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Plithogenic Hesitant Fuzzy Soft Sets

J . Jayasudha [1], R . Nishanthi [2] *

[1]Associate Professor, Department of Mathematics, Nallamuthu Gounder Mahalingam College,

Pollachi-642 001, Tamilnadu, India. E-mail : *jayasudhaj@ngmc.org*

[2] *Assistant Professor, Department of Mathematics, Dr.Mahalingam college of Engineering and Technology, Pollachi-642 003, Tamilnadu, India. E-mail : *raghunisha94@gmail.com*

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Abstract : Plithogenic sets are used extensively in decision-making processes because of their versatility. In this paper we introduced the concept of Plithogenic Hesitant Fuzzy Soft Sets (P-HFSSs) .The paper also discusses other fundamental operations like union, intersection, compliment and some set theoretical operations on Plithogenic Hesitant Fuzzy Soft Sets. Further it investigates a similarity measure between two P-HFSSs.

Keywords : Fuzzy soft set, Hesitant Fuzzy Soft Set, Plithogenic Fuzzy Soft Set, Plithogenic Hesitant Fuzzy Soft Set.

1. Introduction

The notion of fuzzy sets has developed in numerous fields and in numerous ways since its founding in 1965. This theory finds application in a variety of fields, including robotics, artificial intelligence, computer science, medicine, control engineering, decision theory, expert systems, logic, management science, and operations research. The notion of fuzzy sets introduced by Zadeh [1] in 1965, deal with possibilistic uncertainity and it established a place for itself in decision-making issues. Molodtsov introduced the theory of soft sets in 1999 [4]. Maji et al. introduced many operations on soft sets and explored theoretical features of soft sets in [6]. Soft set theory has shown to be helpful in a variety of contexts, including decision-making. The traditional fuzzy sets are characterized by the membership value or the grade of membership value. Maji.al [5] articulated the notion of fuzzy soft sets through the embedding of fuzzy set concepts. Some academics have developed various applications of soft set theory by utilizing this concept of fuzzy soft sets. At times, determining the membership value for fuzzy sets can be exceedingly challenging. As a result, the idea of interval valued fuzzy sets were proposed [2] in order to account for the degree of membership value.

In certain real-world scenarios involving expert systems, belief systems, information fusion, and other domains, we have to consider both the truth-membership and the falsity-membership in order to appropriately represent an item in an ambiguous, uncertain environment. In this case, neither the interval valued fuzzy sets nor the fuzzy sets are appropriate. For this kind of scenario, Atanassov's intuitionistic fuzzy sets [3] make sense. Only incomplete data taking into account both truth-membership (or simply membership) and falsity-membership (or non-membership) values can be handled by intuitionistic fuzzy sets. It does not deal with the ambiguous and contradictory data found in belief systems. The neutrosophic set is a mathematical tool that Smarandache [7] proposed for resolving situations involving inconsistent, imprecise, and indeterminate data. It is a generalization of fuzzy set and intuitionistic fuzzy set. Neutrosopic sets deal with three components such as membership, indeterminacy and non-membership.

An established method for the formal modeling of uncertain data is Hesitant Fuzzy Set Theory. The fact that it has been widely applied to multi-attribute decision-making situations is proof of its effectiveness. The trustworthiness of the decisions that rely on the decision-makers' judgments is enhanced by hesitant fuzzy computations, which provide them greater flexibility and richness. Torra [9] and Torra and Narukawa [8] were the first to offer reluctant fuzzy sets as a generalization of fuzzy sets, allowing the membership to have a set of alternative values. They demonstrated that the envelope of a hesitant fuzzy set is an intuitionistic fuzzy set and talked about the link between hesitant and intuitionistic fuzzy sets. In order to use it in ambiguous scenarios and decision-making, they established the extension concept. Since then, hesitant fuzzy set theory has been used to tackle a number of real-world issues, mostly related to making decisions. A set of aggregation procedures for reluctant fuzzy information was created by Xia and Xu [10] and applied to multi-criteria decision-making issues in an anonymous manner. Using quasi-arithmetic, some ordered aggregation operators and induced ordered aggregation operators were explored by Xia et al. [11]. These operators were used by the authors in group decision-making. A range of distance measures for hesitant fuzzy sets were introduced by Xu and Xia [12, 13], who also examined the features and relationships of these measures as their parameters changed.

Due to real-world issues, ambiguity exists in the majority of decision-making processes. The literature has presented a variety of set theories to address this uncertainty. Among these, the Plithonegic set is the most recent. Smarandache [14] introduced the idea of plithogeny, which is a generalization of logic and neutrosophic sets. The two primary components of the plithogenic set - the appurtenance degrees and contradiction-help to increase the accuracy of the outcomes in an unpredictable setting.

A Plithogenic set, denoted as P , is a set whose components are characterized by one or more attributes, each of which may have many values, according to a definition provided by Smarandache in 2017. Plithogenic sets are a generalization of crisp, fuzzy, intuitionistic, and neutrosophic sets. This is because all four types of sets have a single attribute value (appurtenance), with membership being the only value for crisp and fuzzy sets, non-membership and membership being the values for intuitionistic fuzzy sets, and indeterminacy, membership, and non-membership, and membership being the values for neutrosophic sets. In general, the sets with four or more such qualities can be described as plithogenic sets.

In this paper we introduce a new form of sets namely Plithogenic hesitant fuzzy soft set.

2 Preliminaries

In this section, we recall some basic definitions required to read this paper.

Definition 2.1 [1]

A fuzzy set A in the universal set X is defined as $A = \{x, \mu_{A(x)}, x \in X\}$. Here $\mu_{A(x)}: A \rightarrow [0,1]$ is the grade of the membership function and $\mu_{A(x)}$ is the grade value of $x \in X$ in the fuzzy set A .

Definition 2.2 [4]

A pair (F, E) is called a soft set (over U) if and only if F is a mapping of E into the set of all subsets of the set U. In other words, the soft set is a parameterized family of subsets of the set U. Every set $F(\varepsilon), \varepsilon \in E$, from this family may be considered as the set of ε - elements of the soft set (F, E), or as the set of ε - approximate elements of the soft set.

Definition 2.3 [5]

A fuzzy soft set over U is a pair (F, A), where $F: A \rightarrow P(U)$ is a mapping from A into P(U).

Definition 2.4 [7]

A neutrosophic set A on the universe of discourse X is defined as $A = \{< x, T_A(x), I_A(x), F_A(x) >, x \in X\}$, where $T, I, F: X \rightarrow [0,1]$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2.5 [9]

If X is a fixed set, then a Hesitant Fuzzy Set (HFS) on X is defined as follows: $h_M(x)$ is a function that, when applied to X , returns a subset of [0,1]. The following mathematical expression can be used to express this: M is defined as $\{< x, h_M(x) > | \; x \in X\}$, where $h_M(x)$ is a set of values in [0,1] that represent the various membership degrees of the element $x \in X$ to the set M. We refer to h=$h_M(x)$ as a hesitant fuzzy element (HFE) for simplicity, and H is the set of all HFEs.

Definition 2.6 [13]

Let U be a universal set and E be set of parameters. Let HF(U) denotes the set of all hesitant fuzzy sets defined over U. A pair (F,E) is a hesitant fuzzy soft set if $F(e) \in HF(U)$ for every e in E.

Definition 2.7 [14]

Let U be the discourse universe and P be the non-empty set of elements, such that $P \subseteq U$. Then, a is an attribute (generally multidimensional), V is the attribute values range, d is the degree to which each element x's attribute value belongs to the set P in relation to a given set of criteria ($x \in P$), and c is the degree of contradiction between attribute values. Hence, the set (P, a, V, d, c) is said to as plithogenic.

Definition 2.8 [15]

Assume that U is a discourse universe and that $(\mathrm{U})^{\mathrm{Z}}$ is the Z-power set of U such that

$\mathrm{Z}=\mathrm{C}$ (Power set of U),

$\mathrm{Z}=\mathrm{F}$ (The set of all fuzzy set of U),

$\mathrm{Z}=\mathrm{IF}$ (The set of all intuitionistic fuzzy set of U), or

$\mathrm{Z}=\mathrm{N}$ (The set of all neutrosophic set of U).

Let $a_1, a_2, \cdots, a_n$, for $\mathrm{n} \geq 1$, be n distinct attributes. The corresponding attribute values for each of these sets are $\mathrm{V}_1, \mathrm{V}_2, \cdots, \mathrm{V}_{\mathrm{n}}$, with $\mathrm{Vi} \cap \mathrm{Vj}=\emptyset$, for $j \neq i$, and $i, j \in\{1,2, \cdots, n.\ \}$. Suppose $V_i=\left\{\mathrm{v}_{\mathrm{i}_1}, \mathrm{v}_{\mathrm{i}_2}, \cdots, \mathrm{v}_{\mathrm{i}_{\mathrm{mi}}}\right\}$ and let $\Upsilon=\mathrm{V}_1 \times \mathrm{V}_2 \times \cdots \times \mathrm{V}_{\mathrm{n}}$.Let $\mathrm{c}\left(\mathrm{D}_{\mathrm{i}}, \mathrm{v}_{\mathrm{ij}}\right), \mathrm{i} \in \{1,2, \cdots \mathrm{n}\}, \mathrm{j} \in \{1,2, \cdots m_i\}$ be the attribute value contradiction degree function such that $\mathrm{c}_{\mathrm{i}}: \mathrm{V}_{\mathrm{i}} \times \mathrm{V}_{\mathrm{i}} \rightarrow[0,1]$, and let $\mathrm{D}=\left(\mathrm{D}_1, \mathrm{D}_2, \cdots \mathrm{D}_{\mathrm{n}}\right)$ the dominant attribute element of $\mathrm{A}_{\mathrm{i}}, \forall \mathrm{i}$. Then the pair $\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{Z}}, \Upsilon\right)$, where : $\mathrm{F}_{\mathrm{P}}^{\mathrm{Z}}: \Upsilon \rightarrow[0,1]_{\mathrm{D}} \times(\mathrm{U})^{\mathrm{Z}}$ is called a Plithogenic Soft Set (P-SS In short) over U.

Definition 2.9 [15]

The union of two P-SSs $\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{Z}}, \mathrm{A}\right)$ and $\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{Z}}, \mathrm{B}\right)$ over U , denoted by

$\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{Z}}, \mathrm{A}\right) \vee\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{Z}}, \mathrm{B}\right)$, is the $\mathrm{P}-\mathrm{SS}\left(\mathrm{H}_{\mathrm{P}}^{\mathrm{Z}}, \Omega\right)$ where $\Omega=\mathrm{A} \cup \mathrm{B}$, and $\forall \varepsilon \in \Omega$,

$$H_P^Z(\varepsilon)=\begin{cases}F_P^Z(\varepsilon) & \text{if } \varepsilon \in A-B \\ G_P^Z(\varepsilon) & \text{if } \varepsilon \in B-A \\ F_P^Z(\varepsilon) \vee_{\mathrm{P}}^{\mathrm{H}} G_P^Z(\varepsilon) & \text{if } \varepsilon \in A \cap B\end{cases}$$

where $\vee_{\mathrm{Z}}$ is a $(\mathrm{Z}=\mathrm{F})$-plithogenic union.

Definition 2.10 [15]

The intersection of two P-SSs $\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{Z}}, \mathrm{A}\right)$ and $\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{Z}}, \mathrm{B}\right)$ over U , denoted by

$\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{Z}}, \mathrm{A}\right) \wedge\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{Z}}, \mathrm{B}\right)$, is the $\mathrm{P}-\mathrm{SS}\left(\mathrm{H}_{\mathrm{P}}^{\mathrm{Z}}, \Omega\right)$ where $\Omega=\mathrm{A} \cup \mathrm{B}$, and $\forall \varepsilon \in \Omega$,

$$H_P^Z(\varepsilon)=\begin{cases}F_P^Z(\varepsilon) & \text{if } \varepsilon \in A-B \\ G_P^Z(\varepsilon) & \text{if } \varepsilon \in B-A \\ F_P^Z(\varepsilon) \wedge_{\mathrm{P}}^{\mathrm{H}} G_P^Z(\varepsilon) & \text{if } \varepsilon \in A \cap B\end{cases}$$

where $\wedge_{\mathrm{Z}}$ is a $(\mathrm{Z}=\mathrm{F})$-plithogenic intersection.

Definition 2.11 [15]

Let $\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{F}}, \mathrm{E}\right)$ and $\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{F}}, \mathrm{E}\right)$ be two P-FSSs over U as in Definition 2.8.

Similarity between ($\mathrm{F}_{\mathrm{P}}^{\mathrm{F}}, \mathrm{E}$) and ($\mathrm{G}_{\mathrm{P}}^{\mathrm{F}}, \mathrm{E}$) ,denoted by $\mathrm{S}\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{F}}, \mathrm{G}_{\mathrm{P}}^{\mathrm{F}}\right)$ is defined as

$S(F_P^F, G_P^F) = \frac{1}{|E|}\sum_{k=1}^{|E|} M_k$, where $M_k = 1 - \frac{\sum_{j=1}^{|U|}\sum_{i=1}^{|\varepsilon|}\left|(F_j(e_{ik})) - (G_j(e_{ik}))\right|}{\sum_{j=1}^{|U|}\sum_{i=1}^{|\varepsilon|}\left|(F_j(e_{ik})) + (G_j(e_{ik}))\right|}$, $\varepsilon \in E$.

3 Plithogenic Hesitant Fuzzy Soft Sets

In this section we define the Plithogenic hesitant fuzzy soft set. We also, define its basic operations namely, union, intersection, compliment and discuss some of their properties.

Definition 3.1

Let U be a universe of discourse, $(U)^H$ the power set of U such that H is the set of all hesitant fuzzy soft sets. Let $a_1, a_2, \cdots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $V_1, V_2, \cdots, V_n$, with $V_i \cap Vj = \emptyset$, for $j \neq i$, and $i, j \in \{1,2, \cdots, n\}$. Suppose $V_i = \{v_{i_1}, v_{i_2}, \cdots, v_{i_{mi}}\}$ and let $\mathcal{H} = V_1 \times V_2 \times \cdots \times V_n$.Let $D = (D_1, D_2, \cdots D_n)$ the dominant attribute element of $V_i, \forall i$ and $c(D_i, v_{ij}), i \in \{1,2, \cdots n\}, j \in \{1,2, \cdots m_i\}$ the attribute value contradiction degree function such that $c_i: V_i \times V_i \rightarrow [0,1]$. Then the pair $(F_P^H, \mathcal{H})$, where : $F_P^H: \mathcal{H} \rightarrow [0,1]_D \times (U)^H$ is called a Plithogenic hesitant fuzzy soft set (P-HFSS In short) over U.

Example 3.2

The situation is to select a bike based on the attributes speed, color, model, and price and each attribute has multiple attribute values with multiple memberships. To simplify the explanation, let U represents the set of bikes where $U = \{B_1, B_2, B_3\}$. Let the distinct attributes a_1 = speed, a_2 = color, a_3 = model, a_4 = price and their corresponding attribute values are : speed $\equiv A_1$ = { slow, medium, fast }, color $\equiv A_2$ = { white, yellow, red, black }, model $\equiv A_3$ = { model1, model 2 , model 3 , model 4}, price $\equiv A_4$ = {low, medium, costly }, where D = (medium, black, model 3, costly) denotes the dominant attribute values.

Let $E \subseteq \mathcal{H}$ such that E = {ε_1 = (slow, white, model 1, low), ε_2 = (medium, yellow, model 2, low), ε_3 = (fast, red, model 4, costly) }.

Define a function $F_P^H: E \rightarrow [0,1]_D X(U)^H$ as follows :

$$F_P^H(\varepsilon_1) = \left\{\frac{(B_1,(0,0.2,0.4,1)_D)}{\{(0.6,0.2),(0.3,0.4,0.5),(0.5,0.7),(0.2,0.4,0.6)\}}, \frac{(B_2,(0,0.2,0.4,1)_D)}{\{(0.5,0.2),(0.2,1),(0.6,0.7),(0.6,0.7)\}}, \frac{(B_3,(0,0.2,0.4,1)_D)}{\{(0.6,0.7),(0.4,1),(0.6,0.8),(0.6,0.7)\}}\right\}$$

$$F_P^H(\varepsilon_2) = \left\{\frac{(B_1,(0.5,0,1,1)_D)}{\{(0.2,0.3),(0.4,0.2),(0.6,0.7,0.8),(0.5,0.4)\}}, \frac{(B_2,(0.5,0,1,1)_D)}{\{(0.6,0.3),(0.2,0.1),(0.4,0.6),(0.3,0.5)\}}, \frac{(B_3,(0.5,0,1,1)_D)}{\{(0.5,0.7),(0.7,0.9),(0,1,0.1),(0,1)\}}\right\}$$

$$F_P^H(\varepsilon_3) = \left\{\frac{(B_1,(0,0.5,1,1)_D)}{\{(0.1,0.2),(0,3,0.4),(0.6,0.7,0.9),(0,1)\}}, \frac{(B_2,(0,0.5,1,1)_D)}{\{(0.6,0.7),(0.8,0.9),(0,1,0.2),(0.3,0.5)\}}, \frac{(B_3,(0,0.5,1,1)_D)}{\{(0.1,0.2),(0.5,0.4),(0.6,0.7,0.9),(0.1,1)\}}\right\}$$

Then we have the P- HFSS as follows :

$$(F_P^H, E) = \left\{\varepsilon_1, \left\{\frac{(B_1,(0,0.2,0.4,1)_D)}{\{(0.6,0.2),(0.3,0.4,0.5),(0.5,0.7),(0.2,0.4,0.6)\}}, \frac{(B_2,(0,0.2,0.4,1)_D)}{\{(0.5,0.2),(0.2,1),(0.6,0.7),(0.6,0.7)\}}, \frac{(B_3,(0,0.2,0.4,1)_D)}{\{(0.6,0.7),(0.4,1),(0.6,0.8),(0.6,0.7)\}}\right\}\right.$$

$$\varepsilon_2, \left\{\frac{(B_1,(0.5,0,1,1)_D)}{\{(0.2,0.3),(0.4,0.2),(0.6,0.7,0.8),(0.5,0.4)\}}, \frac{(B_2,(0.5,0,1,1)_D)}{\{(0.6,0.3),(0.2,0.1),(0.4,0.6),(0.3,0.5)\}}, \frac{(B_3,(0.5,0,1,1)_D)}{\{(0.5,0.7),(0.7,0.9),(0,1,0.1),(0,1)\}}\right\},$$

$$\varepsilon_3, \left\{\frac{(B_1,(0,0.5,1,1)_D)}{\{(0.1,0.2),(0,3,0.4),(0.6,0.7,0.9),(0,1)\}}, \frac{(B_2,(0,0.5,1,1)_D)}{\{(0.6,0.7),(0.8,0.9),(0,1,0.2),(0.3,0.5)\}}, \frac{(B_3,(0,0.5,1,1)_D)}{\{(0.1,0.2),(0.5,0.4),(0.6,0.7,0.9),(0.1,1)\}}\right\}\Bigg\}$$

Definition 3.3

The union of two P-HFSSs (F_P^H, A) and (G_P^H, B) over U , denoted by $(F_P^H, A) \vee_P^H (G_P^H, B)$, is the P-HFSS (H_P^H, Ω) , where $\Omega = A \cup B$, and $\forall \varepsilon \in \Omega$,

$$H_P^H(\varepsilon) = \begin{cases} F_P^H(\varepsilon) & \text{if} \quad \varepsilon \in A - B \\ G_P^H(\varepsilon) & \text{if} \quad \varepsilon \in B - A \\ F_P^H(\varepsilon) \vee_P^H G_P^H(\varepsilon) & \text{if} \quad \varepsilon \in A \cap B \end{cases}$$

where $\vee_P^H$ is a plithogenic hesitant union.

Example 3.4

Consider Example 3.2. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$ where $a_1 =$ (slow, white, model 1, low), $a_2 =$ (medium, yellow, model 2 , low), $a_3 =$ (fast, red, model 4 , costly), $b_1 =$ (slow, white, model 1, low), $b_2 =$ (medium, yellow, model 2, low), $b_3 =$ (fast, red, model 4, costly) , $b_4 =$ (fast, black, model 2, medium).

Suppose (F_P^H, A) and (G_P^H, B) are two P - HFSSs over U such that

$(F_P^H, A) =$

$$\Bigg\{a_1, \left\{\frac{(B_1,(0,0.2,0.4,1)_D)}{\{(0.6,0.2),(0.3,0.4,0.5),(0.5,0.7),(0.2,0.4,0.6)\}}, \frac{(B_2,(0,0.2,0.4,1)_D)}{\{(0.5,0.2),(0.2,1),(0.6,0.7),(0.6,0.7)\}}, \frac{(B_3,(0,0.2,0.4,1)_D)}{\{(0.6,0.7),(0.4,1),(0.6,0.8),(0.6,0.7)\}}\right\},$$

$$a_2, \left\{\frac{(B_1,(0.5,0,1,1)_D)}{\{(0.2,0.3),(0.4,0.2),(0.6,0.7,0.8),(0.5,0.4)\}}, \frac{(B_2,(0.5,0,1,1)_D)}{\{(0.6,0.3),(0.2,0.1),(0.4,0.6),(0.3,0.5)\}}, \frac{(B_3,(0.5,0,1,1)_D)}{\{(0.5,0.7),(0.7,0.9),(0,1,0.1),(0,1)\}}\right\}$$

$$a_3, \left\{\frac{(B_1,(0,0.5,1,1)_D)}{\{(0.1,0.2),(0,3,0.4),(0.6,0.7,0.9),(0,1)\}}, \frac{(B_2,(0,0.5,1,1)_D)}{\{(0.6,0.7),(0.8,0.9),(0,1,0.2),(0.3,0.5)\}}, \frac{(B_3,(0,0.5,1,1)_D)}{\{(0.1,0.2),(0.5,0.4),(0.6,0.7,0.9),(0.1,1)\}}\right\}\Bigg\}$$

$(G_P^H, B) =$

$$\Bigg\{b_1, \left\{\frac{(B_1,(0,0.2,0.4,1)_D)}{\{(0.2,0.4),(0.8,0.9,1),(0.6,0.8),(0.7,0.9,1)\}}, \frac{(B_2,(0,0.2,0.4,1)_D)}{\{(0,1),(0.8,0.9),(0.5,0.8),(0.9,1)\}}, \frac{(B_3,(0,0.2,0.4,1)_D)}{\{(0.6,0.8),(0.7,0.9),(0.5,1),(02,0.8)\}}\right\},$$

$$b_2, \left\{\frac{(B_1,(0.5,0,1,1)_D)}{\{(0.5,0.8),(0.2,0.3),(0.5,0.6,0.7),(0.3,0.5)\}}, \frac{(B_2,(0.5,0,1,1)_D)}{\{(0.2,0.3),(0.3,0.5),(0.4,0.6),(0,0.2)\}}, \frac{(B_3,(0.5,0,1,1)_D)}{\{(0.2,0.4),(0.5,0.6),(0.4,0.5,0.7),(0.2,0.3)\}}\right\}$$

$$b_3, \left\{\frac{(B_1,(0,0.5,1,1)_D)}{\{(0.5,0.6),(0,2,0.3),(0,0.5,1),(0.6,0.7)\}}, \frac{(B_2,(0,0.5,1,1)_D)}{\{(0.7,0.9),(0.2,0.1),(0,6,0.7),(0.2,0.6)\}}, \frac{(B_3,(0,0.5,1,1)_D)}{\{(0.7,0.9),(0,0.2),(0.5,0.0,0.6),(0.3,0.6)\}}\right\}$$

$$b_4, \left\{\frac{(B_1,(0.2,0.5,1,1)_D)}{\{(0.5,0.6),(0,2,0.4),(0.5,1),(0.4,0.8)\}}, \frac{(B_2,(0.2,0.5,1,1)_D)}{\{(0.9,1),(0.3,0.4),(0.4,0.5),(0.3,0.4,0.6)\}}, \frac{(B_3,(0.2,0.5,1,1)_D)}{\{(0.3,0.4,0.6),(0.5,0.7,0.9),(0.8,0.6),(0.3,0.6)\}}\right\}\Bigg\}$$

By using fuzzy union(maximum) we have :

$(F_P^H, A) \vee_P^H (G_P^H, B) = (H_P^H, C)$, where

$$\left(\mathrm{H}_{\mathrm{P}}^{\mathrm{H}},\mathrm{C}\right)=\left\{\mathrm{c}_1,\left\{\frac{(\mathrm{B}_1,(0,0.2,0.4,1)_{\mathrm{D}})}{\{(0.6,0.2),(0.7,0.8,0.9),(0.56,0.76),(0.2,0.4,0.6)\}},\frac{(\mathrm{B}_2,(0,0.2,0.4,1)_{\mathrm{D}})}{\{(0,1),(0.32,0.98),(0.54,0.76),(0.6,0.7)\}},\frac{(\mathrm{B}_3,(0,0.2,0.4,1)_{\mathrm{D}})}{\{(0.6,0.8),(0.46,0.98),(0.54,0.92),(02,0.8)\}}\right\},\right.$$

$$\mathrm{c}_2,\left\{\frac{(\mathrm{B}_1,(0.5,0,1,1)_{\mathrm{D}})}{\{(0.35,0.55),(0.56,0.48),(0.5,0.6,0.7),(0.3,0.5)\}},\frac{(\mathrm{B}_2,(0.5,0,1,1)_{\mathrm{D}})}{\{(0.4,0.3),(0.47,0.55),(0.4,0.6),(0,0.2)\}},\frac{(\mathrm{B}_3,(0.5,0,1,1)_{\mathrm{D}})}{\{(0.35,0.46),(0.68,0.87),(0.4,0.5,0.7),(0.2,0.3)\}}\right\},$$

$$\mathrm{c}_3,\left\{\frac{(\mathrm{B}_1,(0,0.5,1,1)_{\mathrm{D}})}{\{(0.5,0.6),(0,3,0.4),(0.6,0.7,0.9),(0.6,0.7)\}},\frac{(\mathrm{B}_2,(0,0.5,1,1)_{\mathrm{D}})}{\{(0.7,0.9),(0.5,0.45),(0,1,0.2),(0.3,0.5)\}},\frac{(\mathrm{B}_3,(0,0.5,1,1)_{\mathrm{D}})}{\{(0.7,0.9),(0.25,0.3),(0.5,0.6,0.8),(0.3,0.6)\}}\right\},$$

$$\left.\mathrm{c}_4,\left\{\frac{(\mathrm{B}_1,(0.2,0.5,1,1)_{\mathrm{D}})}{\{(0.5,0.6),(0,2,0.4),(0.5,1),(0.4,0.8)\}},\frac{(\mathrm{B}_2,(0.2,0.5,1,1)_{\mathrm{D}})}{\{(0.9,1),(0.3,0.4),(0.4,0.5),(0.3,0.4,0.6)\}},\frac{(\mathrm{B}_3,(0.2,0.5,1,1)_{\mathrm{D}})}{\{(0.3,0.4,0.6),(0.5,0.7,0.9),(0.8,0.6),(0.3,0.6)\}}\right\}\right\}$$

Definition 3.5

The intersection of two P-HFSSs $\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)$ and $\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}\right)$ over U , denoted by $\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)\wedge_{\mathrm{P}}^{\mathrm{H}}\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}\right)$, is the P-HFSS $\left(\mathrm{H}_{\mathrm{P}}^{\mathrm{H}},\Omega\right)$ where $\Omega=\mathrm{A}\cup\mathrm{B}$, and $\forall\varepsilon\in\Omega$,

$$H_P^H(\varepsilon)=\begin{cases}F_P^H(\varepsilon) & if \quad \varepsilon\in A-B\\ G_P^H(\varepsilon) & if \quad \varepsilon\in B-A\\ F_P^H(\varepsilon)\wedge_P^H G_P^H(\varepsilon) & if \quad \varepsilon\in A\cap B\end{cases}$$

where $\wedge_{\mathrm{P}}^{\mathrm{H}}$ is a plithogenic hesitant intersection.

Definition 3.6

The compliment of P-HFSS $\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{E}\right)$ denoted by $\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{E}\right)^{\mathrm{C}}$ is defined as

$$\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{E}\right)^{\mathrm{C}}=\left(1-\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{E}\right)\right).$$

Example 3.7

Reconsider Example 3.2, the $\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{E}\right)^{\mathrm{C}}$ can be calculated as follows:

$$\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{E}\right)^{\mathrm{C}}=$$

$$\left\{\varepsilon_1,\left\{\frac{(\mathrm{B}_1,(0,0.2,0.4,1)_{\mathrm{D}})}{\{(0.8,0.4),(0.5,0.6,0.7),(0.3,0.5),(0.4,0.6,0.8)\}}\right\}\frac{(\mathrm{B}_2,(0,0.2,0.4,1)_{\mathrm{D}})}{\{(0.8,0.5),(0,0.8),(0.3,0.4),(0.3,0.4)\}},\frac{(\mathrm{B}_3,(0,0.2,0.4,1)_{\mathrm{D}})}{\{(0.3,0.4),(0,0.6),(0.2,0.4),(0.3,0.4)\}}\right\},$$

$$\varepsilon_2,\left\{\frac{(\mathrm{B}_1,(0.5,0,1,1)_{\mathrm{D}})}{\{(0.7,0.8),(0.8,0.6),(0.2,0.3,0.4),(0.6,0.5)\}},\frac{(\mathrm{B}_2,(0.5,0,1,1)_{\mathrm{D}})}{\{(0.7,0.4),(0.9,0.8),(0.6,0.4),(0.5,0.7)\}},\frac{(\mathrm{B}_3,(0.5,0,1,1)_{\mathrm{D}})}{\{(0.3,0.5),(0.1,0.3),(0.9,0,1),(1,0)\}}\right\}$$

$$\left.\varepsilon_3,\left\{\frac{(\mathrm{B}_1,(0,0.5,1,1)_{\mathrm{D}})}{\{(0.8,0.9),(0,6,0.7),(0.1,0.3,0.4),(1,0)\}},\frac{(\mathrm{B}_2,(0,0.5,1,1)_{\mathrm{D}})}{\{(0.3,0.4),(0.1,0.2),(0.8,0,1),(0.5,0.8)\}},\frac{(\mathrm{B}_3,(0,0.5,1,1)_{\mathrm{D}})}{\{(0.8,0.9),(0.6,0.5),(0.1,0.3,0.4),(0,0.9)\}}\right\}\right\}$$

Definition 3.8

A P-HFSS $\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{E}\right)$ is said to be a null P-HFSS if $\mathrm{F}_{\mathrm{P}}^{\mathrm{H}}(\varepsilon)=0$, where $\varepsilon\in E$.

Definition 3.9

A P-HFSS $\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}}, \mathrm{E}\right)$ is said to be a full P-HFSS if $\mathrm{F}_{\mathrm{P}}^{\mathrm{H}}(\varepsilon)=1$, where $\varepsilon \in \mathrm{E}$.

Definition 3.10

For two P-HFSSs $\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}}, \mathrm{A}\right)$ and $\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}}, \mathrm{B}\right)$ on U, we say that $\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}}, \mathrm{A}\right)$ is a Plithogenic hesitant fuzzy soft subset of $\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}}, \mathrm{B}\right)$ if

(i) $\mathrm{A} \subseteq \mathrm{B}$

(ii) $\mathrm{F}_{\mathrm{P}}^{\mathrm{H}}(\varepsilon) \subseteq \mathrm{G}_{\mathrm{P}}^{\mathrm{H}}(\varepsilon)$, for all $\varepsilon \in \mathrm{E}$.

We define it as $\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}}, \mathrm{A}\right) \subseteq \left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}}, \mathrm{B}\right)$.

Example 3.11

Reconsider Example 3.4.

$$\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}}, \mathrm{A}\right)= \left\{a_1, \left\{\frac{(B_1,(0,0.2,0.4,1)_D)}{\{(0.6,0.2),(0.3,0.4,0.5),(0.5,0.7),(0.2,0.4,0.6)\}}, \frac{(B_2,(0,0.2,0.4,1)_D)}{\{(0.5,0.2),(0.2,1),(0.6,0.7),(0.6,0.7)\}}, \frac{(B_3,(0,0.2,0.4,1)_D)}{\{(0.6,0.7),(0.4,1),(0.6,0.8),(0.6,0.7)\}}\right\}\right.$$

,

$$a_2, \left\{\frac{(B_1,(0.5,0,1,1)_D)}{\{(0.2,0.3),(0.4,0.2),(0.6,0.7,0.8),(0.5,0.4)\}}, \frac{(B_2,(0.5,0,1,1)_D)}{\{(0.6,0.3),(0.2,0.1),(0.4,0.6),(0.3,0.5)\}}, \frac{(B_3,(0.5,0,1,1)_D)}{\{(0.5,0.7),(0.7,0.9),(0,1,0.1),(0,1)\}}\right\}$$

$$\left. a_3, \left\{\frac{(B_1,(0,0.5,1,1)_D)}{\{(0.1,0.2),(0,3,0.4),(0.6,0.7,0.9),(0,1)\}}, \frac{(B_2,(0,0.5,1,1)_D)}{\{(0.6,0.7),(0.8,0.9),(0,1,0.2),(0.3,0.5)\}}, \frac{(B_3,(0,0.5,1,1)_D)}{\{(0.1,0.2),(0.5,0.4),(0.6,0.7,0.9),(0.1,1)\}}\right\} \right\}$$

Then,

$$\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}}, \mathrm{B}\right)= \left\{\mathrm{b}_1, \left\{\frac{(\mathrm{B}_1,(0,0.2,0.4,1)_{\mathrm{D}})}{\{(0.7,0.8),(0.8,0.9),(0.6,0.8),(0.7,0.9,1)\}}, \frac{(\mathrm{B}_2,(0,0.2,0.4,1)_{\mathrm{D}})}{\{(0,1),(0.8,1),(0.8,0.9),(0,1)\}}, \frac{(\mathrm{B}_3,(0,0.2,0.4,1)_{\mathrm{D}})}{\{(0.6,0.8),(0.7,1),(0.9,1),(0.6,0.8)\}}\right\},\right.$$

$$\mathrm{b}_2, \left\{\frac{(\mathrm{B}_1,(0.5,0,1,1)_{\mathrm{D}})}{\{(0.5,0.8),(0.4,0.5,0.6),(0.7,0.8),(0.3,0.7)\}}, \frac{(\mathrm{B}_2,(0.5,0,1,1)_{\mathrm{D}})}{\{(0,1),(0.3,0.8),(0.7,0.9),(0.4,0.2)\}} \frac{(\mathrm{B}_3,(0.5,0,1,1)_{\mathrm{D}})}{\{(0.4,1),(0.6,1),(0.4,0.5,1),(0.2,1)\}}\right\}$$

$$\mathrm{b}_{3,} \left\{\frac{(\mathrm{B}_1,(0,0.5,1,1)_{\mathrm{D}})}{\{(0.5,0.6),(0.3,0.9),(0,0.5,1),(0.6,0.7,1)\}}, \frac{(\mathrm{B}_2,(0,0.5,1,1)_{\mathrm{D}})}{\{(0.7,0.9),(0.2,1),(0,6,0.7),(0.2,0.6)\}}, \frac{(\mathrm{B}_3,(0,0.5,1,)_{\mathrm{D}})}{\{(0.7,0.9),(0,0.2,1),(0.5,0.8,0.9),(0.3,1)\}}\right\}$$

$$\left. \mathrm{b}_4, \left\{\frac{(\mathrm{B}_1,(0.2,0.5,1,1)_{\mathrm{D}})}{\{(0.5,0.6),(0,2,0.4),(0.5,1),(0.4,0.8)\}}, \frac{(\mathrm{B}_2,(0.2,0.5,1,1)_{\mathrm{D}})}{\{(0.9,1),(0.3,0.4),(0,4,0.5),(0.3,0.4,0.6)\}}, \frac{(\mathrm{B}_3,(0.2,0.5,1,1)_{\mathrm{D}})}{\{(0.3,0.4,0.6),(0.5,0.7,0.9),(0.8,0.6),(0.3,0.6)\}}\right\} \right\}$$

.

Definition 3.12

Two hesitant fuzzy soft sets ($\mathrm{F}_{\mathrm{P}}^{\mathrm{H}}, \mathrm{A}$) and ($\mathrm{G}_{\mathrm{P}}^{\mathrm{H}}, B$) are said to be hesitant fuzzy soft equal if ($\mathrm{F}_{\mathrm{P}}^{\mathrm{H}}$, A) is a hesitant fuzzy soft subset of $\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}}, \mathrm{B}\right)$ and $\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}}, \mathrm{B}\right)$ is a hesitant fuzzy soft subset of ($\mathrm{F}_{\mathrm{P}}^{\mathrm{H}}$, A). In this case, we write $\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}}, \mathrm{A}\right)=\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}}, \mathrm{B}\right)$.

Proposition 3.13

Let $\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}}, \mathrm{A}\right)$ and $\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}}, \mathrm{B}\right)$ be two P-HFSS over U . Then,

$(i)\ \left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)\cup\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)=\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)$

$(ii)\ \left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)\cap\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)=\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)$

$(iii)\ \left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)\cup\Phi_{\mathrm{A}}=\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)$

$(iv)\ \left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)\cap\Phi_{\mathrm{A}}=\Phi_{\mathrm{A}}$

$(v)\ \left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)\cup\mathrm{U}_{A}=\mathrm{U}_{\mathrm{A}}$

$(vi)\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)\cap\mathrm{U}_{\mathrm{A}}=\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)$

$(vii)\ \left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)\cup\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}\right)=\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}\right)\cup\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)$

$(viii)\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)\cap\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}\right)=\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}\right)\cap\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)$.

Proposition 3.14

Let $\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)$ be a P-HFSS over U. Then,

$(i)\ \left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}^{-}\right)+\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}^{+}\right)^{\mathrm{C}}=1$

$(ii)\ \left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}^{+}\right)+\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}^{-}\right)^{\mathrm{C}}=1$

Proposition 3.15 (De Morgan's laws)

Let $\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)$ and $\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}\right)$ be two P-HFSSs over U. Then

$(i)\ \left[\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)\cup\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}\right)\right]^{\mathrm{C}}=\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)^{\mathrm{C}}\cap\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}\right)^{\mathrm{C}}$

$(ii)\ \left[\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)\cap\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}\right)\right]^{\mathrm{C}}=\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)^{\mathrm{C}}\cup\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}\right)^{\mathrm{C}}$

Proof : Let $u\in\left[\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)\cup\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}\right)\right]^{C}$, for some $u\in U$.

Then, $1-\mathrm{u}\in\left[\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)\cup\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}\right)\right]$ such that $1-\mathrm{u}\geq\max\left\{\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}^{-}\right),\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}^{-}\right)\right\}$

$$\begin{aligned}
&\Rightarrow 1-\mathrm{u}\geq\max\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}^{-}\right)\mathrm{and}\, 1-\mathrm{u}\geq\max\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}^{-}\right)\\
&\Rightarrow \mathrm{u}\leq 1-\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}^{-}\right)\mathrm{and}\,\mathrm{u}\leq 1-\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}^{-}\right)\\
&\Rightarrow \mathrm{u}\leq\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}^{+}\right)^{\mathrm{C}}\mathrm{and}\,\mathrm{u}\leq\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}^{+}\right)^{\mathrm{C}}\\
&\Rightarrow \mathrm{u}\leq\min\left\{\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}^{+}\right)^{\mathrm{C}},\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}^{+}\right)^{\mathrm{C}}\right\}\\
&\Rightarrow \mathrm{u}\in\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)^{\mathrm{C}}\cap\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}\right)^{\mathrm{C}}
\end{aligned}$$

Therefore, $\left[\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)\cup\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}\right)\right]^{C}\subseteq\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)^{\mathrm{C}}\cap\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}\right)^{\mathrm{C}}$

For the converse part, let $u\in\left(\mathrm{F}_{\mathrm{P}}^{\mathrm{H}},\mathrm{A}\right)^{\mathrm{C}}\cap\left(\mathrm{G}_{\mathrm{P}}^{\mathrm{H}},\mathrm{B}\right)^{\mathrm{C}}$

$$\Rightarrow \mathrm{u} \le \min\left\{\left(\mathrm{F_P^H}, \mathrm{A}^{+}\right)^{\mathrm{C}}, \left(\mathrm{G_P^H}, \mathrm{B}^{+}\right)^{\mathrm{C}}\right\}$$
$$\Rightarrow \mathrm{u} \le 1 - \left(\mathrm{F_P^H}, \mathrm{A}^{-}\right) \text{and} \mathrm{u} \le 1 - \left(\mathrm{G_P^H}, \mathrm{B}^{-}\right)$$
$$\Rightarrow 1 - \mathrm{u} \ge \max\left\{\left(\mathrm{F_P^H}, \mathrm{A}^{-}\right), \left(\mathrm{G_P^H}, \mathrm{B}^{-}\right)\right\}$$
$$\Rightarrow \mathrm{u} \in \left[\left(\mathrm{F_P^H}, \mathrm{A}\right) \cup \left(\mathrm{G_P^H}, \mathrm{B}\right)\right]^{\mathrm{C}}$$

Therefore, $\left(\mathrm{F_P^H}, \mathrm{A}\right)^{\mathrm{C}} \cap \left(\mathrm{G_P^H}, \mathrm{B}\right)^{\mathrm{C}} \subseteq \left[\left(\mathrm{F_P^H}, \mathrm{A}\right) \cup \left(\mathrm{G_P^H}, \mathrm{B}\right)\right]^{\mathrm{C}}$

Thus we have, $\left[\left(\mathrm{F_P^H}, \mathrm{A}\right) \cup \left(\mathrm{G_P^H}, \mathrm{B}\right)\right]^{\mathrm{C}} = \left(\mathrm{F_P^H}, \mathrm{A}\right)^{\mathrm{C}} \cap \left(\mathrm{G_P^H}, \mathrm{B}\right)^{\mathrm{C}}$.

Similarly, we can prove (ii).

4 Plithogenic Hesitant Fuzzy Soft Similarity

This section presents a similarity measure between two P-HFSSs.

Definition 4.1

Let $\left(\mathrm{F_P^H}, \mathrm{E}\right)$ and $\left(\mathrm{G_P^H}, \mathrm{E}\right)$ be two P-HFSSs over U as in Definition 3.1.

The Similarity measure between $\left(\mathrm{F_P^H}, \mathrm{E}\right)$ and $\left(\mathrm{G_P^H}, \mathrm{E}\right)$, denoted by $\mathrm{S_P^H}\left(\left(\mathrm{F_P^H}, \mathrm{E}\right), \left(\mathrm{G_P^H}, \mathrm{E}\right)\right)$. and defined as $\mathrm{S_P^H}\left(\left(\mathrm{F_P^H}, \mathrm{E}\right), \left(\mathrm{G_P^H}, \mathrm{E}\right)\right) = \frac{1}{|\mathrm{E}|}\sum_{k=1}^{|E|} \mathrm{M_k}$, where

$$\mathrm{M_k} = 1 - \frac{\sum_{j=1}^{|U|}\sum_{i=1}^{|\varepsilon|}\left|\max\left(\mathrm{F_j}(\mathrm{e_{ik}})\right) - \max\left(\mathrm{G_j}(\mathrm{e_{ik}})\right)\right|}{\sum_{j=1}^{|U|}\sum_{i=1}^{|\varepsilon|}\left|\max\left(\mathrm{F_j}(\mathrm{e_{ik}})\right) + \max\left(\mathrm{G_j}(\mathrm{e_{ik}})\right)\right|}, \varepsilon \in \mathrm{E}.$$

Proposition 4.2

Let $\left(\mathrm{F_P^H}, \mathrm{E}\right)$ and $\left(\mathrm{G_P^H}, \mathrm{E}\right)$ be two P-HFSS over U such that $\mathrm{F_P^H}$ or $\mathrm{G_P^H}$ a non-zero P-FSS. In that case, the following is true:

(i) $\mathrm{S_P^H}\left(\left(\mathrm{F_P^H}, \mathrm{E}\right), \left(\mathrm{G_P^H}, \mathrm{E}\right)\right) = \mathrm{S_P^H}\left(\left(\mathrm{G_P^H}, \mathrm{E}\right), \left(\mathrm{F_P^H}, \mathrm{E}\right)\right)$

(ii) $0 \le \mathrm{S_P^H}\left(\left(\mathrm{F_P^H}, \mathrm{E}\right), \left(\mathrm{G_P^H}, \mathrm{E}\right)\right) \le 1$

(iii) $\mathrm{F_P^H} = \mathrm{G_P^H} \Rightarrow \mathrm{S_P^H}\left(\left(\mathrm{F_P^H}, \mathrm{E}\right), \left(\mathrm{G_P^H}, \mathrm{E}\right)\right) = 1$.

Example 4.3

Reconsider Example 3.4. Suppose $\left(\mathrm{F_P^H}, \mathrm{A}\right)$ and $\left(\mathrm{G_P^H}, \mathrm{B}\right)$ are two P-HFSS over U

$$\left(\mathrm{F_P^H}, \mathrm{E}\right) = \left\{\varepsilon_1, \left\{\frac{(\mathrm{B_1},(0,0.2,0.4,1)_\mathrm{D})}{\{(0.6,0.2),(0.3,0.4,0.5),(0.5,0.7),(0.2,0.4,0.6)\}}, \frac{(\mathrm{B_2},(0,0.2,0.4,1)_\mathrm{D})}{\{(0.5,0.2),(0.2,1),(0.6,0.7),(0.6,0.7)\}}\right\} \frac{(\mathrm{B_3},(0,0.2,0.4,1)_\mathrm{D})}{\{(0.6,0.7),(0.4,1),(0.6,0.8),(0.6,0.7)\}}\right\},$$

$$\varepsilon_2, \left\{\frac{(\mathrm{B_1},(0.5,0,1,1)_\mathrm{D})}{\{(0.2,0.3),(0.4,0.2),(0.6,0.7,0.8),(0.5,0.4)\}}, \frac{(\mathrm{B_2},(0.5,0,1,1)_\mathrm{D})}{\{(0.6,0.3),(0.2,0.1),(0.4,0.6),(0.3,0.5)\}}, \frac{(\mathrm{B_3},(0.5,0,1,1)_\mathrm{D})}{\{(0.5,0.7),(0.7,0.9),(0,1,0.1),(0,1)\}}\right\}$$

$$\varepsilon_3, \left\{\frac{(\mathrm{B_1},(0,0.5,1,1)_\mathrm{D})}{\{(0.1,0.2),(0,3,0.4),(0.6,0.7,0.9),(0,1)\}}, \frac{(\mathrm{B_2},(0,0.5,1,1)_\mathrm{D})}{\{(0.6,0.7),(0.8,0.9),(0,1,0.2),(0.3,0.5)\}}, \frac{(\mathrm{B_3},(0,0.5,1,1)_\mathrm{D})}{\{(0.1,0.2),(0.5,0.4),(0.6,0.7,0.9),(0.1,1)\}}\right\}\}$$

$\left(\mathrm{G_P^H}, \mathrm{E}\right) =$

$$\left\{\varepsilon_1, \left\{\frac{(\mathrm{B_1},(0,0.2,0.4,1)_\mathrm{D})}{\{(0.2,0.4),(0.8,0.9,1),(0.6,0.8),(0.7,0.9,1)\}}, \frac{(\mathrm{B_2},(0,0.2,0.4,1)_\mathrm{D})}{\{(0,1),(0.8,0.9),(0.5,0.8),(0.9,1)\}}, \frac{(\mathrm{B_3},(0,0.2,0.4,1)_\mathrm{D})}{\{(0.6,0.8),(0.7,0.9),(0.5,1),(02,0.8)\}}\right\},$$

$$\varepsilon_2, \left\{\frac{(B_1,(0.5,0,1,1)_D)}{\{(0.5,0.8),(0.2,0.3),(0.5,0.6,0.7),(0.3,0.5)\}}, \frac{(B_2,(0.5,0,1,1)_D)}{\{(0.2,0.3),(0.3,0.5),(0.4,0.6),(0,0.2)\}}, \frac{(B_3,(0.5,0,1,1)_D)}{\{(0.2,0.4),(0.5,0.6),(0.4,0.5,0.7),(0.2,0.3)\}}\right\}$$

$$\varepsilon_3, \left\{\frac{(B_1,(0,0.5,1,1)_D)}{\{(0.5,0.6),(0,2,0.3),(0,0.5,1),(0.6,0.7)\}}, \frac{(B_2,(0,0.5,1,1)_D)}{\{(0.7,0.9),(0.2,0.1),(0,6,0.7),(0.2,0.6)\}}, \frac{(B_3,(0,0.5,1,1)_D)}{\{(0.7,0.9),(0,0.2),(0.5,0.8,0.6),(0.3,0.6)\}}\right\}\}$$

Then the similarity between $\left(F_P^H, E\right)$ and $\left(G_P^H, E\right)$ is given by :

$$M_1 = 1 - \frac{\sum_{j=1}^{3}\sum_{i=1}^{4}\left|\max\left(F_j(e_{i1})\right) - \max\left(G_j(e_{i2})\right)\right|}{\sum_{j=1}^{3}\sum_{i=1}^{4}\left|\max\left(F_j(e_{i1})\right) + \max\left(G_j(e_{i2})\right)\right|}$$

For $\sum_{i=1}^{4}\left|\max\left(F_j(e_{i1})\right) - \max\left(G_j(e_{i2})\right)\right|$, we calculate the case when $j = 1$. Here we have

$\sum_{i=1}^{4}\left|\max\left(F_1(e_{i1})\right) - \max\left(G_1(e_{i2})\right)\right| = |0.6 - 0.4| + |0.5 - 1| + |0.7 - 0.8| + |0.6 - 1|$
$= 0.2 + 0.5 + 0.1 + 0.4 = 1.2.$

Similarly for $j = 2$, we have

$$\sum_{i=1}^{4}|\max(F_2(e_{i1})) - \max(G_2(e_{i2}))| = 1.0$$

and for $j = 3$,

$$\sum_{i=1}^{4}|\max(F_3(e_{i1})) - \max(G_3(e_{i2}))| = 0.5.$$

Also, for

$$\sum_{i=1}^{4}\left|\max(F_j(e_{i1})) + \max(G_j(e_{i2}))\right|$$

proceeding as above for $j = 1$, we get

$$\begin{aligned}\sum_{i=1}^{4}|\max(F_1(e_{i1})) + \max(G_1(e_{i2}))| &= |0.6 + 0.4| + |0.5 + 1| \\ &\quad + |0.7 + 0.8| + |0.6 + 1| \\ &= 1.0 + 1.5 + 1.5 + 1.6 \\ &= 5.6.\end{aligned}$$

Similarly for $j = 2$, we have

$$\sum_{i=1}^{4}|\max(F_2(e_{i1})) + \max(G_2(e_{i2}))| = 6.6$$

and for $j = 3$, we get

$$\sum_{i=1}^{4}|\max(F_3(e_{i1})) + \max(G_3(e_{i2}))| = 6.7.$$

Here $M_1 = 1 - 0.1428 \approx 0.86$. Similarly, we get $M_2 = 0.82$ and $M_3 = 0.79$. Then the similarity between (F_P^H, E) and (G_P^H, E) is given by,

$$S_P^H((F_P^H, E), (G_P^H, E)) = \frac{1}{|3|}\sum_{k=1}^{3} M_k = 0.82.$$

5 Conclusion

This paper provides the notion of a Plithogenic hesitant fuzzy soft set and some of its properties. Further, we have explained the concepts using simple examples. Also, basic characteristics of these operations and similarity measure between two P-HFSSs were explored.

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