arbitrary constants.

4.1.2. Bifurcation and Phase Portrait of Power-Law Nonlinearity Equation[37]-[38]

The nonlinear autonomous system corresponding to equation (4.23) is given as

$$-V\tilde{H}(\tilde{\zeta}) + a\frac{\tilde{H}^{n+1}(\tilde{\zeta})}{n+1} + b\frac{\tilde{H}^{2n+1}(\tilde{\zeta})}{2n+1} + (c - dV + eV^2)\tilde{H}''(\tilde{\zeta}) = 0 \tag{4.40}$$

where a, b, c, d and e are arbitrary constants.
Introduce notation $H = X, Y = X'$ and if $n = 1$ then the generalized form shown in Equation (4.40) simplifies to an autonomous system.

$$\tilde{X'}(\chi) = \tilde{Y}(\chi), A\tilde{Y'}(\chi) = V.X + a\left(\frac{X^2}{2}\right) + b\left(\frac{X^3}{3}\right) \tag{4.41}$$

The linear equation has (0,0) critical point. Corresponding to the critical point (0, 0) the specified autonomous system takes the form

$$\tilde{x'}(\chi) = \tilde{y}(\chi), A\tilde{y'}(\chi) = Vx + a\left(\frac{x^2}{2}\right) + b\left(\frac{x^3}{3}\right) \tag{4.42}$$

From the given Equation (4.42), we obtain distinct phase portraits at the critical point.
In case (a): when $A > 0$, the characteristic equation of the Jacobian matrix for the system in Equation (4.42) at the critical point (0,0) has two real, unequal eigenvalues with opposite signs. Thus, (0,0) is a saddle point. The corresponding phase portrait is shown in Figure 3.
In case (b): when $A < 0$, the characteristic equation of jacobian matrix for the system Equation (4.42) at the critical point (0,0) has purely imaginary eigen values. Thus, (0,0) is center. The corresponding phase portrait is shown in Figure 4.

5 Conservation Law

Let a kth order system of PDEs with n independent variables $z = (z^1, z^2, ..., z^n)$ and m dependent variabled $v = (v^1, v^2, ..., v^m)$ be defined as,

$$E_\alpha(z, v, v_1, v_2, ..., v_k) = 0 \tag{5.1}$$

where v_i is the collection of ith order partial derivatives of v.
(a) The Euler operator is defined by

$$\frac{\delta}{\delta(v^\alpha)} = \frac{\delta}{\delta(v^\alpha)} - D_i\frac{\delta}{\delta(v_i^\alpha)} + D_iD_j\frac{\delta}{\delta(v_{ij}^\alpha)} - ..., \alpha = 1, 2, ...m. \tag{5.2}$$

where

$$D_i = \frac{\delta}{\delta(z^i)} + v_i^\alpha\frac{\delta}{\delta(v^\alpha)} + v_ij^\alpha\frac{\delta}{\delta(v_j^\alpha)}..., i = 1, 2, ... \tag{5.3}$$

is the total derivative operator w.r.t. z^i
(b) A conserved vector equation (5.1) is an n-tuple $T = (T^1, T^2, ..., T^n)$,$T^i \in A, i = 1, 2, ..., n$ such that

$$D_iT^i = 0 \tag{5.4}$$

holds for all solutions of (5.1). The equation (5.4) is referred to as a local conservation law.
(c) The multipliers Λ^α of the system (5.1) satisfy the property

$$D_i T^i = \Lambda^\alpha E_\alpha \tag{5.5}$$

for arbitrary function v^α [39,40]
(d) The determining equations for the multipliers are derived by applying the variational derivative to (5.5)

$$\frac{\delta}{\delta v^\alpha}[\Lambda^\alpha E_\alpha] = 0 \tag{5.6}$$

Equation (5.6) holds for the arbitrary functions v^α not only for the solutions of system 5.1. Equation (5.6) provides multipliers for all local conservation laws. Once the multipliers are determined, the conserved vectors can be systematically derived using (5.5) as the determining equation. However, in certain cases, constructing the conserved vectors can be straightforward and achieved through elementary manipulations after obtaining the multipliers.

Conservation Law for Gardner's Equation:

Consider the Gardner system

$$q_t + (aq + bq^2)q_x + cq_{xxx} + dq_{xxt} + eq_{xtt} = 0 \tag{5.7}$$

The group-invariant solution of (5.7) was obtained. Now, we aim to construct conservation laws for the system (5.7). Consider simple multipliers of the form $\Lambda(t, x, q, q_x, q_t)$. The determining Equation for the multiplier Λ.

$$E_q = [\Lambda(q_t + (aq + bq^2)q_x + cq_{xxx} + dq_{xxt} + eq_{xtt})] = 0 \tag{5.8}$$

Multipliers Λ_q and for the system (5.7) has the property that

$$\Lambda_q E_q = D_t T^1 \tag{5.9}$$

for all function $q(x, t)$ where the total derivative operators D_t and D_x from 5.3 are

$$D_t = \frac{\delta}{\delta t} + q_t\frac{\delta}{\delta q} + q_{tx}\frac{\delta}{\delta q_x} + q_{tt}\frac{\delta}{\delta q_t} + ...D_x = \frac{\delta}{\delta x} + q_x\frac{\delta}{\delta q} + q_{tx}\frac{\delta}{\delta q_t} + q_{xx}\frac{\delta}{\delta q_x} + ... \tag{5.10}$$

The right-hand side of (5.9) is a divergence expression, and T^1 is the component of the conserved vector $T = T^1$. The determining equations for the multipliers Λ are

$$\frac{\delta}{\delta q} = \frac{\delta}{\delta q} - D_t\frac{\delta}{\delta q_t} - D_x\frac{\delta}{\delta q_x} + D_t^2\frac{\delta}{\delta q_{tt}} + D_t D_x\frac{\delta}{\delta q_{tx}} + D_x^2\frac{\delta}{\delta q_{tx}} + ... \tag{5.11}$$

Separating (5.9), after expansion, with respect to different combinations of q results in the following over-determined of equations:

$$\Lambda_x(1 + a_1 q + a_2 q^2) = 0, \Lambda_{qq} = 0, \Lambda_{xq} = 0, \Lambda_{tq} = 0, \Lambda = f_1(t)q + f_2(x, t) \tag{5.12}$$

where $f_1(t) = c_2$ and $f_2(x, t) = c_1$ so

$$\Lambda = c_1 + c_2 q \tag{5.13}$$

from (5.12) and (5.7), we have

$$D_t[c_1(q+b_2q_xx+b_3qxt)+c_2(\frac{q^2}{2}+b_2qq_{xx}+b_3qq_{xt})]+D_x[c_1(\frac{a_1q^2}{2}+a_2\frac{a_1q^3}{3}+b_1q_xx)+c_2[a_1\frac{q^3}{3}+a_2\frac{q^4}{4}+b_1qq_xx] \tag{5.14}$$

So, the conserved vector for equation (5.7)

$$T_1^1 = q + b_2q_{xx} + b_3q_{xt} \tag{5.15}$$

$$T_1^2 = \frac{a_1q^2}{2} + \frac{a_2q^3}{3} + b_1q_{xx} \tag{5.16}$$

$$T_2^1 = \frac{q^2}{2} + b_2qq_{xx} + b_3qq_{xt} \tag{5.17}$$

$$T_2^2 = \frac{a_1q^3}{3} + \frac{a_2q^4}{4} + b_1qq_{xx} \tag{5.18}$$

6 Conclusion

Current research has recovered traveling wave solutions for KdV-mKdV, which show up as algebraic-type non linearities. Three different forms of dispersion are carried out by the model equations, which combine spatial and temporal types. Incorporating these three dispersion forms, which produce distinct relationships between the wave's amplitude and inverse breadth and velocity, made the work novel. This work also shows the Lie symmetry analysis of nonlinearity power law. Here Lie symmetry analysis gives the trivial symmetries for KdV-mdV and nonlinearities power law form with triplet dispersion and use the $\frac{G'}{G}$-expansion approach to produce distinct types of traveling wave solutions for KdV-mKdV equation.The resulting solutions are newly formed in trigonometric and hyperbolic forms. In the final section, we explored the bifurcation theory of planar dynamical systems to perform a qualitative analysis of the power-law nonlinearity equation, and the corresponding phase portraits were illustrated. The conservation laws of the Gardner system were determined using the multipliers approach, resulting in local conserved vectors. A total of four multipliers were obtained for the Gardner system. The conserved vectors derived here can be used in constructing the solutions of underlying PDE systems and will be considered in the future work. These studies are highly anticipated and will be presented one after the other. [!htb]

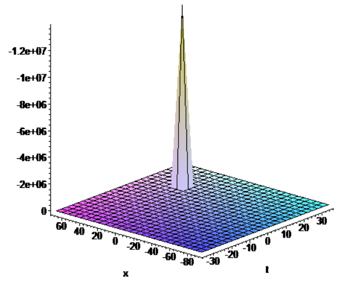

Figure 1: 3D plot for $\tilde{\omega}^2 - 4\tilde{\delta} = 0$ whenever, J_1=2, J_2=3, $\lambda = 2$, $V = 2$

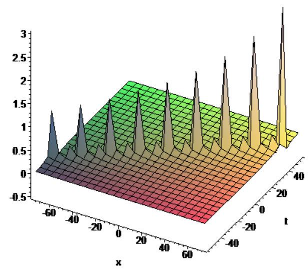

Figure 2: 3D plot for $\tilde{\omega}^2 - 4\tilde{\delta} = 0$ whenever $n = 2$, J_1=2, J_2=3, $K = 1$, $V = 2$

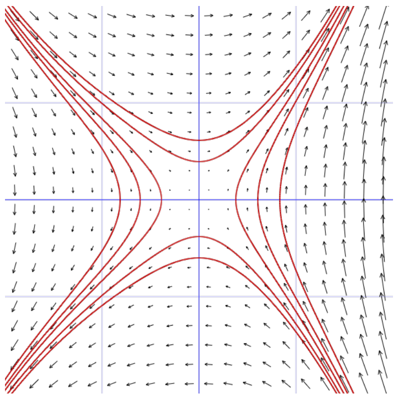

Figure 3: Bifurcation of phase portrait of equation (4.2) for V=1, A=1, a=1,b=1

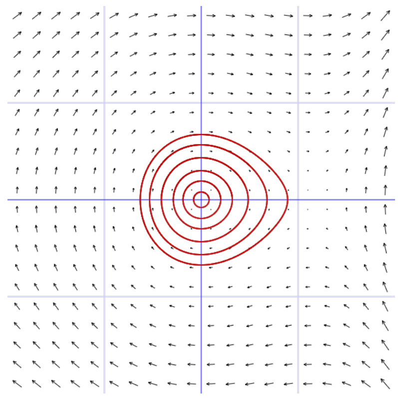

Figure 4: Bifurcation of phase portrait of equation (4.2) for V=1, A=-1,a=1,b=1

References

[1] S.H. Steven, Applications to Physics, Biology, Chemistry and Engineering, Boulder, Nonlinear Dynamics and Chaos, 2001, 44-79.

[2] S.L. Ross, Differential Equation, John Wiley and Sons, New Delhi, 1984.

[3] A.A. Darwish and A. Ramady, Applications of algebraic method to exactly solve some nonlinear partial differential equations, Chaos Solitons Fractals, 2007, 126374.

[4] S.A. Khuri, Exact solutions for a class of nonlinear evolution equations, A unified ansatze approach, Chaos Solitons Fractals, 2008.

[5] A. Bekir1 and O. Unsal, Exact solutions for a class of nonlinear wave equations by using First Integral Method, International Journal of Nonlinear Science, 2013, 15(2):99110.

[6] B. Cantwell, Introduction to Symmetry Analysis, Cambridge University Press, 2002.

[7] R. Kumar, R. Kumar and A. Bansal, Lie symmetry and exact solutions of conformable time fractional Schamel Korteweg De vries equation, International Journal of Applied and Computational Mathematics, 2024, 113.

[8] L.V. Xueqin and J. Gao, Treatment for third-order nonlinear differential equations based on the Adomian decomposition method, LMS Journal of Computation Mathematics, 2017, 110.

[9] S. Kazem, Exact solution of some liner fractional differential equations by laplace transform, International Journal of Nonlinear Sciences, 2013,(16) 3-11.

[10] J. H. He, Variational iteration method-a kind of non-linear analytical technique: some examples, International Journal of Non Linear Mechanics, 1999, (34), 699-708.

[11] R.S. Dubey, P. Goswami, et al., A new analytical method to solve Klein-Gordon equations by using homotopy perturbation Mohand transform method, Malaya Journal of Matematik, 2022, 1-19.

[12] W.X. Ma, A refined invariant subspace method and applications to evolution equations, Sci. China Math., 2012, (55), 1769-1778.

[13] G.W. Bluman and J.D. Cole, Similarity Methods for Differential Equations, Springer Verlag, New York, 1974.

[14] R. Kumar, R.K. Gupta and S.S. Bhatia, Lie symmetry analysis and exact solutions for a variable coefficient generalised Kuramoto-Sivashinsky equation, Romanian Reports in Physics, 2014, 66(4), 923-928.

[15] G. W. Bluman and S. Kumei, Symmetries and differential equations, Springer Science and Business Media, 2013.

[16] T. T. Zhang, On Lie symmetry analysis, conservation laws and solitary waves to a longitudinal wave motion equation, Applied Mathematics Letters, 2019, (98) 199-205.

[17] R. Kumar, R. Kumar, A. Bansal, A. Biswas, Y. Yildirim, S. P. Moshokoa, A. Asiri, Optical solitons and group invariants for Chen-Lee-Liu equation with time-dependent chromatic dispersion and nonlinearity by Lie symmetry, Ukrainian Journal of Physical Optics, 2023, 24(4), 1816-2002.

[18] A. Bansal, A. Biswas, H. Triki, Q. Zhou, S. P. Moshokoa, M. Belic, Optical solitons and group invariant solutions to LakshmananPorsezianDaniel model in optical fibers and PCF, Optik, 2018, 160, 86-91.

[19] A. Bansal, A. Biswas, Q. Zhou, M.M. Babatin, Lie symmetry analysis for cubicquartic nonlinear Schrödinger's equation, Optik, 2018, 169, 12-15.

[20] P.J. Olver, Applications of Lie Groups to Differential Equations, Springer Verlag, 2000.

[21] A. Biswas, N. Coleman, A. H. Kara, S. Khan, L. Moraru, S. Moldovanu, C. Iticescu and Y. Yld1r1m, Shallow water waves and conservation laws with dispersion triplet, Applied Sciences 2022, 12(7) 3647.

[22] N.A. Kudryashov, A note on solutions of the Korteweg-de Vries hierarchy, Communications in Nonlinear Science Numerical Simulation, 2011, (16) 1703-1705.

[23] A. Biswas, A.H. Kara, 1-soliton solution and conservation laws for nonlinear wave equation in semiconductors, Applied Mathathematics and Computation, 2010, (217) 4289-4292.

[24] K.A. Gepreel, Exact soliton solutions for nonlinear perturbed Schrödinger equations with nonlinear optical media, Applied Sciences, 2020, (10) 8929.

[25] A. Biswas, E.V. Krishnan, P. Suarez, A.H. Kara, S. Kumar, Solitary waves and conservation laws of Bona-Chen equation, Indian Journal of Physics, 2013, (87) 169175.

[26] G Yel, H. M. Baskonus and W. Gao, New dark-bright soliton in the shallow water wave model, AIMS Mathematics 2020, 5(4) 4027-4044.

[27] M. Antonova, A. Biswas, Adiabatic parameter dynamics of perturbed solitary waves, Communications in Nonlinear Science Numerical Simulation, 2009, (14) 734-748.

[28] E. Barthélémy, Nonlinear shallow water theories for coastal waves, Surveys in Geophysics, 2004, (25) 315-337.

[29] D. Mitsotakis, A simple introduction to water waves, HAL open sciences, 2013.

[30] L. Girgis and A. Biswas, A study of solitary waves by He's semi-invers variational principle, Waves Random Complex Media, 2011, (21), 96-104.

[31] R.I. Joseph, R. Egri, Another possible model equation for long waves in nonlinear dispersive system, Physics Letter A, 1977, (61) 429-430.

[32] S. Lie, On integration of a class of linear partial differential equations by means of definite integrals. CRC Handbook of Lie Group Analysis Differential Equations, 1881, 473-508.

[33] R. Kumar, R. Kumar, A. Bansal, S. Kumar, Symmetry reductions and qualitative analysis of time fractional K(m,1) equation, Partial Differential Equations in Applied Mathematics, 2023, 9(1), 100603.

[34] A. M. Shahoot, K. A. E. Alurrfi, M. O. M. Elmrid, A. M. Almsiri, A. M. H. Arwiniya, The $(\frac{G'}{G})$-expansion method for solving a nonlinear PDE describing the nonlinear low-pass electrical lines, Journal of Taibah university for Science, 2019, (13) 1, 63-70.

[35] E.M.E. Zayed, M.A.S. EL-Malky, The $\frac{G'}{G}$-expansion method for solving nonlinear Klein-Gordon equations, Numerical Analysis and Applied Mathematics ICNAAM 2011 AIP Conference Proceedings, 2011, 1389(1), 2020-2024.

[36] A. Bansal, R.K. Gupta, Modified $\frac{G'}{G}$-expansion method for finding exact wave solutions of the coupled Klein-Gordon-Schrödinger equation, Mathematical Methods in the Applied Sciences, 2012, 35(10), 1175-1187.

[37] A.K. Yuri, Elements of Applied Bifurcation Theory, New York, Spinger, 112, 1998.

[38] R. Kumar, A. Bansal and S.Saini, Travelling wave solutions and bifuracation anaysis of Chaffee-infante equation,In Proceedings of International Conference on Trrends in Computational and Cognitive Engineering, TCCE 2019, 2020, 153-161.

[39] H. Steudel, Uber die Zuordnung zwischen Invarianzeigenschaaften and Erhaltungssatzen, Zeeitschrift fiir Naturforchung, 1962, 129-132.

[40] P.J. Olver, Applications of Lie Groups to Differential Equations, Gradute Texts in Mathematics, Springer, New York, 1993.