Advances in Nonlinear Variational Inequalities
ISSN: 1092-910X
Vol 28 No. 7s (2025)

Solution of a Triangular Intuitionistic Fuzzy Optimal Subdivision Problem - A Novel Approach

T. Nagalakshmi[1], I. Nandhini[2], M. Karthikeyan[3], M. Priyanka[4]

[1] Assistant Professor, Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology, Avadi, Chennai-62, Tamilnadu, India.
[2] Assistant Professor, Department of ECE, Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology, Avadi, Chennai-62, Tamilnadu, India.
[3] Associate Professor, Deparment of Mathematics, St.Joseph College of Engineering, Sriperumbudur, Chennai-602117, Tamilnadu, India.
[4] Assistant Professor, Department of Mathematics, Department of Science and Humanities, Vel Tech Multi Tech Dr. Rangarajan Dr. Sakunthala Engineering College, Chennai, Tamilnadu, India.
E-mail: drnagalakshmit@veltech.edu.in , drnandhinii@veltech.edu.in , kmkmkarthikeyan@gmail.com , priyankamahalingam31@gmail.com

---

Article History:

Received: 18-12-2024

Revised:28-1-2025

Accepted:6-2-2025

Abstract:

Introduction: Dynamic Programming is the mathematical technique of optimizing a series of connected decisions over a given amount of time. The process of making decisions in many real-world scenarios involves choosing a set of plans from a wide range of possible combinations in unclear circumstances.

Objectives: In this article, a novel approach is developed to solve a triangular intuitionistic fuzzy optimal subdivision problem in which a positive quantity which is to be partitioned is taken as triangular intuitionistic fuzzy number.

Methods: The mathematical induction technique is applied in the process of obtaining the optimal solution with fuzzy approach. The unique characteristics of the proposed approach is that the fuzziness and ambiguity in optimal subdivision models is eradicated by applying the technique of fuzzy dynamic programming.

Results: A new approach has been proposed in this paper for solving fuzzy optimal subdivision problem in which the positive quantity which is to be divided into ‘n’ factors is taken as trapezoidal fuzzy number. The solution is obtained by the method of Mathematical Induction. The optimal solution is obtained by using the fuzzy recursive equations.

Conclusions: Many real-world problems involve sequential or multistage decision making. Sometimes, the parameters may not be known precisely due to some uncontrollable factors. If the obtained results are crisp values then it might lose some helpful information. Fuzzy Dynamic Programming is a powerful optimization procedure that is particularly applicable to many complex problems requiring a sequence of interrelated decisions in a fuzzy environment and hence it has wide range of applications in the future.

Keywords: Intuitionistic Fuzzy dynamic Programming, Triangular Intuitionistic Fuzzy numbers, Mathematical Induction, Optimal Subdivision problem.

1. Introduction

The decision-making process often involves several decisions to taken at different times. For example, problems of inventory control, evaluation of investment opportunities, long-term corporate planning, and problems of optimal subdivision require sequential decision-making. The mathematical method in which the sequence of inter-related decisions are optimized over a period of time is called Dynamic Programming. If fuzziness is involved in the process, it is termed as fuzzy dynamic programming. It used the method of recursion to solve a complex problem which is broken down into a series of inter-related decision stages where the outcome of a decision at one stage affects the decision at each of the following stages. Dynamic Programming has wide range of applications in the fields of inventory, Markovian decision models, equipment replacement, advertising media, salesman allocation etc.,

Multistage decision making is still used in all of our daily actions as of right now. DP is employed to address those issues. However, a lot of variables including ambiguity and uncertainty are common in the field of MCDM challenges. Zadeh [1] presented the idea of fuzzy theory. Later on, it has several uses across various industries. Bellman and Zadeh [2] used fuzzy set theory as a tool for decision-making in 1970. Fuzzy Dynamic Programming is the process of using fuzzy set theory in the context of dynamic programming. Numerous researchers made contributions to the topic of FDP.

Kacpryzk [3] used FDP approach to solve the source problem with fuzzy states and fuzzy controls. He obtained optimal reference fuzzy control policies relating to above problem. He used interpolative reasoning approach to attain optimal fuzzy controls and fuzzy states. It is observed that they are not necessarily the reference ones. Phu and Tri [4] applied generalized Hukuhara differentiability to fuzzy functions. They also derived necessary and sufficient conditions for Bellman's principle with the help of fuzzy collocation and fuzzy product. They also discussed some examples of FDPPs in the field of fuzzy metric space. Nagalakshmi and Uthra [5] proposed an approach to find an optimal solution of least cost route problem with a fuzzy approach using generalized trapezoidal fuzzy numbers.

Kaliyaperumal [6] has considered a dynamic programming problem with single additive constraint along with additive seperable return. In his work, he used trapezoidal fuzzy membership functions. He also obtained solutions for numerical examples of both linear and non-linear problems. Mohanaselvi and Suparna [7] also followed the same approach developed by Nagalakshmi and Uthra [5] to obtain optimal solution to their fuzzy least cost route problem. Krishnaveni and Ganesan [8] proposed a technique based on DP approach to solve a fuzzy travelling salesman problem where parameters are considered as Triangular or Trapezoidal fuzzy numbers.

Zhu [9] applied geometric extremum theory to estimate the range. He considered a-gon (Ha) with unit edges. He divided it into 'b' parts by using optimal subdivision. His method is mainly used to setup the base station points for 5G so that the number of base stations can be minimum. This method can also be applied in the placement of sprinklers for irrigating the farms in the field of agriculture. Druti and Smita [10] developed a new approach to solve a shortest route problem using FDP technique. They considered trapezoidal fuzzy numbers in their approach. They used matlab software for their analysis along with fuzzy tool box. Two input parameters such as distance and weight were taken into account. Fuzzy rules were framed and applied to fuzzy inference system in order to determine optimum time values required to travel from one node to other. Khan and Aftab [11] introduced AFDP technique to LPP problems with fuzzy constraints. Nagalakshmi [12] solved a capital budgeting problem with neutrosophic fuzzy parameters.

This article is developed to solve a triangular intuitionistic fuzzy optimal subdivision problem in which a positive quantity $\tilde{c}$ which is to be partitioned is taken as triangular intuitionistic fuzzy number. The mathematical induction technique is applied in the process of obtaining the optimal solution with fuzzy approach.

2. Objectives & Preliminaries

This article is developed to solve a triangular intuitionistic fuzzy optimal subdivision problem in which a positive quantity $\tilde{c}$ which is to be partitioned is taken as triangular intuitionistic fuzzy number. The mathematical induction technique is applied in the process of obtaining the optimal solution with fuzzy approach.

2.1 Fuzzy set

Let X denotes a universal set. Then, the membership function μ_A by which a fuzzy set A is usually defined has the form $\mu_A: X \to [0,1]$, where [0,1] denotes the interval of real numbers from 0 to 1, both inclusive.

2.2 Intuitionistic Fuzzy Set:

Let X be the universe of discourse. Then an intuitionistic fuzzy set $\tilde{A}_{INT}$ in X is denoted by $\tilde{A}_{INT} = \{(x, \mu_{\tilde{A}_{INT}}(x), \nu_{\tilde{A}_{INT}}(x))/x\epsilon X\}$ where $\mu_{\tilde{A}_{INT}}(x): X \to [0,1]$ and $\nu_{\tilde{A}_{INT}}(x): X \to [0,1]$ such that $0 \le \mu_{\tilde{A}_{INT}}(x) + \nu_{\tilde{A}_{INT}}(x) \le 1$. Here, the degree of membership is denoted as $\mu_{\tilde{A}_{INT}}(x)$ and the degree of non-membership is denoted as $\nu_{\tilde{A}_{INT}}(x)$.

2.3 Intuitionistic Fuzzy Number:

Consider an intuitionistic fuzzy subset $\tilde{A}_{INT} = \{(x, \mu_{\tilde{A}_{INT}}(x), \nu_{\tilde{A}_{INT}}(x))/x\epsilon X\}$ of the real line R. It is said to be an intuitionistic fuzzy number if the following condition holds.

- There exists $d \in R$ such that $\mu_{\tilde{A}_{INT}}(d) = 1$ and $\nu_{\tilde{A}_{INT}}(d) = 0$. Here, d is called the mean value of $\tilde{A}_{INT}$.
- $\mu_{\tilde{A}_{INT}}$ is a continuous mapping from $R \to [0,1]$ and for all $x \in R$, the relation $0 \le \mu_{\tilde{A}_{INT}}(x) + \nu_{\tilde{A}_{INT}}(x) \le 1$ holds.

2.4 Triangular Intuitionistic Fuzzy Number:

A triangular Intuitionistic Fuzzy Number $\tilde{A}_{TIFN}$ is denoted by $\tilde{A}_{TIFN} = (j, k, l; j', k, l')$ where $j' \leq j \leq k \leq l \leq l'$. Its membership function $\mu_{\tilde{A}_{TIFN}}(x)$ and non-membership function $\nu_{\tilde{A}_{TIFN}}(x)$ are discussed below:

$$\mu_{\tilde{A}_{TIFN}}(x) = \begin{cases} \dfrac{x-j}{k-j}, & j \leq x \leq k \\ \dfrac{l-x}{l-k}, & k \leq x \leq l \\ 0, & otherwise \end{cases}$$

$$\nu_{\tilde{A}_{TIFN}}(x) = \begin{cases} \dfrac{k-x}{k-j^{'}}, & j^{'} \leq x \leq k \\ \dfrac{x-k}{l^{'}-k}, & k \leq x \leq l^{'} \\ 1, & otherwise \end{cases}$$

2.5 Arithmetic Operations of Triangular Intuitionistic Fuzzy Numbers:

Let $\tilde{A}_{TIFN} = (j_1, k_1, l_1; {j_1}', k_1, {l_1}')$ and $\tilde{B}_{TIFN} = (j_2, k_2, l_2; {j_2}', k_2, {l_2}')$ be two trapezoidal fuzzy numbers. Then,

(i) $\tilde{A}_{TIFN} + \tilde{B}_{TIFN} = (j_1 + j_2, k_1 + k_2, l_1 + l_2; {j_1}' + {j_2}', k_1 + k_2, {l_1}' + {l_2}')$

(ii) $\tilde{A}_{TIFN} - \tilde{B}_{TIFN} = (j_1 - l_2, k_1 - k_2, l_1 - j_2; {j_1}' - {l_2}', k_1 - k_2, {l_1}' - {j_2}')$

(iii) $\tilde{A}_{TIFN} * \tilde{B}_{TIFN} = (j_1 * j_2, k_1 * k_2, l_1 * l_2; {j_1}' * {j_2}', k_1 * k_2, {l_1}' * {l_2}')$

(iv) $\alpha\tilde{A}_{TIFN} = (\alpha j_1, \alpha k_1, \alpha l_1; \alpha {j_1}', \alpha k_1, \alpha {l_1}')\ for\ \alpha \geq 0$

(v) $\alpha\tilde{A}_{TIFN} = (\alpha l_1, \alpha k_1, \alpha j_1; \alpha {l_1}', \alpha k_1, \alpha {j_1}')\ for\ \alpha < 0$

3. Methods

3.1 Fuzzy Optimal Subdivision Problem

In this article, a fuzzy optimal subdivision problem is considered. This article is proposed to factorize a positive quantity $\tilde{c}$ which is considered as Triangular Intuitionistic Fuzzy Number into ‘n’ factors in such a manner that their sum is minimum. Its LPP model is given by

Min $\tilde{z} = \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 + \cdots + \tilde{x}_n$

such that $\tilde{x}_1\tilde{x}_2\tilde{x}_3 \ldots \tilde{x}_n = \tilde{c}$

and $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots x_n \geq 0$

A recursive equation that connects the optimal decision function for n-stages with n-1 stave optimal decision function is developed. This article deals with a problem where a positive quantity $\tilde{c}$ is factorized to n factors such that its sum is minimum. Therefore, it is considered as n-stage FDP. Let the ith part of $\tilde{c}$ be $\tilde{x}_i$. Here, each i is taken as a stage. Then $\tilde{x}_i$ is assumed any positive value satisfying the condition $\tilde{x}_1\tilde{x}_2\tilde{x}_3 \ldots \tilde{x}_n = \tilde{c}$, which has infinite alternative values at every stage. This implies that $\tilde{x}_i$ is continuous. Therefore, the method of classical differentiation is used to find optimal decisions at every stage.

Let $\widetilde{f_n}(\tilde{c})$ be the minimum attainable sum $\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 + \cdots + \tilde{x}_n$when the positive quantity '$\tilde{c}$' is factorized into 'n' factors $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots \tilde{x}_n$. Let $\tilde{c} = (j_1, k_1, l_1; {j_1}', k_1, {l_1}')$ where $j' \le j \le k \le l \le l'$. and $\tilde{x} = (x, x, x; x, x, x)$

3.2 Solution of a Triangular Intuitionistic Fuzzy Optimal Subdivision Problem:

Stage 1:

In stage 1 solution for case 1 is obtained for $n = 1$.

$\tilde{x}_1 = \tilde{c}$ i.e., $\tilde{x}_1 = (j_1, k_1, l_1; {j_1}', k_1, {l_1}')$ is the only factor. Then,

$$\widetilde{f_1}(\tilde{c}) = \min_{\tilde{x}_1 = \tilde{c}} (j_1, k_1, l_1; {j_1}', k_1, {l_1}')$$

$$\Rightarrow \widetilde{f_1}(\tilde{c}) = (j_1, k_1, l_1; {j_1}', k_1, {l_1}')$$

$$\Rightarrow \widetilde{f_1}(\tilde{c}) = \tilde{c} \text{ --[3.2.1]}$$

This is considered as a trivial case.

Stage 2:

In stage 2 solution for case 2 is obtained for $n = 2$.

We take $\tilde{x}_1 = (x, x, x; x, x, x)$ and $\tilde{x}_2 = \left(\frac{j_1}{x}, \frac{k_1}{x}, \frac{l_1}{x}; \frac{{j_1}'}{x}, \frac{k_1}{x}, \frac{{l_1}'}{x}\right)$

Next, $\tilde{c}$ is factorized into following two factors $\tilde{x}_1$and $\tilde{x}_2$ such that $\tilde{x}_1\tilde{x}_2 = \tilde{c}$

The following method is used to prove the above claim.

$$\tilde{x}_1\tilde{x}_2 = (x, x, x; x, x, x)\left(\frac{j_1}{x}, \frac{k_1}{x}, \frac{l_1}{x}; \frac{{j_1}'}{x}, \frac{k_1}{x}, \frac{{l_1}'}{x}\right)$$

The above product is obtained by the property of Triangular Intuitionistic Fuzzy Numbers.

$$\Rightarrow \tilde{x}_1\tilde{x}_2 = \left(x\frac{j_1}{x}, x\frac{k_1}{x}, x\frac{l_1}{x}; x\frac{{j_1}'}{x}, x\frac{k_1}{x}, x\frac{{l_1}'}{x}\right)$$

$$\tilde{x}_1\tilde{x}_2 = (j_1, k_1, l_1; {j_1}', k_1, {l_1}')$$

$$\Rightarrow \tilde{x}_1\tilde{x}_2 = \tilde{c}$$

$$\widetilde{f_2}(\tilde{c}) = \min_{0 \le \tilde{x} \le \tilde{c}} \{\tilde{x}_1 + \tilde{x}_2\}$$

$$\Rightarrow \widetilde{f_2}(\tilde{c}) = \min_{0 \le \tilde{x} \le \tilde{c}} \left\{(x, x, x; x, x, x) + \left(\frac{j_1}{x}, \frac{k_1}{x}, \frac{l_1}{x}; \frac{{j_1}'}{x}, \frac{k_1}{x}, \frac{{l_1}'}{x}\right)\right\} \qquad \text{[From 3.2.1]}$$

$\Rightarrow \tilde{f}_2(\tilde{c}) = \min\limits_{0\le\tilde{x}\le\tilde{c}} \left\{(x,x,x;x,x,x) + \tilde{f}_1\left(\frac{j_1}{x},\frac{k_1}{x},\frac{l_1}{x};\frac{j_1{}'}{x},\frac{k_1}{x},\frac{l_1{}'}{x}\right)\right\}$ ----------------------[3.2.2]

Stage 3:

In stage 3 solution for case 3 is obtained for $n = 3$.

We take $\tilde{x}_1 = (x,x,x;x,x,x)$ and $\tilde{x}_2\tilde{x}_3 = \left(\frac{j_1}{x},\frac{k_1}{x},\frac{l_1}{x};\frac{j_1{}'}{x},\frac{k_1}{x},\frac{l_1{}'}{x}\right)$

Then, $\tilde{c}$ can be factorized into three factors $\tilde{x}_1, \tilde{x}_2$ and $\tilde{x}_3$ such that $\tilde{x}_1\tilde{x}_2\tilde{x}_3 = \tilde{c}$

i.e., $\left(\frac{j_1}{x},\frac{k_1}{x},\frac{l_1}{x};\frac{j_1{}'}{x},\frac{k_1}{x},\frac{l_1{}'}{x}\right)$ is again factorized into two factors such that their minimum attainable sum is $\tilde{f}_2\left(\frac{j_1}{x},\frac{k_1}{x},\frac{l_1}{x};\frac{j_1{}'}{x},\frac{k_1}{x},\frac{l_1{}'}{x}\right)$.

Then $\tilde{f}_3(\tilde{c}) = \min\limits_{0\le\tilde{x}\le\tilde{c}}\{\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3\}$

$\Rightarrow \tilde{f}_3(\tilde{c}) = \min\limits_{0\le\tilde{x}\le\tilde{c}} \left\{(x,x,x;x,x,x) + \tilde{f}_2\left(\frac{j_1}{x},\frac{k_1}{x},\frac{l_1}{x};\frac{j_1{}'}{x},\frac{k_1}{x},\frac{l_1{}'}{x}\right)\right\}$ [From 3.2.2]

Stage n:

In general stage n, the solution for case n is obtained for $n = n$. Its recursive equation for the stage-n problem is given as

$\tilde{f}_n(\tilde{c}) = \min\limits_{0\le\tilde{x}\le\tilde{c}} \left\{(x,x,x;x,x,x) + \tilde{f}_{n-1}\left(\frac{j_1}{x},\frac{k_1}{x},\frac{l_1}{x};\frac{j_1{}'}{x},\frac{k_1}{x},\frac{l_1{}'}{x}\right)\right\}$ -------------------------[3.2.3]

The process of Mathematical Induction is used to solve the above recursive equation of stage-n.

When $n = 2$, Eq. [3.2.3] gives

$\tilde{f}_2(\tilde{c}) = \min\limits_{0\le\tilde{x}\le\tilde{c}} \left\{(x,x,x;x,x,x) + \tilde{f}_1\left(\frac{j_1}{x},\frac{k_1}{x},\frac{l_1}{x};\frac{j_1{}'}{x},\frac{k_1}{x},\frac{l_1{}'}{x}\right)\right\}$

$\Rightarrow \tilde{f}_2(\tilde{c}) = \min\limits_{0\le\tilde{x}\le\tilde{c}} \left\{(x,x,x;x,x,x) + \left(\frac{j_1}{x},\frac{k_1}{x},\frac{l_1}{x};\frac{j_1{}'}{x},\frac{k_1}{x},\frac{l_1{}'}{x}\right)\right\}$ [From 3.2.1]

$\Rightarrow \tilde{f}_2(\tilde{c}) = \min\limits_{0\le\tilde{x}\le\tilde{c}} \left\{x+\frac{j_1}{x}, x+\frac{k_1}{x}, x+\frac{l_1}{x}; x+\frac{j_1{}'}{x}, x+\frac{k_1}{x}, x+\frac{l_1{}'}{x}\right\}$

Let $p_1 = x + \frac{j_1}{x}$. To check whether p_1 is maximum or minimum, it is considered that $\frac{dp_1}{dx} = 0$. The solution is obtained as $x = \sqrt{j_1}$

Similarly, if we consider p_2, p_3, p_4, p_5, p_6 as $x+\frac{k_1}{x}, x+\frac{l_1}{x}; x+\frac{j_1{}'}{x}, x+\frac{k_1}{x}, x+\frac{l_1{}'}{x}$ respectively, then the solutions are obtained as $x = \sqrt{k_1}, x = \sqrt{l_1}, x = \sqrt{j_1{}'}, x = \sqrt{k_1}, x = \sqrt{l_1{}'}$

This function $\left\{x+\frac{j_1}{x}, x+\frac{k_1}{x}, x+\frac{l_1}{x}; x+\frac{j_1{}'}{x}, x+\frac{k_1}{x}, x+\frac{l_1{}'}{x}\right\}$ will be minimum only if

$\tilde{x} = \left(\sqrt{j_1}, \sqrt{k_1}, \sqrt{l_1}; \sqrt{{j_1}'}, \sqrt{k_1}, \sqrt{{l_1}'}\right)$

Therefore, $\tilde{f}_2(\tilde{c}) = \min\limits_{0 \le \tilde{x} \le \tilde{c}} \left\{(x, x, x; x, x, x) + \left(\frac{j_1}{x}, \frac{k_1}{x}, \frac{l_1}{x}; \frac{{j_1}'}{x}, \frac{k_1}{x}, \frac{{l_1}'}{x}\right)\right\}$

$$\Rightarrow \tilde{f}_2(\tilde{c}) = \min_{0 \le \tilde{x} \le \tilde{c}} \left\{\left(\sqrt{j_1}, \sqrt{k_1}, \sqrt{l_1}; \sqrt{{j_1}'}, \sqrt{k_1}, \sqrt{{l_1}'}\right) + \left(\frac{j_1}{\sqrt{j_1}}, \frac{k_1}{\sqrt{k_1}}, \frac{l_1}{\sqrt{l_1}}; \frac{{j_1}'}{\sqrt{{j_1}'}}, \frac{k_1}{\sqrt{k_1}}, \frac{{l_1}'}{\sqrt{{l_1}'}}\right)\right\}$$

$$\Rightarrow \tilde{f}_2(\tilde{c}) = \min_{0 \le \tilde{x} \le \tilde{c}} \left\{\sqrt{j_1} + \frac{j_1}{\sqrt{j_1}}, \sqrt{k_1} + \frac{k_1}{\sqrt{k_1}}, \sqrt{l_1} + \frac{l_1}{\sqrt{l_1}}; \sqrt{{j_1}'} + \frac{{j_1}'}{\sqrt{{j_1}'}}, \sqrt{k_1} + \frac{k_1}{\sqrt{k_1}}, \sqrt{{l_1}'} + \frac{{l_1}'}{\sqrt{{l_1}'}}\right\}$$

$$\Rightarrow \tilde{f}_2(\tilde{c}) = \min_{0 \le \tilde{x} \le \tilde{c}} \left\{\left(2\sqrt{j_1}, 2\sqrt{k_1}, 2\sqrt{l_1}; 2\sqrt{{j_1}'}, 2\sqrt{k_1}, 2\sqrt{{l_1}'}\right)\right\}$$

$$\Rightarrow \tilde{f}_2(\tilde{c}) = 2 \min_{0 \le \tilde{x} \le \tilde{c}} \left\{\left(\sqrt{j_1}, \sqrt{k_1}, \sqrt{l_1}; \sqrt{{j_1}'}, \sqrt{k_1}, \sqrt{{l_1}'}\right)\right\}$$

$$\Rightarrow \tilde{f}_2(\tilde{c}) = 2\left({j_1}^{\frac{1}{2}}, {k_1}^{\frac{1}{2}}, {l_1}^{\frac{1}{2}}; {j_1}'^{\frac{1}{2}}, {k_1}^{\frac{1}{2}}, {l_1}'^{\frac{1}{2}}\right)$$

The optimal policy is given by

$\left[\left({j_1}^{\frac{1}{2}}, {k_1}^{\frac{1}{2}}, {l_1}^{\frac{1}{2}}; {j_1}'^{\frac{1}{2}}, {k_1}^{\frac{1}{2}}, {l_1}'^{\frac{1}{2}}\right), \left({j_1}^{\frac{1}{2}}, {k_1}^{\frac{1}{2}}, {l_1}^{\frac{1}{2}}; {j_1}'^{\frac{1}{2}}, {k_1}^{\frac{1}{2}}, {l_1}'^{\frac{1}{2}}\right)\right]$ and

$$\tilde{f}_2(\tilde{c}) = 2\left({j_1}^{\frac{1}{2}}, {k_1}^{\frac{1}{2}}, {l_1}^{\frac{1}{2}}; {j_1}'^{\frac{1}{2}}, {k_1}^{\frac{1}{2}}, {l_1}'^{\frac{1}{2}}\right)$$

For $n = 3$, Equation [3.2.3] becomes

$$\tilde{f}_3(\tilde{c}) = \min_{0 \le \tilde{x} \le \tilde{c}} \left\{(x, x, x; x, x, x) + \tilde{f}_2\left(\frac{j_1}{x}, \frac{k_1}{x}, \frac{l_1}{x}; \frac{{j_1}'}{x}, \frac{k_1}{x}, \frac{{l_1}'}{x}\right)\right\}$$

$$\Rightarrow \tilde{f}_3(\tilde{c}) = \min_{0 \le \tilde{x} \le \tilde{c}} \left\{(x, x, x; x, x, x) + 2\left(\left(\frac{j_1}{x}\right)^{\frac{1}{2}}, \left(\frac{k_1}{x}\right)^{\frac{1}{2}}, \left(\frac{l_1}{x}\right)^{\frac{1}{2}}; \left(\frac{{j_1}'}{x}\right)^{\frac{1}{2}}, \left(\frac{k_1}{x}\right)^{\frac{1}{2}}, \left(\frac{{l_1}'}{x}\right)^{\frac{1}{2}}\right)\right\}$$

$$\Rightarrow \tilde{f}_3(\tilde{c}) = \min_{0 \le \tilde{x} \le \tilde{c}} \left\{(x, x, x; x, x, x) + \left(2\left(\frac{j_1}{x}\right)^{\frac{1}{2}}, 2\left(\frac{k_1}{x}\right)^{\frac{1}{2}}, 2\left(\frac{l_1}{x}\right)^{\frac{1}{2}}; 2\left(\frac{{j_1}'}{x}\right)^{\frac{1}{2}}, 2\left(\frac{k_1}{x}\right)^{\frac{1}{2}}, 2\left(\frac{{l_1}'}{x}\right)^{\frac{1}{2}}\right)\right\}$$

$$\Rightarrow \tilde{f}_3(\tilde{c}) = \min_{0 \le \tilde{x} \le \tilde{c}} \left\{x + 2\left(\frac{j_1}{x}\right)^{\frac{1}{2}}, x + 2\left(\frac{k_1}{x}\right)^{\frac{1}{2}}, x + 2\left(\frac{l_1}{x}\right)^{\frac{1}{2}}; x + 2\left(\frac{{j_1}'}{x}\right)^{\frac{1}{2}}, x + 2\left(\frac{k_1}{x}\right)^{\frac{1}{2}}, x + 2\left(\frac{{l_1}'}{x}\right)^{\frac{1}{2}}\right\}$$

Let $p_1 = x + 2\left(\frac{j_1}{x}\right)^{\frac{1}{2}}$. To check whether p_1 is maximum or minimum, it is considered that $\frac{dp_1}{dx} = 0$. The solution is obtained as $x = \sqrt[3]{j_1}$

Similarly, if we consider p_2, p_3, p_4, p_5, p_6 as $x+2\left(\frac{k_1}{x}\right)^{\frac{1}{2}}, x+2\left(\frac{l_1}{x}\right)^{\frac{1}{2}}; x+2\left(\frac{j_1'}{x}\right)^{\frac{1}{2}}, x+2\left(\frac{k_1}{x}\right)^{\frac{1}{2}}, x+2\left(\frac{l_1'}{x}\right)^{\frac{1}{2}}$ respectively, then the solutions are obtained as $x=\sqrt[3]{k_1}, x=\sqrt[3]{l_1}, x=\sqrt[3]{j_1'}, x=\sqrt[3]{k_1}, x=\sqrt[3]{l_1'}$

This function $\left\{x+2\left(\frac{j_1}{x}\right)^{\frac{1}{2}}, x+2\left(\frac{k_1}{x}\right)^{\frac{1}{2}}, x+2\left(\frac{l_1}{x}\right)^{\frac{1}{2}}; x+2\left(\frac{j_1'}{x}\right)^{\frac{1}{2}}, x+2\left(\frac{k_1}{x}\right)^{\frac{1}{2}}, x+2\left(\frac{l_1'}{x}\right)^{\frac{1}{2}}\right\}$ will be minimum only if $\tilde{x}=\left(\sqrt[3]{j_1}, \sqrt[3]{k_1}, \sqrt[3]{l_1}; \sqrt[3]{j_1'}, \sqrt[3]{k_1}, \sqrt[3]{l_1'}\right)$.

Further it is proceeded as below:

$$\tilde{f}_3(\tilde{c})=\min_{0\leq\tilde{x}\leq\tilde{c}}\left\{(x,x,x;x,x,x)+2\left(\left(\frac{j_1}{x}\right)^{\frac{1}{2}},\left(\frac{k_1}{x}\right)^{\frac{1}{2}},\left(\frac{l_1}{x}\right)^{\frac{1}{2}};\left(\frac{j_1'}{x}\right)^{\frac{1}{2}},\left(\frac{k_1}{x}\right)^{\frac{1}{2}},\left(\frac{l_1'}{x}\right)^{\frac{1}{2}}\right)\right\}$$

$$\Rightarrow \tilde{f}_3(\tilde{c})=\min_{0\leq\tilde{x}\leq\tilde{c}}\left\{{j_1}^{\frac{1}{3}}+2\left(\frac{j_1}{{j_1}^{\frac{1}{3}}}\right)^{\frac{1}{2}}, {k_1}^{\frac{1}{3}}+2\left(\frac{k_1}{{k_1}^{\frac{1}{3}}}\right)^{\frac{1}{2}}, {l_1}^{\frac{1}{3}}+2\left(\frac{l_1}{{l_1}^{\frac{1}{3}}}\right)^{\frac{1}{2}}; {j_1'}^{\frac{1}{3}}+2\left(\frac{j_1'}{{j_1'}^{\frac{1}{3}}}\right)^{\frac{1}{2}}, {k_1}^{\frac{1}{3}}+ 2\left(\frac{k_1}{{k_1}^{\frac{1}{3}}}\right)^{\frac{1}{2}}, {l_1'}^{\frac{1}{3}}+2\left(\frac{l_1'}{{l_1'}^{\frac{1}{3}}}\right)^{\frac{1}{2}}\right\}$$

$$\Rightarrow \tilde{f}_3(\tilde{c})=\min_{0\leq\tilde{x}\leq\tilde{c}}\left\{{j_1}^{\frac{1}{3}}+2\left({j_1}^{\frac{2}{3}}\right)^{\frac{1}{2}}, {k_1}^{\frac{1}{3}}+2\left({k_1}^{\frac{2}{3}}\right)^{\frac{1}{2}}, {l_1}^{\frac{1}{3}}+2\left({l_1}^{\frac{2}{3}}\right)^{\frac{1}{2}}; {j_1'}^{\frac{1}{3}}+2\left({j_1'}^{\frac{2}{3}}\right)^{\frac{1}{2}}, {k_1}^{\frac{1}{3}}+ 2\left({k_1}^{\frac{2}{3}}\right)^{\frac{1}{2}}, {l_1'}^{\frac{1}{3}}+2\left({l_1'}^{\frac{2}{3}}\right)^{\frac{1}{2}}\right\}$$

$$\Rightarrow \tilde{f}_3(\tilde{c})=\min_{0\leq\tilde{x}\leq\tilde{c}}\left\{3{j_1}^{\frac{1}{3}}, 3{k_1}^{\frac{1}{3}}, 3{l_1}^{\frac{1}{3}}; 3{j_1'}^{\frac{1}{3}}, 3{k_1}^{\frac{1}{3}}, 3{l_1'}^{\frac{1}{3}}\right\}$$

$$\Rightarrow \tilde{f}_3(\tilde{c})=3\min_{0\leq\tilde{x}\leq\tilde{c}}\left\{{j_1}^{\frac{1}{3}}, {k_1}^{\frac{1}{3}}, {l_1}^{\frac{1}{3}}; {j_1'}^{\frac{1}{3}}, {k_1}^{\frac{1}{3}}, {l_1'}^{\frac{1}{3}}\right\}$$

$$\Rightarrow \tilde{f}_3(\tilde{c})=3\left\{{j_1}^{\frac{1}{3}}, {k_1}^{\frac{1}{3}}, {l_1}^{\frac{1}{3}}; {j_1'}^{\frac{1}{3}}, {k_1}^{\frac{1}{3}}, {l_1'}^{\frac{1}{3}}\right\}$$

The optimal policy is obtained as follows:

$\left[\left({j_1}^{\frac{1}{3}}, {k_1}^{\frac{1}{3}}, {l_1}^{\frac{1}{3}}; {j_1'}^{\frac{1}{3}}, {k_1}^{\frac{1}{3}}, {l_1'}^{\frac{1}{3}}\right), \left({j_1}^{\frac{1}{3}}, {k_1}^{\frac{1}{3}}, {l_1}^{\frac{1}{3}}; {j_1'}^{\frac{1}{3}}, {k_1}^{\frac{1}{3}}, {l_1'}^{\frac{1}{3}}\right), \left({j_1}^{\frac{1}{3}}, {k_1}^{\frac{1}{3}}, {l_1}^{\frac{1}{3}}; {j_1'}^{\frac{1}{3}}, {k_1}^{\frac{1}{3}}, {l_1'}^{\frac{1}{3}}\right)\right]$ and $\tilde{f}_3(\tilde{c})=3\left({j_1}^{\frac{1}{3}}, {k_1}^{\frac{1}{3}}, {l_1}^{\frac{1}{3}}; {j_1'}^{\frac{1}{3}}, {k_1}^{\frac{1}{3}}, {l_1'}^{\frac{1}{3}}\right)$

The process of mathematical induction is applied for the problem to find the optimal policy.

It is assumed that the optimal policy when considering $n=m$ is given by

$$\left[\left(j_1^{\frac{1}{m}}, k_1^{\frac{1}{m}}, l_1^{\frac{1}{m}}; {j_1'}^{\frac{1}{m}}, k_1^{\frac{1}{m}}, {l_1'}^{\frac{1}{m}}\right), \left(j_1^{\frac{1}{m}}, k_1^{\frac{1}{m}}, l_1^{\frac{1}{m}}; {j_1'}^{\frac{1}{m}}, k_1^{\frac{1}{m}}, {l_1'}^{\frac{1}{m}}\right), \ldots \left(j_1^{\frac{1}{m}}, k_1^{\frac{1}{m}}, l_1^{\frac{1}{m}}; {j_1'}^{\frac{1}{m}}, k_1^{\frac{1}{m}}, {l_1'}^{\frac{1}{m}}\right)\right]$$

and $\tilde{f}_m(\tilde{c}) = m\left(j_1^{\frac{1}{m}}, k_1^{\frac{1}{m}}, l_1^{\frac{1}{m}}; {j_1'}^{\frac{1}{m}}, k_1^{\frac{1}{m}}, {l_1'}^{\frac{1}{m}}\right)$

When the value of n is taken as $m+1$, Eq.3 is transformed as

$$\tilde{f}_{m+1}(\tilde{c}) = \min_{0\le\tilde{x}\le\tilde{c}}\left\{(x,x,x;x,x,x) + \tilde{f}_m\left(\frac{j_1}{x},\frac{k_1}{x},\frac{l_1}{x};\frac{j_1'}{x},\frac{k_1}{x},\frac{l_1'}{x}\right)\right\}$$

$$\Rightarrow \tilde{f}_{m+1}(\tilde{c}) = \min_{0\le\tilde{x}\le\tilde{c}}\left\{(x,x,x;x,x,x) + m\left(\left(\frac{j_1}{x}\right)^{\frac{1}{m}},\left(\frac{k_1}{x}\right)^{\frac{1}{m}},\left(\frac{l_1}{x}\right)^{\frac{1}{m}};\left(\frac{j_1'}{x}\right)^{\frac{1}{m}},\left(\frac{k_1}{x}\right)^{\frac{1}{m}},\left(\frac{l_1'}{x}\right)^{\frac{1}{m}}\right)\right\}$$

$$\Rightarrow \tilde{f}_{m+1}(\tilde{c}) = \min_{0\le\tilde{x}\le\tilde{c}}\left\{(x,x,x;x,x,x) + \left(m\left(\frac{j_1}{x}\right)^{\frac{1}{m}},m\left(\frac{k_1}{x}\right)^{\frac{1}{m}},m\left(\frac{l_1}{x}\right)^{\frac{1}{m}};m\left(\frac{j_1'}{x}\right)^{\frac{1}{m}},m\left(\frac{k_1}{x}\right)^{\frac{1}{m}},m\left(\frac{l_1'}{x}\right)^{\frac{1}{m}}\right)\right\}$$

$$\Rightarrow \tilde{f}_{m+1}(\tilde{c}) = \min_{0\le\tilde{x}\le\tilde{c}}\left\{x + m\left(\frac{b_1}{x}\right)^{\frac{1}{m}}, x + m\left(\frac{b_2}{x}\right)^{\frac{1}{m}}, x + m\left(\frac{b_3}{x}\right)^{\frac{1}{m}}, x + m\left(\frac{b_4}{x}\right)^{\frac{1}{m}}\right\}$$

Let $p_1 = x + m\left(\frac{j_1}{x}\right)^{\frac{1}{m}}$. To check whether p_1 is maximum or minimum, it is considered that $\frac{dp_1}{dx} = 0$. The solution is obtained as $x = \sqrt[m+1]{j_1}$

Similarly, if we consider p_2, p_3, p_4, p_5, p_6 as $x + m\left(\frac{k_1}{x}\right)^{\frac{1}{m}}, x + m\left(\frac{l_1}{x}\right)^{\frac{1}{m}}; x + m\left(\frac{j_1'}{x}\right)^{\frac{1}{m}}, x + m\left(\frac{k_1}{x}\right)^{\frac{1}{m}}, x + m\left(\frac{l_1'}{x}\right)^{\frac{1}{m}}$ respectively, then the solutions are obtained as $x = \sqrt[m+1]{k_1}, x = \sqrt[m+1]{l_1}, x = \sqrt[m+1]{j_1'}, x = \sqrt[m+1]{k_1}, x = \sqrt[m+1]{l_1'}$

This function $\left\{x + m\left(\frac{j_1}{x}\right)^{\frac{1}{m}}, x + m\left(\frac{k_1}{x}\right)^{\frac{1}{m}}, x + m\left(\frac{l_1}{x}\right)^{\frac{1}{m}}; x + m\left(\frac{j_1'}{x}\right)^{\frac{1}{m}}, x + m\left(\frac{k_1}{x}\right)^{\frac{1}{m}}, x + m\left(\frac{l_1'}{x}\right)^{\frac{1}{m}}\right\}$ will be minimum only if $\tilde{x} = \left(j_1^{\frac{1}{m+1}}, k_1^{\frac{1}{m+1}}, l_1^{\frac{1}{m+1}}; {j_1'}^{\frac{1}{m+1}}, k_1^{\frac{1}{m+1}}, {l_1'}^{\frac{1}{m+1}}\right)$.

Further it is proceeded as follows:

$$\tilde{f}_{m+1}(\tilde{c}) = \min_{0\le\tilde{x}\le\tilde{c}}\left\{x + m\left(\frac{j_1}{x}\right)^{\frac{1}{m}}, x + m\left(\frac{k_1}{x}\right)^{\frac{1}{m}}, x + m\left(\frac{l_1}{x}\right)^{\frac{1}{m}}; x + m\left(\frac{j_1'}{x}\right)^{\frac{1}{m}}, x + m\left(\frac{k_1}{x}\right)^{\frac{1}{m}}, x + m\left(\frac{l_1'}{x}\right)^{\frac{1}{m}}\right\}$$

$$\Rightarrow \tilde{f}_{m+1}(\tilde{c}) = \min_{0\le\tilde{x}\le\tilde{c}}\left\{j_1^{\frac{1}{m+1}} + m\left(\frac{j_1}{j_1^{\frac{1}{m+1}}}\right)^{\frac{1}{m}}, k_1^{\frac{1}{m+1}} + m\left(\frac{k_1}{k_1^{\frac{1}{m+1}}}\right)^{\frac{1}{m}}, l_1^{\frac{1}{m+1}} + \right.$$

$$\left. m\left(\frac{l_1}{l_1^{\frac{1}{m+1}}}\right)^{\frac{1}{m}}; j_1'^{\frac{1}{m+1}} + m\left(\frac{j_1'}{j_1'^{\frac{1}{m+1}}}\right)^{\frac{1}{m}}, k_1^{\frac{1}{m+1}} + m\left(\frac{k_1}{k_1^{\frac{1}{m+1}}}\right)^{\frac{1}{m}}, l_1'^{\frac{1}{m+1}} + m\left(\frac{l_1'}{l_1'^{\frac{1}{m+1}}}\right)^{\frac{1}{m}}\right\}$$

$$\Rightarrow \tilde{f}_{m+1}(\tilde{c}) = \min_{0\le\tilde{x}\le\tilde{c}}\left\{j_1^{\frac{1}{m+1}} + m\left(j_1^{\frac{m}{m+1}}\right)^{\frac{1}{m}}, k_1^{\frac{1}{m+1}} + m\left(k_1^{\frac{m}{m+1}}\right)^{\frac{1}{m}}, l_1^{\frac{1}{m+1}} + \right.$$

$$\left. m\left(l_1^{\frac{m}{m+1}}\right)^{\frac{1}{m}}; j_1'^{\frac{1}{m+1}} + m\left(j_1'^{\frac{m}{m+1}}\right)^{\frac{1}{m}}, k_1^{\frac{1}{m+1}} + m\left(k_1^{\frac{m}{m+1}}\right)^{\frac{1}{m}}, l_1'^{\frac{1}{m+1}} + m\left(l_1'^{\frac{m}{m+1}}\right)^{\frac{1}{m}}\right\}$$

$$\Rightarrow \tilde{f}_{m+1}(\tilde{c}) = \min_{0\le\tilde{x}\le\tilde{c}}\left\{(m+1)j_1^{\frac{1}{m+1}}, (m+1)k_1^{\frac{1}{m+1}}, (m+1)l_1^{\frac{1}{m+1}}; (m+1)j_1'^{\frac{1}{m+1}}, (m+\right.$$

$$\left.1)k_1^{\frac{1}{m+1}}, (m+1)l_1'^{\frac{1}{m+1}}\right\} \quad \text{Since } m+1>0$$

$$\Rightarrow \tilde{f}_{m+1}(\tilde{c}) = (m+1)\left\{j_1^{\frac{1}{m+1}}, k_1^{\frac{1}{m+1}}, l_1^{\frac{1}{m+1}}; j_1'^{\frac{1}{m+1}}, k_1^{\frac{1}{m+1}}, l_1'^{\frac{1}{m+1}}\right\}$$

i.e., It is observed that the result generated is true when n takes the value $m+1$.

Hence, the optimal policy is obtained by the process of mathematical induction.

$$\left[\left(j_1^{\frac{1}{n}}, k_1^{\frac{1}{n}}, l_1^{\frac{1}{n}}; j_1'^{\frac{1}{n}}, k_1^{\frac{1}{n}}, l_1'^{\frac{1}{n}}\right), \left(j_1^{\frac{1}{n}}, k_1^{\frac{1}{n}}, l_1^{\frac{1}{n}}; j_1'^{\frac{1}{n}}, k_1^{\frac{1}{n}}, l_1'^{\frac{1}{n}}\right), \dots \left(j_1^{\frac{1}{n}}, k_1^{\frac{1}{n}}, l_1^{\frac{1}{n}}; j_1'^{\frac{1}{n}}, k_1^{\frac{1}{n}}, l_1'^{\frac{1}{n}}\right)\right]$$

and $\tilde{f}_n(\tilde{c}) = n\left(j_1^{\frac{1}{n}}, k_1^{\frac{1}{n}}, l_1^{\frac{1}{n}}; j_1'^{\frac{1}{n}}, k_1^{\frac{1}{n}}, l_1'^{\frac{1}{n}}\right)$.

4. Results

It is observed that the result generated is true when n takes the value $m+1$. Hence, the optimal policy is obtained by the process of mathematical induction.

$$\left[\left(j_1^{\frac{1}{n}}, k_1^{\frac{1}{n}}, l_1^{\frac{1}{n}}; j_1'^{\frac{1}{n}}, k_1^{\frac{1}{n}}, l_1'^{\frac{1}{n}}\right), \left(j_1^{\frac{1}{n}}, k_1^{\frac{1}{n}}, l_1^{\frac{1}{n}}; j_1'^{\frac{1}{n}}, k_1^{\frac{1}{n}}, l_1'^{\frac{1}{n}}\right), \dots \left(j_1^{\frac{1}{n}}, k_1^{\frac{1}{n}}, l_1^{\frac{1}{n}}; j_1'^{\frac{1}{n}}, k_1^{\frac{1}{n}}, l_1'^{\frac{1}{n}}\right)\right]$$

and $\tilde{f}_n(\tilde{c}) = n\left(j_1^{\frac{1}{n}}, k_1^{\frac{1}{n}}, l_1^{\frac{1}{n}}; j_1'^{\frac{1}{n}}, k_1^{\frac{1}{n}}, l_1'^{\frac{1}{n}}\right)$.

5. Discussion

A new approach has been proposed in this paper for solving fuzzy optimal subdivision problem in which the positive quantity which is to be divided into ‘n’ factors is taken as trapezoidal fuzzy number. The solution is obtained by the method of Mathematical Induction. The optimal solution is obtained by using the fuzzy recursive equations. Many real-world problems involve sequential or multistage decision making. Sometimes, the parameters may not be known precisely due to some uncontrollable factors. If the obtained results are crisp values then it might lose some helpful information. Fuzzy Dynamic Programming is a powerful optimization procedure that is particularly applicable to many complex problems requiring a sequence of interrelated decisions in a fuzzy environment and hence it has wide range of applications in the future.

Refrences

[1] Zadeh, L.A.: Fuzzy sets, Information and control, 8 (1965), 338 – 353.

[2] Bellman, R.E., Zadeh, L.A.: Decision – making in a fuzzy environment, Management Science, 17 (1970), B141 – B164.

[3] Kacprzyk, J. (2013) 'Fuzzy dynamic programming: interpolative reasoning for an efficient derivation of optimal control policies', Control and Cybernetics, Vol.42, No.1, pp.63–84.

[4] Phu, N.D., Tri, P.V. (2016) 'The fuzzy dynamic programming problems', Journal of Intelligent & Fuzzy Systems, Vol.30, pp.1663–1674.

[5] Nagalakshmi, T., Uthra, G. (2017) 'An approach of finding an optimal solution for a Fuzzy Least cost route problem by using Generalized Trapezoidal Fuzzy Numbers', Advances in Fuzzy Mathematics, Vol.12, No.3, pp.737-745.

[6] Kaliyaperumal, P. (2017) 'Fuzzy Dynamic Programming Problem for Single Additive Constraint with Additively Separable Return by Means of Trapezoidal Membership Functions', In A. Sangaiah, X. Gao, & A. Abraham (Eds.), Handbook of Research on Fuzzy and Rough Set Theory in Organizational Decision Making, IGI Global, pp. 168-187. https://doi.org/10.4018/978-1-5225-1008-6.ch008

[7] Mohanaselvi, S., Suparna, S.M. (2019) 'A Fuzzy Dynamic Programming approach to Fuzzy Least Cost Route Problem', Journal of Physics: Conference Series, Vol.1377, No.012042, IOP Publishing. doi:10.1088/1742-6596/1377/1/012042

[8] Krishnaveni,G., Ganesan, K. (2019) 'Fuzzy dynamic programming technique for the solution of fuzzy travelling salesman problems', AIP Conference Proceedings, Vol.2112, No.020150. https://doi.org/10.1063/1.5112335.

[9] Zhu, Y. (2023) 'Some Conclusions of an Optimal Subdivision Problem', Journal of Physics: Conference Series, IOP Publishing, Vol.2470, No.012011, doi:10.1088/1742-6596/2470/1/012011.

[10] Druti, M., Smita, V. (2023) 'Fuzzy Dynamic Programming to Determine the Shortest Route Problem', Turkish Journal of Computer and Mathematics Education , Vol.14, No.2, pp.23 – 28.

[11] Khan, I.U., Aftab, M. (2023) 'Adaptive fuzzy dynamic programming (AFDP) technique for linear programming problems lps with fuzzy constraints', Soft Computing,Vol.27, pp.13931–13949.

[12] Nagalakshmi, T. (2023) ‘A neutrosophic approach in the optimal solution of a neutrosophic fuzzy capital budgeting problem’, Int. J. Mathematics in Operational Research, Vol. 24, No. 4, 491–509.