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A Variational Approach for Finding Initial Approximation in Numerical Solution of an Equation

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Abstract: An attempt to estimate the initial approximation in numerical solution of equation is introduced. In this paper, the aim is to find strong starting approximation from favorable initial approximation in the field of both real and complex equations. We have established that the strong initial approximation convergent to the root of the equation under iteration.

Introduction: Each numerical method needs a good starting approximation to diminish the frequency of iteration and to converge the root of a function. Generally intermediate value theorem (IVT) is widely used to estimate an interval containing the root of a real function and as it is well known that the theorem depends on the alternative sign of a function. When the range of a function is non-negative or non-positive in its natural domain, IVT does not suggest any interval containing root.

Sometimes the linear interpolation method is also recommending to estimate the initial approximation of a root, but here we need prior information about the interval containing root; that is, the linear interpolation technique is depends on IVT. In some cases, graph of the corresponding functions are also used to estimate initial approximation as an alternative. But tracing of graph is not always an easy task without using mathematical tool. Supposed that $F(x) = 0$ is a real valued function and it can be expressed as other two real valued function such as $f(x) = g(x)$ (say), then their intersection point can be used as an initial approximation of the function $F(x)$. But here difficulty is that, the solution of the expression $f(x) = g(x)$ may not be calculate easily.

The content of this paper is organized as follows: we briefly review the literature associated with initial approximation in Section-2. The main results on proposed technique are studied in both real and complex field with examples in Section-3 and Section-4 respectively. The paper end with a conclusion in Section-5.

Objectives: The prime objective of this paper is proposing a technique to estimate a better initial approximation to a root of a function. In this technique our main focus is to overcome the aforementioned limitations and make it user friendly.

Methods: The Wiersma basin attraction root point Theorem**,** Schr*ö*der Fixed Point Theorem are used. The concept of variational inenequality problem approach is used to find the root interval of the basin attraction root point of a function.

Results: The basin of attraction of the equations of real function and complex function are established. The results are established with the concrete examples.

Conclusions: In this piece of work, we suggested a new strategy to estimate the real and complex initial approximation for both real and complex equations respectively. Any standard iteration by function will definitely convergent to the root by choosing these initial approximation and it will also diminish the frequency of iteration.

Keywords: Favorable initial approximation; Newton's basin of attraction; Supper strong initial approximation.

2010 Mathematics Subject Classification. 65-XX, 74G15, 74H15.

1. Introduction

Each numerical methods needs a good starting approximation to diminish the frequency of iteration and to converge the root of a function. Generally intermediate value theorem (IVT) is widely used to estimate an interval containing the root of a real function and as it is well known that the theorem depends on the alternative sign of a function. When the range of a function is non-negative or non-positive in its natural domain, IVT does not suggest any interval containing root. A real function $\mathrm{F}(x) = (x-1)^2$ or $-(x-1)^2$ always attained non-negative or non-positive value respectively for all $x \in \mathbb{R}$, having second order root at 1. In such cases IVT fails to suggest a conclusion over the roots of the function. Sometimes the linear interpolation method is also recommend to estimate the initial approximation of a root, but here we need prior information about the interval containing root; that is, the linear interpolation technique is depends on IVT. In some cases, graph of the corresponding functions are also used to estimate initial approximation as an alternative. But tracing of graph is not always an easy task without using mathematical tool. Supposed that $\mathrm{F}(x) = 0$ is a real valued function and it can be expressed as other two real valued function such as $f(x) = g(x)$(say), then their intersection point can be used as an initial approximation of the function $\mathrm{F}(x)$. But here difficulty is that, the solution of the expression $f(x) = g(x)$ may not be calculate easily. Let us consider a function $\mathrm{F}(x) = \cos x \cdot \cosh x - 1$ and it can be expressed as other two such as $\cosh x = \sec x$, but to find the intersection point of the expression is not an easy task.

The prime objective of this paper is propose a technique to estimate a better initial approximation to a root of a function. In this technique our main focus is to overcome the aforementioned limitations and make it user friendly. The content of this paper is organized as follows: we briefly review the literature associated with initial approximation in Section-3. The main results on proposed technique are studied in both real and complex field with examples in Section-4 &5 respectively. The paper end with a conclusion in Section-6.

2. Preliminaries

In 2006, Kornerup [6] studied for finding the best possible seed values when computing $a^{\frac{1}{p}}$ using the Newton-Raphson iteration in a given interval. A natural choice of the seed value would be the one that best approximates the expected result. It turns out that in most cases, the best seed value can be quite far from this natural choice. When a monotone function $f(a)$ is evaluated in the interval $[a_{min}, a_{max}]$, by building the sequence x_n defined by the Newton-Raphson (NR) iteration, the natural choice consists in choosing x_0 equal to the arithmetic mean of the endpoint values. This minimizes the maximum possible distance between x_0 and $f(a)$. And yet, if it perform n iterations, what matters is to minimize the maximum possible distance between x_n and $f(a)$. Thus the study of Kornerup [6] is confined on the equation $x^p - a = 0$. In 2010, Muheadden [7] modified NR and Secant methods to complex form, then from that they get new iterative formula to them, named Newton (Secant) real iteration formula and Newton (Secant) imaginary iteration formula, real iteration used to find real roots while imaginary iteration one will be used to find complex roots of nonlinear equations such as $x^2 + a = 0$. In 2017, Kafash and Hosseini [5] have attempted to approximate the real and complex roots of nonlinear equations, by considering the convergence conditions of Adomian Decomposition (AD) Method for solving functional equations. It can be computed suitable approximate real and complex roots of a given function more efficient than Maple software. The problem of specifying the Newton basins was posed by Cayley [4] in 1879. Cayley (like Schrӧder before him) could not go beyond quadratic polynomials because of the chaotic global behavior of the complex Newton method. The study of iterations with rational functions (such as the Newton iteration for a polynomial) was greatly advanced by Fatou and Julia in 1918–1920; see [3] and the history in [1]. We revisited some results to make this work self contained.

Definition 2.1 (Wiersma) If x_* is a root of f, the basin of attraction of x_* is the set of all numbers x_0 such that Newton's method starting at x_0 converges to x_*. In symbols,

$$\mathrm{B}_* = \{x_0 \colon x_n = \mathrm{N}^n(x_0) \text{ convergent to } x_*\}.$$

Theorem 2.2 (Alexandar et al.) (Schrӧder Fixed Point Theorem) If f is an analytic function with $f(\lambda) = \lambda$ and $|f'(\lambda)| < 1$, then there is neighborhood D of the fixed point λ on which all points converge to λ under iteration by f, that is, for any $z \in \mathrm{D}$, $\lim_{n\to\infty} f^{(n)}(z) = \lambda$.

Fixed points are often classified according to the following definition:

Definition 2.3 (Wiersma) Suppose that a map $f \colon \mathrm{X} \to \mathrm{X}$ is differentiable at a fixed point x_*. Then

(a) x_* is attracting if and only if $|f'(x_*)| < 1$,
(b) x_* is supper-attracting if and only if $|f'(x_*)| = 0$,
(c) x_* is repelling if and only if $|f'(x_*)| > 1$.

If there exist an element $\epsilon_X \in X$ such that $|1_X - \epsilon_X| < 1$, then the Definition 3.3 can be defined as a Variational Inequality Problem (see [9]) to find $x_* \in X$ such that for all $x \in X$,

$$\langle \nabla f(x_*) - 1_X, x - x_* \rangle = \begin{cases} -\langle \epsilon_X, x - x_* \rangle, & \text{if } x < x_*; \\ -\langle 1_X, x - x_* \rangle, & \text{if } x = x_*; \\ \langle \epsilon_X, x - x_* \rangle, & \text{if } x > x_*. \end{cases}$$

3. Variational Approach for finding Initial approximation in numerical method of function with real roots

Let us consider a real polynomial $\mathrm{F}(x)$ of n degree, define in the domain $\mathrm{D} \subseteq \mathbb{R}$

$$\mathrm{F}(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3 + \cdots + a_{n-1} \cdot x^{n-1} + a_n \cdot x^n,\ a_0, a_n \neq 0 \quad (3.1)$$

which can be represented by the polynomial equation

$$\mathrm{F}(x) = 1 + \left(\frac{a_1}{a_0}\right)x + \left(\frac{a_2}{a_0}\right)x^2 + \left(\frac{a_3}{a_0}\right)x^3 + \cdots + \left(\frac{a_{n-1}}{a_0}\right)x^{n-1} + \left(\frac{a_n}{a_0}\right)x^n = 0 \quad (3.2)$$

For our need, we define the concept of favorable initial approximation of the real equation.

Definition 3.1 *Let $F(x)$ be a real function defined in the domain $D \subseteq \mathbb{R}$ and $f(x)$ be the two lowest degree terms (with unity constant if any) or single term of degree ≥ 3 of $F(x)$. Let x_0 be the root of the equation $f(x) \mp x = 0$. Then, a point $x \in D$ is said to be favorable initial approximation of the real equation $F(x) = 0$ if and only if $\lim_{x \to x_0} \left|\frac{f(x)}{\pm x}\right| = 1$ or 0.*

In other words, a point $x \in \mathrm{D}$ is said to be favorable initial approximation of the real equation $\mathrm{F}(x) = 0$, if and only if, for any $\epsilon > 0$ there exist $\delta > 0$ depending on ϵ such that $0 < |x - x_0| < \delta, \Rightarrow \left|\frac{f(x)}{\pm x} - \gamma\right| < \epsilon$ where γ is 1 or 0 and x_0 is the root of the equation $f(x) \mp x = 0$ in the domain $\mathrm{D} \subseteq \mathbb{R}$ of the real function $\mathrm{F}(x)$, where $f(x)$ is the two lowest degree terms (with unity constant if any) or single term of degree ≥ 3 of $\mathrm{F}(x)$.

Remark 3.2 The choice of the number ϵ is arbitrary, but as small as one wishes. The root of the equation $f(x) \mp x = 0$ may be either real definite or complex or indefinite values. The number 1 or 0 provides some information on the behavior of the function in the neighborhood of the point x_0. It conveys that when $f(x)$ assume its value closer to x, x is the favorable initial approximation of $\mathrm{F}(x) = 0$. The definition also guess the favorable initial approximation of complex roots of the real or complex equation.

Let us consider $f(x) = 1 + \left(\frac{a_1}{a_0}\right)x$ two lowest degree terms (with unity constant) of $\mathrm{F}(x) = 0$, according to the Definition 3.1, it follows

$$\left|\frac{f(x)}{\pm x} - 1\right| < \epsilon \Rightarrow \left|\frac{1 + \left(\frac{a_1}{a_0}\right)x}{x} - 1\right| < \epsilon,$$

$$\Rightarrow \left|x - \frac{a_0}{a_0 - a_1}\right| < \delta,\ \textit{by choosing}\ \left(\frac{a_1 - a_0}{a_0}\right)\epsilon x < \delta,$$

$$\Rightarrow |x - x_0| < \delta,\ x_0 = \frac{a_0}{a_0 - a_1} \text{ is the root of } f(x) - x = 0,$$

$$\Rightarrow x \in (x_0 - \delta, x_0 + \delta) = \mathrm{N}_\delta(x_0)$$

where $\mathrm{N}_\delta(x_0)$ is the neighborhood of favorable initial approximation x_0. Similarly, we can consider $f(x) = \left(\frac{a_3}{a_0}\right)x^3$, term of degree 3 in $\mathrm{F}(x) = 0$, according to the Definition 3.1, the following

$$\left|\frac{f(x)}{\pm x} - 1\right| < \epsilon \Rightarrow \left|\frac{\left(\frac{a_3}{a_0}\right)x^3}{x} - 1\right| < \epsilon,$$

$$\Rightarrow \left|x^2 - \frac{a_0}{a_3}\right| < \left(\frac{a_0\epsilon}{a_3}\right) = \delta,$$

$$\Rightarrow |(x - x_0)(x - x_1)| < \delta, x_0 = \sqrt{\frac{a_0}{a_3}}, x_1 = -\sqrt{\frac{a_0}{a_3}},$$

x_0 and x_1 are roots of $f(x) - x = 0$,

$$\Rightarrow |x - x_0| \cdot |x - x_1| < \delta,$$

$$\Rightarrow x \in (x_0 - \delta, x_0 + \delta) = \mathrm{N}_\delta(x_0) \text{ and } x \in (x_1 - \delta, x_1 + \delta) = \mathrm{N}_\delta(x_1),$$

where $\mathrm{N}_\delta(x_0)$ and $\mathrm{N}_\delta(x_1)$ are the neighborhoods of favorable initial approximation x_0 & x_1 respectively. Proceeding similarly we can estimate other favorable initial approximation of the equation $\mathrm{F}(x) = 0$. Here each neighborhoods $\mathrm{N}_\delta(x_i), i = 0, 1, \cdots$ are the subsets of the Newton's basin of attraction for corresponding roots. We can represent the above argument in the following Figure: 1.

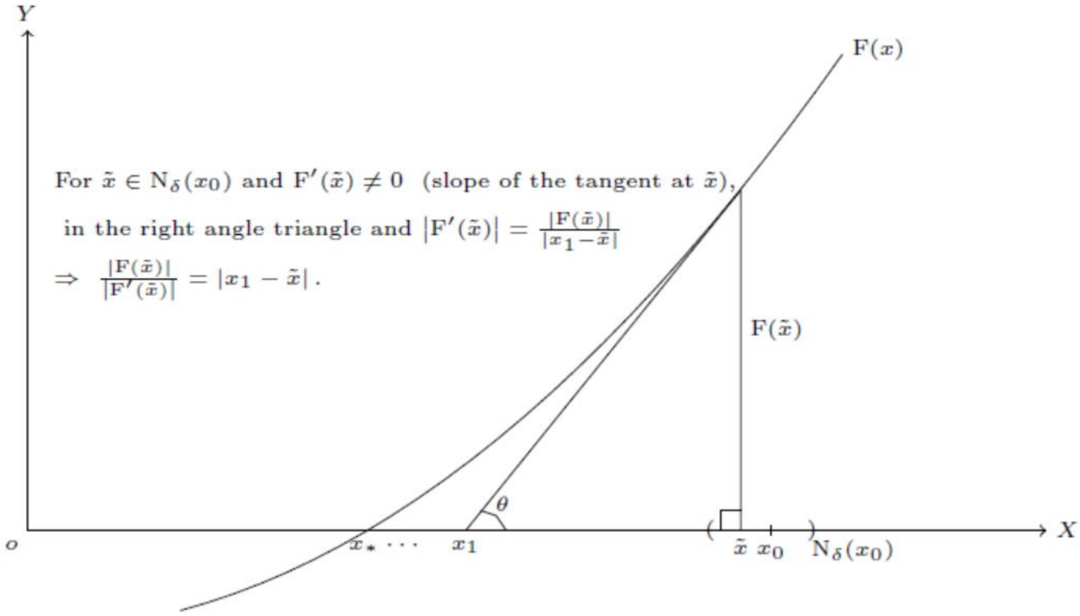

Figure 1: First approximation of any $\tilde{x} \in \mathrm{N}_\delta(x_0)$ convergent to x_1 under iteration by F. In symbols, $\mathrm{I_F}(\tilde{x}) = x_1$.

On the basis of these arguments, we can classified the favorable initial approximation in the following definition.

Definition 3.3 *Suppose that a function* $F: \mathbb{R} \to \mathbb{R}$, *is differentiable at a favorable initial approximation* $\tilde{x} \in N_\delta(x_0)$ *and* $F'(\tilde{x}) \neq 0$. *Then*

- $\tilde{x}$ is root if and only if $\left|\frac{\mathrm{F}(\tilde{x})}{\mathrm{F}'(\tilde{x})}\right| = 0$,

- $\tilde{x}$ is supper strong attracting if and only if $0 < \left|\frac{\mathrm{F}(\tilde{x})}{\mathrm{F}'(\tilde{x})}\right| \leq \frac{1}{2}$,
- $\tilde{x}$ is strong attracting if and only if $\frac{1}{2} < \left|\frac{\mathrm{F}(\tilde{x})}{\mathrm{F}'(\tilde{x})}\right| < 1$,
- $\tilde{x}$ is useless if and only if $\left|\frac{\mathrm{F}(\tilde{x})}{\mathrm{F}'(\tilde{x})}\right| \geq 1$.

Let us consider an exponential function $\mathrm{F}(x) = e^{ax}$. For $|a| \leq 1$, we have $\left|\frac{\mathrm{F}}{\mathrm{F}'}\right| \geq 1$ for all values of $x \in \mathbb{R}$. This implies that, all values of $x \in \mathbb{R}$ are useless in terms of setting for an initial approximation for the equation $\mathrm{F}(x) = 0$, because the function does not have any root in $\mathbb{R}$. Similarly for, $|a| > 1$ we have $0 < \left|\frac{\mathrm{F}}{\mathrm{F}'}\right| < 1$, for all values of $x \in \mathbb{R}$. In this case the Definition 3.3 suggest strong initial approximation but that will diverge to $-\infty$, after $n > \mathrm{M}$ iteration where M is a large positive integer and $-\infty$ is the root of the exponential function in $\mathbb{R}^*$(extended real line). Now let us consider a trigonometric function $\mathrm{F}(x) = \sec x$ and its Maclaurin's series is

$$\mathrm{F}(x) = \sec x = \mathrm{E}_0 - \frac{\mathrm{E}_2}{2!}x^2 + \frac{\mathrm{E}_4}{4!}x^4 - \cdots$$

where $\mathrm{E}_0 = 1, \mathrm{E}_2 = -1, \mathrm{E}_4 = 5, \mathrm{E}_6 = -61, \cdots$ are Euler's number. Setting, $f(x) = 1 + \frac{x^2}{2!}$ (see Definition 3.1). The roots of $f(x) \mp x = 0$ are complex conjugate, implying the complex conjugate are not suitable for favorable initial approximation of a real function. Similarly, by setting, $f(x) = \frac{5}{4!}x^4$ (see Definition 3.1), we obtain the roots of $f(x) \mp x = 0$ are real or complex conjugates as $\left|\frac{\mathrm{F}}{\mathrm{F}'}\right| \geq 1$ for the real root by Definition 3.3. Thus there will be no suitable value for the initial approximation of the equation $\mathrm{F}(x) = \sec x = 0$. On the basis of the above discussion, we obtain the following result.

Theorem 3.4 *If a function $F(x)$ is differentiable in its natural domain D and x_* is a root of the equation $F(x) = 0$, then there is a neighborhood of a strong attracting point (that is, $N_\delta \subset B_*$) on which all points converges to x_* under iteration by F, where B_* is the Newton's basin of attraction for the root x_*, i.e., for any $\tilde{x} \in N_\delta$ and $I_F^n(\tilde{x}) = x_n, \Rightarrow \lim_{n\to\infty} x_n = x_*$.*

Proof. A function $\mathrm{F}(x)$ is differentiable in its natural domain D and x_* is a root of the equation $\mathrm{F}(x) = 0$. Let x_0 be a strong attraction point and there are two numbers $0 < \delta, \lambda < 1$, such that $\mathrm{N}_\delta(x_0) = (x_0 - \delta, x_0 + \delta)$ on which all points satisfy $\left|\frac{\mathrm{F}}{\mathrm{F}'}\right| \leq \lambda < 1$, by Definition- 3.3. Now let $\tilde{x} \in \mathrm{N}_\delta(x_0)$, then by Lagrange's Mean Value Theorem (MVT) for finite increment ($h > 0$),

$$\mathrm{F}(x_* + h) - \mathrm{F}(x_*) = [(x_* + h) - (x_*)]\mathrm{F}'(x_* + \theta h),\ 0 < \theta < 1,$$

$$\text{since } x_* + \theta h \approx x_* + h = \tilde{x} \text{ (say)}, as\ 0 < h < 1,$$

$$\Rightarrow \mathrm{F}(\tilde{x}) - \mathrm{F}(x_*) = (\tilde{x} - x_*)\mathrm{F}'(\tilde{x}),$$

$$\Rightarrow \tilde{x} - x_* = \frac{\mathrm{F}(\tilde{x})}{\mathrm{F}'(\tilde{x})} - \frac{\mathrm{F}(x_*)}{\mathrm{F}'(\tilde{x})}, (as\ x_* \text{ is a root of } \mathrm{F}(x) = 0 \text{ and by Definition 3.3,})$$

$$\Rightarrow |\tilde{x} - x_*| \leq \left|\frac{\mathrm{F}(\tilde{x})}{\mathrm{F}'(\tilde{x})}\right| \leq \lambda,$$

$$\Rightarrow |\tilde{x} - x_*| \leq \lambda,$$

implies that, the distance between strong initial approximation $\tilde{x}$ and the root x_* is smaller than unity. So we repeat the same argument under iteration by F, i.e., $\mathrm{I}_\mathrm{F}(\tilde{x}) = x_1$ is the first approximation and using Lagrange's MVT given as follows:

$$\mathrm{F}\left(x_* + \frac{h}{2}\right) - \mathrm{F}(x_*) = \left[\left(x_* + \frac{h}{2}\right) - x_*\right] \mathrm{F}'\left(x_* + \frac{\theta h}{2}\right),\ 0 < \theta < 1,$$

because $x_* + \frac{\theta h}{2} \approx x_* + \frac{h}{2} = x_1$ (say),

$\mathrm{F}(x_1) - \mathrm{F}(x_*) = (x_1 - x_*)\mathrm{F}'(x_1)$,

$\Rightarrow x_1 - x_* = \frac{\mathrm{F}(x_1)}{\mathrm{F}'(x_1)} - \frac{\mathrm{F}(x_*)}{\mathrm{F}'(x_1)}$, (as x_* is a roof of $\mathrm{F}(x) = 0$ and by Definition-4.3,)

$\Rightarrow |x_1 - x_*| \le \left|\frac{\mathrm{F}(x_1)}{\mathrm{F}'(x_1)}\right| \le \lambda^2$ $\Rightarrow |x_1 - x_*| \le \lambda^2$.

Since $0 < \lambda < 1$, implying $\lambda^2 < 1$, i.e., the distance between first approximation x_1 and the root x_* is more smaller than before. So we repeat the same argument under iteration by F for $n > 0$, i.e., $\mathrm{I}_{\mathrm{F}}^{n}(\tilde{x}) = x_n$ and we have $|x_n - x_*| \le \lambda^{n+1}$. Since $0 < \lambda < 1, \lambda^{n+1} \to 0$ as $n \to \infty$. Therefore, $x_n \to x_*$ as $n \to \infty$. This completes the proof.

A good number of examples on algebraic and transcendental equations are studied to illustrated the process of calculation. In the following example we have taken the problem studied by Philip [8] and Wiersma [10].

Example 3.5 The function $\mathrm{F}(x) = x^3 - x$ has three real roots $x = \pm 1$ and 0. A simple calculation gives critical points $x = \pm\frac{1}{\sqrt{3}}$, so we can conclude the basin of attraction of as $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \subset \mathrm{B}(-1), \left(-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \subset \mathrm{B}(0)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right) \subset \mathrm{B}(1)$, where $\mathrm{B}(-1), \mathrm{B}(0)$ and $\mathrm{B}(1)$ are Newton's basin of attraction for the roots $-1, 0$ and 1 respectively. We establish the result by our proposed method in the following Table-1.

Choose $f(x) = x^3 - x$					
	Roots of $f(x) \mp x = 0$	$\lim\limits_{x \to x_i}\left\|\frac{f(x)}{\pm x}\right\|$	$\left\|\frac{\mathrm{F}(x_i)}{\mathrm{F}'(x_i)}\right\|$	Nature	$\mathrm{N}_\delta(x_i) \subset \mathrm{B}(x_*)$
	$x_0 = 0$	1	0	root	$\mathrm{N}_\delta(x_0) \subset \left(\frac{-1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$
	$x_1 = \sqrt{2}$	1	0.28	supper	$\mathrm{N}_\delta(x_1) \subset \left(\frac{1}{\sqrt{3}}, \infty\right)$
	$x_2 = -\sqrt{2}$	1	0.28	supper	$\mathrm{N}_\delta(x_2) \subset \left(-\infty, \frac{-1}{\sqrt{3}}\right)$
	$x_3 = x_0 = 0$	1	0	root	$\mathrm{N}_\delta(x_0) \subset \left(\frac{-1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$
Choose $f(x) = x^3$					
	Roots of $f(x) \mp x = 0$	$\lim\limits_{x \to x_i}\left\|\frac{f(x)}{\pm x}\right\|$	$\left\|\frac{\mathrm{F}(x_i)}{\mathrm{F}'(x_i)}\right\|$	Nature	$\mathrm{N}_\delta(x_i) \subset \mathrm{B}(x_*)$
	$x_0 = 0$	0	0	root	$\mathrm{N}_\delta(x_0) \subset \left(\frac{-1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$
	$x_1 = 1$	1	0	root	$\mathrm{N}_\delta(x_1) \subset \left(\frac{1}{\sqrt{3}}, \infty\right)$
	$x_2 = -1$	1	0	root	$\mathrm{N}_\delta(x_2) \subset \left(-\infty, \frac{-1}{\sqrt{3}}\right)$
	$x_3 = x_0 = 0$	0	0	root	$\mathrm{N}_\delta(x_0) \subset \left(\frac{-1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$
	$x_4 = \pm i \notin \mathbb{D}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Table 1: Example: 3.5

Example 3.6 Let us consider a transcendental equation $\mathrm{F}(x) = x - 2\sin x = 0$ with roots such as $0, \pm 1.895494, \cdots$ etc. and $\mathrm{F}'(x) = 1 + 2\cos x$. Using the Maclaurin's series of $\sin x$, we rewrite the equation as $\mathrm{F}(x) = -x + \frac{x^3}{3} - \frac{2x^5}{5!} \cdots = 0$, setting $f(x) = \frac{x^3}{3}$. Now we summarize the proposed method of calculation in the following Table-2.

Table 2: Example: 3.6

Choose $f(x) = \frac{x^3}{3}$					
	Roots of $f(x) - x = 0$	$\lim_{x \to x_i} \left\lvert \frac{f(x)}{x} \right\rvert$	$\left\lvert \frac{\mathrm{F}(x_i)}{\mathrm{F}'(x_i)} \right\rvert$	Nature	$\mathrm{N}_\delta(x_i) \subset \mathrm{B}(x_*)$
	$x_0 = 0$	0	0	root	$\mathrm{N}_\delta(x_0) \subset \mathrm{B}(0)$
	$x_1 = \sqrt{3}$	1	0.35	supper	$\mathrm{N}_\delta(x_1) \subset \mathrm{B}(1.895)$
	$x_2 = -\sqrt{3}$	1	0.35	supper	$\mathrm{N}_\delta(x_2) \subset \mathrm{B}(-1.895)$

Clearly, the initial approximation $x_1 = \sqrt{3} = 1.732$ is fairly close to the least positive root 1.895.

Example 3.7 Let us consider a transcendental equation $\mathrm{F}(x) = \cos x \cdot \cosh x - 1 = 0$ with roots such as $0, 4.73,$ etc. Using Cauchy product of series on the Maclaurin's series of $\cos x$ and $\cosh x$, we can rewrite the equation as follows

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\mathrm{F}(x) = \cos x \cdot \cosh x - 1 = \frac{-5 \cdot x^4}{2 \cdot 3!} + \frac{x^8}{3 \cdot 7 \cdot 5!} + \cdots$$

Now we summarize the proposed method of calculation in the following Table-3.

Choose $f(x) = \frac{-5 \cdot x^4}{12}$ or $\frac{x^8}{3 \cdot 7 \cdot 5!}$					
	Roots of $f(x) - x = 0$	$\lim_{x \to x_i} \left\lvert \frac{f(x)}{x} \right\rvert$	$\left\lvert \frac{\mathrm{F}(x_i)}{\mathrm{F}'(x_i)} \right\rvert$	Nature	$\mathrm{N}_\delta(x_i) \subset \mathrm{B}(x_*)$
	$x_0 = 0$	0	0	root	$\mathrm{N}_\delta(x_0) \subset \mathrm{B}(0)$
	$x_1 = \pm 1.33$	1	0.35	supper	$\mathrm{N}_\delta(x_1) \subset \mathrm{B}(0)$
	$x_2 = 3.06$	1	0.41	supper	$\mathrm{N}_\delta(x_2) \subset \mathrm{B}(4.73)$

Table 3: Example: 3.7

The imaginary roots of $f(x) \pm x = 0$ for $f(x) = \frac{-5 \cdot x^4}{12}$, are not included in the Table-3. Similarly, other initial approximation can be estimate.

Now we are extending our argument in the field of complex number, in the following section.

4. Initial approximation in numerical method of function with complex roots

Let us consider a complex polynomial $\mathrm{F}(z)$ of n degree, define in the domain $\mathrm{D} \subseteq \mathbb{C}$

$$\mathrm{F}(z) = a_0 + a_1 \cdot z + a_2 \cdot z^2 + a_3 \cdot z^3 + \ \cdots \ + a_{n-1} \cdot z^{n-1} + a_n \cdot z^n, \ a_0, a_n \neq 0 \quad (4.1)$$

and its polynomial equation can be represent as

$$\mathrm{F}(z) = 1 + \left(\frac{a_1}{a_0}\right) z + \left(\frac{a_2}{a_0}\right) z^2 + \left(\frac{a_3}{a_0}\right) z^3 + \ \cdots \ + \left(\frac{a_{n-1}}{a_0}\right) z^{n-1} + \left(\frac{a_n}{a_0}\right) z^n = 0 \qquad (4.2)$$

Definition 4.1 *Let $F(z)$ be a map defined in the domain $D \subseteq \mathbb{C}$ and $f(z)$ be the two lowest degree terms (with unity constant if any) or single term of degree ≥ 3 of $F(z)$. Let z_0 be the root of the equation $f(z) \mp z = 0$. Then, a point $z \in D$ is said to be favorable initial approximation of the equation $F(z) = 0$ if and only if $\lim_{z \to z_0} \left|\frac{f(z)}{\pm z}\right| = 1$ or 0.*

In other words, A point $z \in \mathrm{D}$ is said to be favorable initial approximation of the equation $\mathrm{F}(z) = 0$, if and only if, for any $\epsilon > 0$ there exist $\delta > 0$ depending on ϵ such that $0 < |z - z_0| < \delta, \Rightarrow \left|\frac{f(z)}{\pm z} - (1 \text{ or } 0)\right| < \epsilon$ where z_0 is the root of the equation $f(z) \mp z = 0$ in the domain $\mathrm{D} \subseteq \mathbb{C}$ of the map $\mathrm{F}(z)$, where $f(z)$ is the two lowest degree terms (with unity constant if any) or single term of degree ≥ 3 of $\mathrm{F}(z)$.

Let us consider $f(z) = 1 + \left(\frac{a_1}{a_0}\right) z$ two lowest degree terms (with unity constant) of $\mathrm{F}(z) = 0$, according to the Definition- 4.1, the following

$$\left|\frac{f(z)}{\pm z} - 1\right| < \epsilon \Rightarrow \left|\frac{1 + \left(\frac{a_1}{a_0}\right) z}{z} - 1\right| < \epsilon,$$

$$\Rightarrow \left|z - \frac{a_0}{a_0 - a_1}\right| < \delta, \ \delta = z \cdot \epsilon \left(\frac{a_1 - a_0}{a_0}\right),$$

$$\Rightarrow |z - z_0| < \delta, \ z_0 = \frac{a_0}{a_0 - a_1} \text{ is the root of } f(z) - z = 0,$$

$$\Rightarrow z \in \mathrm{N}_\delta(z_0) = \{z: |z - z_0| < \delta\}.$$

Where $\mathrm{N}_\delta(z_0)$ is the neighborhood of favorable initial approximation z_0. Similarly, we can repeat the argument for the other favorable initial approximation of $\mathrm{F}(z) = 0$.

Definition 4.2 *Suppose that a map $F: \mathbb{C} \to \mathbb{C}$, is analytic at a favorable initial approximation $\tilde{z} \in N_\delta(z_0)$ and $|F'(\tilde{z})| \neq 0$. Then*

(a) $\tilde{z}$ is root if and only if $\left|\frac{\mathrm{F}(\tilde{z})}{\mathrm{F}'(\tilde{z})}\right| = 0$,

(b) $\tilde{z}$ is supper strong attracting if and only if $0 < \left|\frac{\mathrm{F}(\tilde{z})}{\mathrm{F}'(\tilde{z})}\right| \leq \frac{1}{2}$;

(c) $\tilde{z}$ is strong attracting if and only if $\frac{1}{2} < \left|\frac{\mathrm{F}(\tilde{z})}{\mathrm{F}'(\tilde{z})}\right| < 1$;

(d) $\tilde{z}$ is useless if and only if $\left|\frac{\mathrm{F}(\tilde{z})}{\mathrm{F}'(\tilde{z})}\right| \geq 1$.

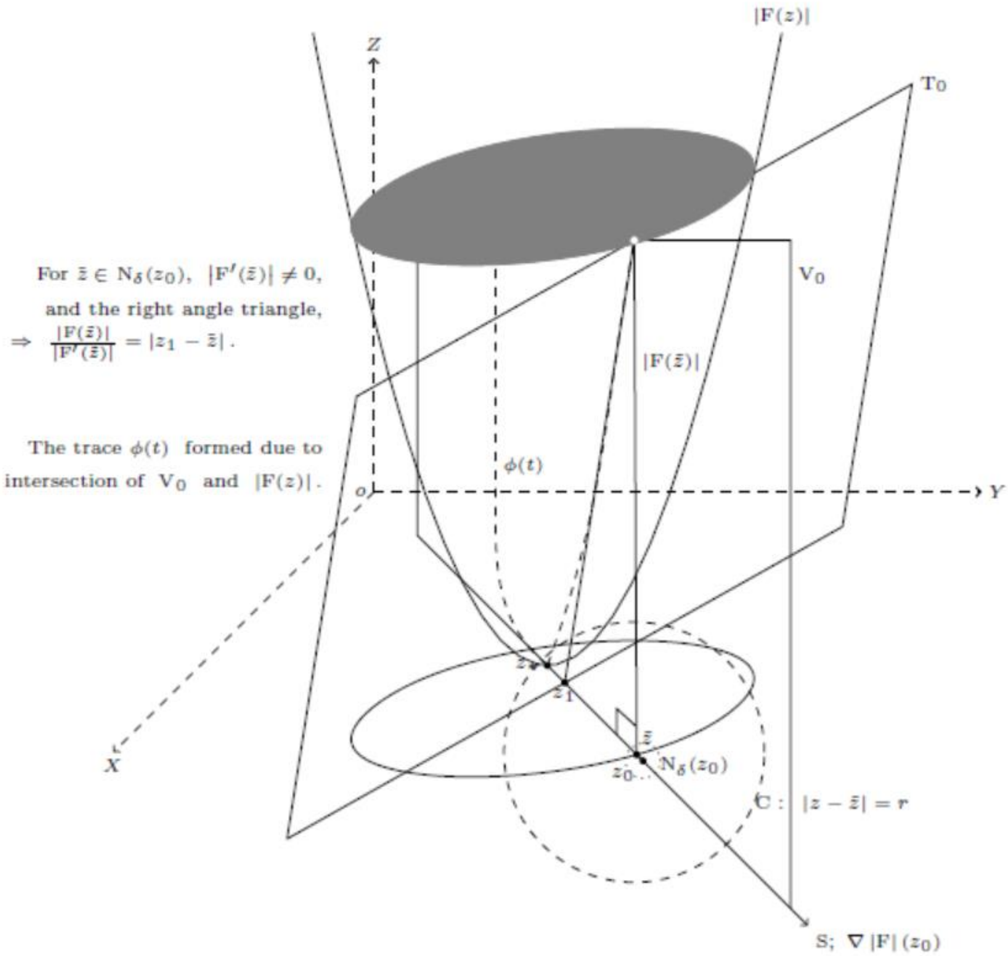

Figure 2: Favorable initial approximation z_0 and first iteration $z_1 = \mathrm{I}_\mathrm{F}(\tilde{z})$: T_0 is the tangent plane at z_0; V_0 is the vertical plane through z_0; $\phi(t)$ is a real valued function defined over the line S in the direction of $\nabla|\mathrm{F}|(z_0)$.

Theorem 4.3 *If a map $F(z)$ is analytic in its natural domain $D \subseteq \mathbb{C}$ and z_* is a zero of the equation $F(z) = 0$, then there is a neighborhood of a strong attracting point (that is, $N_\delta \subset B_*$) on which all points converges to z_* under iteration by F, where B_* is the Newton's basin of attraction for the root z_*; that is, for any $\tilde{z} \in N_\delta(z_0)$ and $I_F^n(\tilde{z}) = z_n, \Rightarrow \lim_{n\to\infty} z_n = z_*$.*

Proof. A map $\mathrm{F}(z)$ is analytic in its natural domain $\mathrm{D} \subseteq \mathbb{C}$ and z_* is a zero of the equation $\mathrm{F}(z) = 0$. Let z_0 be a strong attraction point and there are two numbers $0 < \delta, \lambda < 1$, such that $\mathrm{N}_\delta(z_0) = \{z : |z - z_0| \leq \delta\}$ on which all points satisfy $\left|\frac{\mathrm{F}}{\mathrm{F}'}\right| \leq \lambda < 1$, by Definition- 4.3. Now let us consider a circle $\mathrm{C}(i.e., |z - z_0| \leq r) \subset \mathrm{D}$, center is at z_0 with radius $r > 0$ such that it contain the point z_* (see Figure: 2). Applying Taylor's series expansion of $\mathrm{F}(z)$ at $\tilde{z} \in \mathrm{N}_\delta(z_0)$ with definite increment $h > 0$, we have the following

$\mathrm{F}(\tilde{z}+h)-\mathrm{F}(\tilde{z})=[(\tilde{z}+h)-(\tilde{z})]\cdot \mathrm{F}'(\tilde{z})$. Neglecting the higher order terms. set

$$\tilde{z}+h=z_*\ \text{(say)},$$

$$\mathrm{F}(\tilde{z})-\mathrm{F}(z_*)=(\tilde{z}-z_*)\cdot \mathrm{F}'(\tilde{z}),$$

$$\Rightarrow \tilde{z}-z_*=\frac{\mathrm{F}(\tilde{z})}{\mathrm{F}'(\tilde{z})}-\frac{\mathrm{F}(z_*)}{\mathrm{F}'(\tilde{z})},\because\ z_* \text{ is a root of } \mathrm{F}(z)=0 \text{ and by Definition}-4.2,$$

$$\Rightarrow |\tilde{z}-z_*|\le\left|\frac{\mathrm{F}(z_0)}{\mathrm{F}\prime(\tilde{z})}\right|\le\lambda,$$

$$\Rightarrow |\tilde{z}-z_*|\le\lambda.$$

This implies that, distance between strong initial approximation $\tilde{z}$ and the root z_* is smaller than unity. So we repeat the same argument under the iteration by F, i.e., $\mathrm{I}_{\mathrm{F}}(\tilde{z})=z_1$ is the first approximation and using Taylor's series at z_1, as follows

$$\Rightarrow \mathrm{F}\left(z_1+\frac{h}{2}\right)-\mathrm{F}(z_1)=\left[\left(z_1+\frac{h}{2}\right)-(z_1)\right]\cdot \mathrm{F}'^{(z_1)},\ \text{neglecting higher order terms,}$$

$$\text{set } z_1+\frac{h}{2}=z_*\ \text{(say)},$$

$$\Rightarrow \mathrm{F}(z_1)-\mathrm{F}(z_*)=(z_1-z_*)\cdot \mathrm{F}'(z_1),$$

$$\Rightarrow z_1-z_*=\frac{\mathrm{F}(z_1)}{\mathrm{F}'(z_1)}-\frac{\mathrm{F}(z_*)}{\mathrm{F}'(z_1)},\because\ z_* \text{ is a root of } \mathrm{F}(z)=0 \text{ and by Definition}-4.2,$$

$$\Rightarrow |z_1-z_*|\le\left|\frac{\mathrm{F}(z_1)}{\mathrm{F}\prime(z_1)}\right|\le\lambda,$$

$$\Rightarrow |z_1-z_*|\le\lambda^2.$$

Since $0<\lambda<1$, we have $\lambda^2<1$. This implies that, distance between first approximation z_1 and the root z_* is more smaller than before. So we repeat the same argument under the iteration by F for $n>0$, i.e., $\mathrm{I}_{\mathrm{F}}^n(\tilde{z})=z_n$ and we have $|z_n-z_*|\le\lambda^{n+1}$. Since $0<\lambda<1, \lambda^{n+1}\to 0$ as $n\to\infty$. Therefore, $z_n\to z_*$ as $n\to\infty$. This completes the proof.

Corollary 4.4 *Strong initial approximation of complex function coincides with strong initial approximation of real function in a particular direction.*

Proof. A map $\mathrm{F}(z)$ is analytic in its natural domain $\mathrm{D}\subseteq\mathbb{C}$ and $z_*=(x_*,y_*)$ is a zero of the equation $\mathrm{F}(z)=0$. Let $z_0=(x_0,y_0)$ be a strong attraction point for the root z_* such that $\mathrm{N}_\delta(z_0)\subset \mathrm{B}(z_*)$. Let T_0 be the tangent plane to the surface $|\mathrm{F}(z)|=f(x,y)$(say) at the point z_0 (see Figure: 2) and V_0 vertical plane in the direction of z_0. Let S be the trace of the line containing z_*, z_0 and in the direction of $\nabla|\mathrm{F}|(z_0)$. Let $\phi\colon \mathrm{S}\to\mathbb{R}$ be the real valued function defined over S, is formed due to the intersection of vertical plane V_0 and $|\mathrm{F}(z)|$. The trace S can be parameterized as $x=t, y=mt+c$ where m is the slope and c is the y −intercept. Using these parameter we can rewrite the surface as

$$|\mathrm{F}(z)|=f(x,y)=f(t,mt+c)=\phi(t),$$

$$\Rightarrow |\mathrm{F}(z_0)|=f(x_0,y_0)=\phi(x_0), \text{at } z_0=(x_0,y_0). \tag{4.3}$$

Hence, the strong initial approximation $\tilde{z}\in \mathrm{N}_\delta(z_0)$ in the complex function coincides with the $\tilde{x}\in \mathrm{N}_\delta(x_0)$ of real function, in the trace S.

This established the statement.

Example 4.5 Let us consider an example studied by Yau-$Yau98$, $\mathrm{F}(z) = z^4 - 1 = 0$ has four roots such as $\pm 1, \pm i$. Several sets of absolute value of $|\mathrm{F}(z)|$ are plotted for the roots of the equation, see $-$ $Yau98$. Now we summarize the proposed method of calculation in Table-4.

Table 4: Example: 4.5

Choose $f(z) = z^4$					
	Roots of $f(z) \mp z = 0$	$\lim_{z \to z_i} \left\|\frac{f(z)}{\pm z}\right\|$	$\left\|\frac{\mathrm{F}(z_i)}{\mathrm{F}'(z_i)}\right\|$	Nature	$\mathrm{N}_\delta(z_i) \subset \mathrm{B}(z_*)$
	$z_0 = 0$	0	∞	critical point	$\cdots$
	$z_1 = 1$	1	0	root	$\mathrm{N}_\delta(z_1) \subset \mathrm{B}(1)$
	$z_2 = \frac{-1}{2} + i\frac{\sqrt{3}}{2} = e^{\frac{2\pi i}{3}}$	1	0.75	strong	$\mathrm{N}_\delta(z_2) \subset \mathrm{B}(i)$
	$z_3 = \frac{-1}{2} - i\frac{\sqrt{3}}{2} = e^{\frac{4\pi i}{3}}$	1	0.75	strong	$\mathrm{N}_\delta(z_3) \subset \mathrm{B}(-i)$
	$z_4 = -1$	1	0	root	$\mathrm{N}_\delta(z_4) \subset \mathrm{B}(-1)$
	$z_5 = \frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{\frac{\pi i}{3}}$	1	0.75	strong	$\mathrm{N}_\delta(z_5) \subset \mathrm{B}(i)$
	$z_6 = \frac{1}{2} - i\frac{\sqrt{3}}{2} = e^{\frac{5\pi i}{3}}$	1	0.75	strong	$\mathrm{N}_\delta(z_6) \subset \mathrm{B}(-i)$

Example 4.6 Let us consider an example studied by Kafash [5],

$$\mathrm{F}(z) = (-4.5 + 2.5i) + (2 - i)z + e^{iz^2} = 0$$

with roots $z_* \in \mathrm{Z}_*$, where Z_* is the set of roots such that

$$\mathrm{Z}_* = \{(2.17 + 0.12i), (0.8 - 0.81i), (-0.39 + 2.77i), (-0.8 + 1.23i), (3 - 0.07i), \cdots\}$$

and $\mathrm{F}'(z) = (2 - i) + 2ize^{iz^2}$. Using the Maclaurin's series of e^{iz^2}, we can rewrite the equation as $\mathrm{F}(z) = (-3.5 + 2.5i) + (2 - i)z + iz^2 - \frac{z^4}{2} - \frac{iz^6}{3!} + \cdots = 0$. Now we summarize the proposed method of calculation in the following Table- 5.

Table 5: Example: 4.6

Choose $f(z) = (-3.5 + 2.5i) + (2 - i)z$					
	Roots of $f(z) \mp z = 0$	$\lim_{z \to z_i} \left\|\frac{f(z)}{\pm z}\right\|$	$\left\|\frac{\mathrm{F}(z_i)}{\mathrm{F}'(z_i)}\right\|$	Nature	$\mathrm{N}_\delta(z_i) \subset \mathrm{B}(z_*)$

	$z_1 = 3 + 0.5i$	1	0.62	strong	$\mathrm{N}_\delta(z_1) \subset \mathrm{B}(2.17 + 0.12i)$
	$z_2 = 1.3 - 0.4i$	1	0.31	supper	$\mathrm{N}_\delta(z_2) \subset \mathrm{B}(0.8 - 0.81i)$

Similarly by repeating the same argument, we can estimate the other initial approximation as well.

5. Conclusion

In this piece of work, we suggested a new strategy to estimate the real and complex initial approximation for both real and complex equations respectively. Any standard iteration by function will definitely convergent to the root by choosing these initial approximation and it will also diminish the frequency of iteration.

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