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Applications of Second-Order Differential Subordination Considering Mittag-Leffler Poisson Distribution Series to a Class of Analytic Functions

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Abstract:

In this paper, we establish an innovative subclass of analytic functions by utilising the Mittag-Leffler poisson distribution series and acquire some number of Differential Subordination (DSO) results.

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1. Introduction

Let $\mathbb{C}$ be complex plane and $\mathbb{D} = \{\mathfrak{w}: \mathfrak{w} \in \mathbb{C}\ and\ |\mathfrak{w}| < 1\} = \mathbb{D}\backslash\{0\}$ exists as an open unit circle inside $\mathbb{C}$. As well as, assume $H(\mathbb{D})$ represent a class of functions which are analytic in $\mathbb{D}$. For $n \in \mathbb{N} = \{1,2,3, \dots ,\}$ as well as $a \in \mathbb{C}$, let $H[a, n]$ be a subclass of $H(\mathbb{D})$ constituted as functions having of the form

$$l(\mathfrak{w}) = \mathfrak{w} + a_n\mathfrak{w}^n + a_{n+1}\mathfrak{w}^{n+1} + \cdots$$

Through $H_0 \cong H[0,1]$ along with $H \cong H[1,1]$. Assume A_n is the class, which is a group of every holomorphic mappings of the form

$$l(\mathfrak{w}) = \mathfrak{w} + \sum_{k=n+1}^{\infty} a_n\mathfrak{w}^n \qquad (1.1)$$

within a unit circle that is open in $\mathbb{D}$ through $A_1 = A$. A mapping $l \in H(\mathbb{D})$ is injective if it satisfies the one-one correspondence mapping within $\mathbb{D}$. The subclass of A formed by univalent functions in $\mathbb{D}$, which is indicated by S . Suppose there is a function $l \in A$, which takes the mapping $\mathbb{D}$ onto a $convex$ domain as well as l injective, then l represented as $convex$ mapping and denoted the class by

$$K = \{l \in A: \Re\left\{1 + \frac{\mathfrak{w}l''(\mathfrak{w})}{l'(\mathfrak{w})}\right\} > 0,\ \mathfrak{w} \in\ \mathbb{D}\},$$

The class of all $convex$ functions specified in $\mathbb{D}$ and are normalized by the conditions ``$l(0) = 0\ and\ l'(0) = 1$''.

Assume l *and* f are elements in $H(\mathbb{D})$. A mapping l is said to be *subordinate* to f if ∃ a *Schwartz function belongs to* $\mathbb{D}$, which is analytic among

$$g(0) = 0 \text{ as well as } |g(\mathfrak{w})| < 1, \qquad \mathfrak{w} \in \mathbb{D},$$

So as to

$$l(\mathfrak{w}) = f\big(g(\mathfrak{w})\big).$$

In this instance, we write

$$l(\mathfrak{w}) \prec f(\mathfrak{w}) \text{ or } l \prec f.$$

Moreover, if the mapping f is injective in $\mathbb{D}$, after that we obtain the following equivalence

$$l(\mathfrak{w}) \prec f(\mathfrak{w}) \Leftrightarrow l(0) = f(0) \quad and \quad l(\mathbb{D}) \prec f(\mathbb{D}).$$

The technique of DSO (also called as the method of admissible functions) was early introduced through Miller and Mocanu [7] in the year 1978, in addition to the progress of the theory was started in 1981[see. [8]]. The book by Miller and Mocanu [9] in 2000 contained the total information about the DSO. In the last several years, frequently the researchers deliberate the attributes of DSO (refer [1-3,11]).

Letting $\Psi: \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ as well as allow h be one-one in $\mathbb{D}$. Suppose that the analytic function p within $\mathbb{D}$ and fulfills the second order of differential subordination

$$\Psi(\mathrm{p}(\mathfrak{w}), \mathfrak{w}p'(\mathfrak{w}), \mathfrak{w}p''(\mathfrak{w}); \mathfrak{w}) \prec \mathrm{h}(\mathfrak{w}). \qquad (1.2)$$

Subsequently p is referred to as the DSO solution. Now the injective mapping q also named as a dominant of the DSO solution or, equivalently, a dominant if $p \prec q$ for all p fulfilling (1.2). The dominant q_1 fulfilling $q_1 \prec q$ to each and every dominants q of (1.2) and then q_1 named as the best dominant of (1.2).

In the last several years, Binomial, Pascal and Poisson distribution series etc., are plays significant role in geometric function theory. For Star-like(ST), Uniformly Convexity(UCV) and for some special functions in the geometric function theory, there was innovated the sufficient conditions and the proofs. By the motivation of the works [1, 4, 8, 9], we develop this work.

In [12], Porwal, Poisson distribution series and provides the cordial submission on analytic functions; which gives a new way of research exposure in Geometric Function Theory. Afterwards, some researchers moved on the distribution series of confluent hyper-geometric, hyper geometric, Binomial, Pascal and prevails essential and adequate requirement for certain classes of Injective mappings.

Recently, Porwal and Dixit [13] create Mittag-Leffler type Poisson distribution with prominent instants, mgf, that is a generalization of Poisson distribution using the definition. For this, the probability mass function is as

$$P(\lambda, \mu, s; \mathscr{k})(\mathfrak{w}) = \frac{\lambda^{\mathscr{k}}}{E_{\mu,s}(\lambda)\Gamma(\mu\mathscr{k} + s)}, \qquad (\mathscr{k} = 0,1,2, \ldots), \qquad (1.3)$$

where

$$E_{\mu,s}(\mathfrak{w}) = \sum_{k=0}^{\infty} \frac{\mathfrak{w}^{k}}{\Gamma(\mu k + s)}, \qquad (\mu, s \in \mathbb{C}, \mathscr{R}(\mu) > 0, \mathscr{R}(s) > 0). \quad (1.4)$$

The Mittag-Leffler type Poisson distribution series was innovated by Porwal and Dixit [17] along with specification as

$$K(\lambda,\mu,s)(\mathfrak{w}) = \mathfrak{w} + \sum_{k=2}^{\infty} \frac{\lambda^{k-1}}{\Gamma(\mu(k-1)+s)E_{\mu,s}(\lambda)} \mathfrak{w}^{k}, \qquad (1.5)$$

Which is a standardization mapping in $S, since\ K(\lambda,\mu,s)(0) = 0\ and\ K^{'}(\lambda,\mu,s)(0) = 1$. After that, the geometric properties of (1.5) were investigated by Porwal et al. in [14].

For $l \in A\ according\ to\ (1.1) and\ j(\mathfrak{w})$ given by

$$j(\mathfrak{w}) = \mathfrak{w} + \sum_{k=2}^{\infty} b_{k} \mathfrak{w}^{k}, \qquad (1.6)$$

the convolution of $j\&l$ is represented as $(j * l)$, which is exemplified with

$$(j * l)(\mathfrak{w}) = \mathfrak{w} + \sum_{k=2}^{\infty} a_{k} b_{k} \mathfrak{w}^{k} = (l * j)(\mathfrak{w}), \ (\mathfrak{w} \in \Upsilon) \qquad (1.7)$$

Note that $j * l \in A$. Further, we establish the operator which is convolution product as

$$\mathscr{J}_{\lambda,\mu}^{s} l(\mathfrak{w}) = K(\lambda,\mu,s) * j(\mathfrak{w})$$

$$= \mathfrak{w} + \sum_{k=2}^{\infty} \varphi_{\lambda}^{k}(\mu,s) a_{k} \mathfrak{w}^{k}, \qquad (1.8)$$

where $\varphi_{\lambda}^{k}(\mu,s) = \frac{\lambda^{k-1}}{\Gamma(\mu(k-1)+s)E_{\mu,s}(\lambda)}$.

Definition 1.1. Let $\mathfrak{L}_{\lambda,\mu,s}(\varrho) be\ a\ class\ of\ function\ l \in A$ fulfilling the inequality

$$\Re\left(\mathscr{J}_{\lambda,\mu}^{s} l(\mathfrak{w})\right)^{'} \geq \varrho, where\ \mathfrak{w} \in \mathbb{D}, \ 0 \leq \varrho < 1.$$

Lemma 1.2. [6]Given h be a $convex$ function with $h(0) = a$ and suppose $\wp \in \mathbb{C}^{*} \coloneqq \mathbb{C}\backslash\{0\}, \wp$, which is complex among $\Re\{\wp\} \geq 0. Suppose\ if\ p\ belong\ to\ H[a,n]\ as\ well\ as$

$$p(\mathfrak{w}) + \frac{1}{\wp} \mathfrak{w} p^{'}(\mathfrak{w}) \prec h(\mathfrak{w}), \qquad (1.9)$$

then

$$p(\mathfrak{w}) \prec q(\mathfrak{w}) \prec h(\mathfrak{w}),$$

$$where \qquad q(\mathfrak{w}) = \frac{\wp}{n\mathfrak{w}^{\frac{\wp}{n}}} \int_{0}^{\mathfrak{w}} t^{\frac{\wp}{n-1}} h(t) dt, \quad \mathfrak{w} \in \mathbb{D}.$$

The mapping q is $convex$, which acts as best $dominant$ for subordination (1.9).

Lemma 1.3. [10] Let $\Re\{\mu\} > 0, n \in \mathbb{N}$ $and\ writ\ w = \frac{n^2+|\mu|^2-|n^2-\mu^2|}{4n\Re\{\mu\}}$, in addition let the holomorphic function $h \in \mathbb{D}$ $with\ h(0) = 1$. Assuming

$$\Re\left\{1 + \frac{\mathfrak{w}h''(\mathfrak{w})}{h'(\mathfrak{w})}\right\} > -\mathfrak{w}.$$

If $p(\mathfrak{w}) = 1 + p_n\mathfrak{w}^n + p_{n+1}\mathfrak{w}^{n+1} + \cdots$ is always differentiable in $\mathbb{D}$ and

$$p(\mathfrak{w}) + \frac{1}{\mu}\mathfrak{w}p'(\mathfrak{w}) \prec h(\mathfrak{w}), \qquad (1.10)$$

then

$$p(\mathfrak{w}) \prec q(\mathfrak{w}),$$

$where\ q\ is\ the\ solution\ of\ differential\ equation\ q(\mathfrak{w}) + \frac{n}{\mu}\mathfrak{w}q'(\mathfrak{w}) = h(\mathfrak{w}),\ \ q(0) = 1,$

Provided by

$$q(\mathfrak{w}) = \frac{\mu}{n\mathfrak{w}^{\frac{\mu}{n}}}\int_0^{\mathfrak{w}} t^{\frac{\mu}{n-1}}h(t)dt, \quad \mathfrak{w} \in \mathbb{D}.$$

q is therefore $best\ dominant$ for the DSO (1.10).

Lemma 1.4. [15] Suppose $r\ be\ as\ convex\ function\ in\ \mathbb{D}$ as well as let

$$h(\mathfrak{w}) = r(\mathfrak{w}) + n\varrho\mathfrak{w}r'(\mathfrak{w}), \qquad \mathfrak{w} \in \mathbb{D}, as\ long\ as\ n \in \mathbb{N}\ and\ \varrho > 0.$$

In case

$$p(\mathfrak{w}) = r(0) + p_n\mathfrak{w}^n + p_{n+1}\mathfrak{w}^{n+1} + \cdots \quad , \mathfrak{w} \in \mathbb{D},$$

Which is holomorphic in $\mathbb{D}$ and

$$p(\mathfrak{w}) + \varrho\mathfrak{w}p'(\mathfrak{w}) \prec h(\mathfrak{w}), \qquad \mathfrak{w} \in \mathbb{D},$$

then

$$p(\mathfrak{w}) \prec q(\mathfrak{w})$$

in addition to this outcome is pointed.

In the current work, we make use the $subordination\ results$ from [6] and [10] to demonstrate our main results.

2. Main Results

Theorem 2.1. Prove that $\mathfrak{L}_{\lambda,\mu,s}(\varrho)$ is convex set.

proof. Suppose

$$f_j(\mathfrak{w}) = \mathfrak{w} + \sum_{k=2}^{\infty} a_{k,j}\mathfrak{w}^k, \mathfrak{w} \in \mathbb{D}, j = 1, \dots, m\ be\ in\ the\ class\ \mathfrak{L}_{\lambda,\mu,s}(\varrho).$$

Then we obtained by Definition 1.1,

$$\mathfrak{R}\left\{(\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w}))'\right\} = \mathfrak{R}\left\{1 + \sum_{k=2}^{\infty} \varphi_{\lambda}^{k}(\mu, s) a_{k,j} k \mathfrak{w}^{k-1}\right\} > \varrho \qquad (2.1)$$

For any positive numbers $\varsigma_1, \varsigma_2, \varsigma_3, \ldots, \varsigma_m$ $such\ that$ $\sum_{j=1}^{m} \varsigma_j = 1,$

Now we essential to demonstrate the mapping

$$h(\mathfrak{w}) = \sum_{j=1}^{m} \varsigma_j f_j(\mathfrak{w})$$

is a member of $\mathfrak{L}_{\lambda,\mu,s}(\varrho), i.e.,$

$$\mathfrak{R}\left\{(\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w}))'\right\} > \varrho. \qquad (2.2)$$

Thus, we have

$$\mathscr{I}_{\lambda,\mu}^{s} h(\mathfrak{w}) = \mathfrak{w} + \sum_{k=2}^{\infty} \varphi_{\lambda}^{k}(\mu, s) \left\{\sum_{j=1}^{m} \varsigma_j a_{k,j}\right\} \mathfrak{w}^{k}. \qquad (2.3)$$

Suppose differentiating (13) with respect to $\mathfrak{w}$, after that we acquire

$$(\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w}))' = 1 + \sum_{k=2}^{\infty} k \varphi_{\lambda}^{k}(\mu, s) \left\{\sum_{j=1}^{m} \varsigma_j a_{k,j}\right\} \mathfrak{w}^{k-1}.$$

Thus, we have

$$\mathfrak{R}\left\{(\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w}))'\right\} = 1 + \sum_{j=1}^{m} \varsigma_j \, \mathfrak{R}\left\{\sum_{k=2}^{\infty} k \varphi_{\lambda}^{k}(\mu, s) a_{k,j} \mathfrak{w}^{k-1}\right\}$$

$$> 1 + \sum_{j=1}^{m} \varsigma_j (\varrho - 1), \quad by\ (2.2)$$

$$= \varrho.$$

Hence, the equation (2.1) is accurate as well as we arrive the desired result. Hence the theorem demonstrated.

Theorem 2.2 Suppose q can be $convex\ function$ in $\mathbb{D}$ $having\ q(0) = 1\ as\ well\ as$

$$h(\mathfrak{w}) = q(\mathfrak{w}) + \frac{1}{\wp + 1} \mathfrak{w} q'(\mathfrak{w}), \qquad \mathfrak{w} \in \mathbb{D},$$

Where $\wp\ is\ a\ complex\ number\ with\ \mathfrak{R}\{\wp\} > -1.$

$If\ l \in \mathfrak{L}_{\lambda,\mu,s}(\varrho)\ and\ \mathcal{N} = \Upsilon_{\wp} f$, where $\mathcal{N}\ defined\ as$

$$\mathcal{N}(\mathfrak{w}) = \Upsilon_{\wp} l(\mathfrak{w}) = \frac{\wp + 1}{\mathfrak{w}^{\wp}} \int_{0}^{\mathfrak{w}} t^{\wp - 1} l(t) dt, \qquad (2.4)$$

then

$$\left(\mathscr{J}_{\lambda,\mu}^{s} l(\mathfrak{w})\right)' \prec h(\mathfrak{w}) \qquad (2.5)$$

implies that

$$\left(\mathscr{J}_{\lambda,\mu}^{s} \mathcal{N}(\mathfrak{w})\right)' \prec q(\mathfrak{w})$$

along with this outcome is pointed.

Proof. With regard of equality (2.4), we can write

$$\mathfrak{w}^{\wp} \mathcal{N}(\mathfrak{w}) = (\wp + 1) \int_{0}^{\mathfrak{w}} t^{\wp - 1}\, l(t) dt. \qquad (2.6)$$

Now, we obtained by differentiating (2.6) with respect to $\mathfrak{w}$

$$(\wp)\mathcal{N}(\mathfrak{w}) + \mathfrak{w}\mathcal{N}'(\mathfrak{w}) = (\wp + 1) l(\mathfrak{w}).$$

Additionally, by utilising the operator $\mathscr{J}_{\lambda,\mu}^{s}$ to the above equality, we obtain

$$(\wp)\mathscr{J}_{\lambda,\mu}^{s}\mathcal{N}(\mathfrak{w}) + \mathfrak{w}\left(\mathscr{J}_{\lambda,\mu}^{s}\mathcal{N}(\mathfrak{w})\right)' = (\wp + 1)\mathscr{J}_{\lambda,\mu}^{s} l(\mathfrak{w}). \qquad (2.7)$$

Now, we obtained by differentiating (2.7) with respect to $\mathfrak{w}$

$$\left(\mathscr{J}_{\lambda,\mu}^{s}\mathcal{N}(\mathfrak{w})\right)' + \frac{1}{\wp + 1}\mathfrak{w}\left(\mathscr{J}_{\lambda,\mu}^{s} l(\mathfrak{w})\right)'' = \left(\mathscr{J}_{\lambda,\mu}^{s} l(\mathfrak{w})\right)' \qquad (2.8)$$

Through the use of $the\ differential\ subordination$, which is given by (2.5)and in equality (2.8), we attain

$$\left(\mathscr{J}_{\lambda,\mu}^{s}\mathcal{N}(\mathfrak{w})\right)' + \frac{1}{\wp + 1}\mathfrak{w}\left(\mathscr{J}_{\lambda,\mu}^{s} l(\mathfrak{w})\right)'' \prec h(\mathfrak{w}). \qquad (2.9)$$

We define

$$p(\mathfrak{w}) = \left(\mathscr{J}_{\lambda,\mu}^{s}\mathcal{N}(\mathfrak{w})\right)'. \qquad (2.10)$$

Therefore, as a outcome of simple calculations, we obtain

$$p(\mathfrak{w}) = \left\{\mathfrak{w} + \sum_{k=2}^{\infty} \varphi_{\lambda}^{k}(\mu, s)\frac{\wp + 1}{\wp + k}\, a_k \mathfrak{w}^k\right\}'$$

$$= 1 + p_1\mathfrak{w} + p_2\mathfrak{w}^2 + \cdots, p \in H[1,1].$$

Through the use of (2.10) in $subordination$ (2.9), we acquire

$$p(\mathfrak{w}) + \frac{1}{\wp + 1}\mathfrak{w}p'(\mathfrak{w}) \prec h(\mathfrak{w}) = q(\mathfrak{w}) + \frac{1}{\wp + 1}\mathfrak{w}q'(\mathfrak{w}), \qquad \mathfrak{w} \in \mathbb{D}.$$

By utilizing Lemma 1.3, after that we write as

$$p(\mathfrak{w}) \prec q(\mathfrak{w}).$$

We thereby achieved the preferred q is the best dominant.

Thus the theorem is demonstrated.

Example 2.3. If we prefer in Theorem 2.2,

$$\wp = i + 1 \text{ and } q(\mathfrak{w}) = \frac{1 + \mathfrak{w}}{1 - \mathfrak{w}},$$

then obtain

$$h(\mathfrak{w}) = \frac{(i + 2) - \big((i + 2)\mathfrak{w} + 2\big)\mathfrak{w}}{(i + 2)(1 - \mathfrak{w})^2}.$$

Suppose $l \in \mathfrak{L}_{\lambda,\mu,s}(\varrho)$ *and* $\mathcal{N}$ *is written as*

$$\mathcal{N}(\mathfrak{w}) = \Upsilon_i l(\mathfrak{w}) = \frac{i + 2}{\mathfrak{w}^{i+1}} \int_0^{\mathfrak{w}} t^i\, l(t)dt,$$

then, by virtue of Theorem 2.2, we determine

$$\left(\mathcal{J}_{\lambda,\mu}^{s} l(\mathfrak{w})\right)' \prec h(\mathfrak{w}) = \frac{(i + 2) - \big((i + 2)\mathfrak{w} + 2\big)\mathfrak{w}}{(i + 2)(1 - \mathfrak{w})^2}.$$

$$\Rightarrow \left(\mathcal{J}_{\lambda,\mu}^{s} l(\mathfrak{w})\right)' \prec \frac{1 + \mathfrak{w}}{1 - \mathfrak{w}}.$$

Theorem 2.4. *let* $\Re\{\wp\} > -1$ and

$$\mathfrak{w} = \frac{1 + |\wp + 1|^2 - |\wp^2 + 2\wp|}{4\Re\{\wp + 1\}}.$$

Assume that $h \in \mathbb{D}$ is a *holomorphic mapping with* $h(0) = 1$ with to

$$\Re\left\{1 + \frac{\mathfrak{w}h''(\mathfrak{w})}{h'(\mathfrak{w})}\right\} > -\mathfrak{w}.$$

If $l \in \mathfrak{L}_{\lambda,\mu,s}(\varrho)$ *and* $\mathcal{N} = \Upsilon_{\mu}^{s} f$, *where* $\mathcal{N}$ *is specified by* (2.4), *then*

$$\left(\mathcal{J}_{\lambda,\mu}^{s} l(\mathfrak{w})\right)' \prec h(\mathfrak{w})$$

gives that

$$\left(\mathcal{J}_{\lambda,\mu}^{s} \mathcal{N}(\mathfrak{w})\right)' \prec q(\mathfrak{w}),$$

where q is the primitive of the *differential equation*

$$h(\mathfrak{w}) = q(\mathfrak{w}) + \frac{1}{\wp + 1}\mathfrak{w}q'(\mathfrak{w}), \qquad q(0) = 1,$$

given by

$$q(\mathfrak{w}) = \frac{\wp + 1}{\mathfrak{w}^{\wp+1}}\int_0^{\mathfrak{w}} t^{\wp}\, l(t)dt.$$

Moreover q , $the\ best\ dominant\ for\ subordination (2.11)$.

Proof. Suppose we prefer $n = 1\ in\ addition\ \mu = \wp + 1\ in\ Lemma 1.2,$ then the proof is obtained by means of the proof of Theorem 2.4.

Theorem 2.5. Let $h(\mathfrak{w}) = \frac{1+(2\varrho-1)\mathfrak{w}}{1+\mathfrak{w}},\ \ 0 \le \varrho < 1$ $\qquad (2.12)$

be $convex$ in $\mathbb{D}$ $with\ h(0) = 1. if\ \ l \in A$ also verifies the differential subordination

$$\left(\mathscr{I}^{s}_{\lambda,\mu} l(\mathfrak{w})\right)' \prec h(\mathfrak{w}),$$

then

$$\left(\mathscr{I}^{s}_{\lambda,\mu} \mathcal{N}(\mathfrak{w})\right)' \prec q(\mathfrak{w}) = (2\,\varrho - 1) + \frac{2(1-\varrho)(\wp+1)\tau(\wp)}{\mathfrak{w}^{\wp+1}},$$

for which the formula provides $\mathcal{T}$ as

$$\tau(\wp) = \int_0^{\mathfrak{w}} \frac{t^{\wp}}{t+1}dt \qquad (2.13)$$

Also $\mathcal{N}$ was known by equation (2.4). The mapping q is $convex$ and is the $best\ dominant$.

Proof: Suppose $h(\mathfrak{w}) = \frac{1+(2\varrho-1)\mathfrak{w}}{1+\mathfrak{w}},\ \ 0 \le \varrho < 1$,

afterwards h is $convex$ in addition to, considering Theorem 2.4, we can write

$$\left(\mathscr{I}^{s}_{\lambda,\mu} \mathcal{N}(\mathfrak{w})\right)' \prec q(\mathfrak{w}).$$

by using Lemma1.2, we get

$$\begin{aligned} q(\mathfrak{w}) &= \frac{\wp+1}{\mathfrak{w}^{\wp+1}}\int_0^{\mathfrak{w}} t^{\wp} h(t)\, dt \\ &= \frac{\wp+1}{\mathfrak{w}^{\wp+1}}\int_0^{\mathfrak{w}} t^{\wp}\left\{\frac{1+(2\varrho-1)t}{1+t}\right\}dt \\ &= (2\varrho - 1) + \frac{2(1-\varrho)(\wp+1)}{\mathfrak{w}^{\wp+1}}\,\mathcal{T}(\wp), \end{aligned}$$

wherever τ is specified through (2.13). Hence, we obtain

$$\left(\mathscr{I}^{s}_{\lambda,\mu} \mathcal{N}(\mathfrak{w})\right)' \prec q(\mathfrak{w}) = (2\varrho - 1) + \frac{2(1-\varrho)(\wp+1)}{\mathfrak{w}^{\wp+1}}\,\tau(\wp).$$

The mapping q is $convex$. Furthermore, it is the $best\ dominant$.

The theorem was proven.

Theorem 2.6. If $0 < \varrho < 1, 0 \leq \mu < 1, \delta \geq 0, \Re\{\wp\} > -1, and\ \mathcal{N} = \Upsilon_{\wp} f$ is defined by (2.4), then

$$\Upsilon_{\wp}\left(\mathfrak{L}_{\lambda,\mu,s}(\varrho)\right) \subset \mathfrak{L}_{\lambda,\mu,s}(\rho),$$

where

$$\rho = \min_{|\mathfrak{w}|=1} \Re\{q(\mathfrak{w})\} = \rho(\wp, \varrho) = (2\varrho - 1) + 2(1-\varrho)(\wp + 1)\tau(\wp) \qquad (2.14)$$

and $\tau\ is\ given\ by\ (2.13)$.

proof: Let us assuming that h is provided through equation(2.12), $l \in \mathfrak{L}_{\lambda,\mu,s}(\varrho)\ and\ \mathcal{N} = \Upsilon_{\wp} f$ is defined by (2.4). Then h is convex and, by Theorem 2.4, we deduce

$$\left(\mathscr{I}_{\lambda,\mu}^{s} \mathcal{N}(\mathfrak{w})\right)' \prec q(\mathfrak{w}) = (2\varrho - 1) + \frac{2(1-\varrho)(\wp + 1)}{\mathfrak{w}^{\wp+1}}\,\tau(\wp) \qquad (2.15)$$

where τ is given by (2.13). Since q is $convex$, $q(\mathbb{D})$ is symmetric about the real axis, and $\Re\{\wp\} > -1$, we determine

$$\Re\left\{\left(\mathscr{I}_{\lambda,\mu}^{s} \mathcal{N}(\mathfrak{w})\right)'\right\} \geq \min_{|\mathfrak{w}|=1} \Re\{q(\mathfrak{w})\} = \Re\{q(1)\} = \rho(\wp, \varrho)$$

$$= (2\varrho - 1) + 2(1-\varrho)(\wp + 1)(1-\varrho)\tau(\wp).$$

Which obey from inequality (2.15) that

$$\Upsilon_{\wp}\left(\mathfrak{L}_{\lambda,\mu,s}(\varrho)\right) \subset \mathfrak{L}_{\lambda,\mu,s}(\rho), where\ \rho\ is\ given\ by\ (2.14).$$

Hence theorem proved.

Theorem 2.7. Let q be a $convex$ mapping with $q(0) = 1$ and h be a function such that

$$h(\mathfrak{w}) = q(\mathfrak{w}) + \mathfrak{w}q^{1}(\mathfrak{w}), \quad \mathfrak{w} \in \mathbb{U}.$$

If $l \in A$, then the subordination

$$\left(\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w})\right)' \prec h(\mathfrak{w}) \qquad (2.16)$$

Implies that

$$\frac{\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w})}{\mathfrak{w}} \prec q(\mathfrak{w}),$$

and the result is sharp.

Proof: Let $p(\mathfrak{w}) = \frac{\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w})}{\mathfrak{w}}$ (2.17)

Differentiating (2.17), we find

$$\left(\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w})\right)' = p(\mathfrak{w}) + \mathfrak{w}p'(\mathfrak{w}).$$

We now compute $p(\mathfrak{w})$.This gives

$$p(\mathfrak{w}) = \frac{\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w})}{\mathfrak{w}} = 1 + \frac{\sum_{k=2}^{\infty} \varphi_{\lambda}^{k}(\mu,s)\ a_k \mathfrak{w}^k}{\mathfrak{w}}$$

$$= 1 + p_1\mathfrak{w} + p_2\mathfrak{w}^2 + \cdots, \qquad p \in H[1,1]. \qquad (2.18)$$

By utilizing (2.18) in $subordination$(2.16), we obtain

$$p(\mathfrak{w}) + \mathfrak{w}p'(\mathfrak{w}) \prec h(\mathfrak{w}) = q(\mathfrak{w}) + \mathfrak{w}q'(\mathfrak{w}).$$

Hence, by applying Lemma 1.4, we determine the result that

$$p(\mathfrak{w}) \prec q(\mathfrak{w})$$

i.e.,

$$\frac{\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w})}{\mathfrak{w}} \prec q(\mathfrak{w}).$$

This outcome is pointed with q is the $best\ dominant$.

Hence the theorem is validated.

Theorem2.8. Let

$$h(\mathfrak{w}) = \frac{1 + (2\varrho - 1)\mathfrak{w}}{1 + \mathfrak{w}}, \quad \mathfrak{w} \in \mathbb{D}$$

be $convex$ in $\mathbb{D}$ satisfying $h(0) = 1\ and\ 0 < \varrho < 1. If\ l \in A$ obeys $differential\ subordination$ property

$$\left(\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w})\right)' \prec h(\mathfrak{w}) \qquad (2.19)$$

then

$$\frac{\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w})}{\mathfrak{w}} \prec q(\mathfrak{w}) = (2\varrho - 1) + \frac{2(1-\varrho)\ln(1+\mathfrak{w})}{\mathfrak{w}}.$$

Now, the mapping q is $convex$ and, more over it is the $best\ dominant$.

Proof: Assume

$$p(\mathfrak{w}) = \frac{\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w})}{\mathfrak{w}} = 1 + p_1\mathfrak{w} + p_2\mathfrak{w}^2 + \cdots, \qquad p \in H[1,1]. \qquad (2.20)$$

Differentiating (2.20), we find

$$\left(\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w})\right)' = p(\mathfrak{w}) + \mathfrak{w}p'(\mathfrak{w}). \qquad (2.21)$$

Considering (2.21), the differential subordination (2.19) becomes

$$\left(\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w})\right)' \prec h(\mathfrak{w}) = \frac{1 + (2\varrho - 1)\mathfrak{w}}{1 + \mathfrak{w}}.$$

By using Lemma1.2, we deduce

$$p(\mathfrak{w}) \prec q(\mathfrak{w}) = \frac{1}{\mathfrak{w}} \int h(t)dt = (2\varrho - 1) + \frac{2(1 - \varrho)\, ln(1 + \mathfrak{w})}{\mathfrak{w}}.$$

By equation (2.20) we achieved the intended outcome.

Thus the theorem obtained.

Carollary 2.9. $If\ l \in \mathfrak{L}_{\lambda,\mu,s}(\varrho), then$

$$\Re\left\{\frac{\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w})}{\mathfrak{w}}\right\} > (2\varrho - 1) + 2(1 - \varrho)\ln(2).$$

Proof: $If\ l \in \mathfrak{L}_{\lambda,\mu,s}(\varrho),$ then it follows from Definition 1.1 that

$$\Re\left\{\left(\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w})\right)'\right\} > \varrho, \qquad \mathfrak{w} \in \mathbb{D},$$

which is equivalent to

$$\left(\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w})\right)' \prec h(\mathfrak{w}) = \frac{1 + (2\varrho - 1)\mathfrak{w}}{1 + \mathfrak{w}}.$$

Utilising Theorem 2.8, we acquire

$$\frac{\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w})}{\mathfrak{w}} \prec q(\mathfrak{w}) = (2\varrho - 1) + \frac{2(1 - \varrho)\, ln(1 + \mathfrak{w})}{\mathfrak{w}}.$$

Since $q(\mathbb{D})$ is symmetric about the real axis as well as q is $convex$, we finalize that

$$\Re\left\{\frac{\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w})}{\mathfrak{w}}\right\} > \Re\big(q(1)\big) = (2\varrho - 1) + 2(1 - \varrho)\, ln(2).$$

Theorem 2.10. Let q be a $convex$ mapping to the extent that $q(0) = 1\ and\ h$ is the mapping given by the formula

$$h(\mathfrak{w}) = q(\mathfrak{w}) + \mathfrak{w}q'(\mathfrak{w}), \quad \mathfrak{w} \in \mathbb{D}.$$

If $l \in A$ with confirms $the\ differential\ subordination$

$$\left\{\frac{\mathfrak{w}\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w})}{\mathscr{I}_{\lambda,\mu}^{s} \mathcal{N}(\mathfrak{w})}\right\}' \prec h(\mathfrak{w}), \quad \mathfrak{w} \in \mathbb{D}, \qquad (2.22)$$

Then

$$\frac{\mathscr{I}_{\lambda,\mu}^{s} l(\mathfrak{w})}{\mathscr{I}_{\lambda,\mu}^{s} \mathcal{N}(\mathfrak{w})} \prec q(\mathfrak{w}), \quad \mathfrak{w} \in \mathbb{D},$$

with this the outcome is pointed.

Proof: For function $l \in A$, given by equation (1.1), we get

$$\mathscr{J}_{\lambda,\mu}^{s}\mathcal{N}(\mathfrak{w}) = \mathfrak{w} + \sum_{k=2}^{\infty} \varphi_{\lambda}^{k}(\mu, s)\frac{\wp + 1}{k + \wp} a_k b_k \mathfrak{w}^k, \quad \mathfrak{w} \in \mathbb{D}.$$

We now consider the function

$$p(\mathfrak{w}) = \frac{\mathscr{J}_{\lambda,\mu}^{s} l(\mathfrak{w})}{\mathscr{J}_{\lambda,\mu}^{s}\mathcal{N}(\mathfrak{w})} = \frac{\mathfrak{w} + \sum_{k=2}^{\infty} \varphi_{\lambda}^{k}(\mu, s)\, a_k b_k \mathfrak{w}^k}{\mathfrak{w} + \sum_{k=2}^{\infty} \varphi_{\lambda}^{k}(\mu, s)\frac{\wp + 1}{k + \wp} a_k b_k \mathfrak{w}^k}$$

$$= \frac{1 + \sum_{k=2}^{\infty} \varphi_{\lambda}^{k}(\mu, s)\, a_k b_k \mathfrak{w}^{k-1}}{1 + \sum_{k=2}^{\infty} \varphi_{\lambda}^{k}(\mu, s)\frac{\wp + 1}{k + \wp} a_k b_k \mathfrak{w}^{k-1}}$$

This instance, we obtain

$$(p(\mathfrak{w}))' = \frac{\left(\mathscr{J}_{\lambda,\mu}^{s} l(\mathfrak{w})\right)'}{\mathscr{J}_{\lambda,\mu}^{s}\mathcal{N}(\mathfrak{w})} - p(\mathfrak{w})\frac{\left(\mathscr{J}_{\lambda,\mu}^{s}\mathcal{N}(\mathfrak{w})\right)'}{\mathscr{J}_{\lambda,\mu}^{s}\mathcal{N}(\mathfrak{w})}.$$

Then

$$p(\mathfrak{w}) + \mathfrak{w}p'(\mathfrak{w}) = \left\{\frac{\mathfrak{w}\mathscr{J}_{\lambda,\mu}^{s} l(\mathfrak{w})}{\mathscr{J}_{\lambda,\mu}^{s}\mathcal{N}(\mathfrak{w})}\right\}', \quad \mathfrak{w} \in \mathbb{D}, \qquad (2.23)$$

By using the relation (2.23) in inequality (2.22), we achieve

$$p(\mathfrak{w}) + \mathfrak{w}p'(\mathfrak{w}) \prec h(\mathfrak{w}) = q(\mathfrak{w}) + \mathfrak{w}q'(\mathfrak{w})$$

and, by considering Lemma 1.4,

$$p(\mathfrak{w}) \prec q(\mathfrak{w}),$$

i.e.,

$$\frac{\mathscr{J}_{\lambda,\mu}^{s} l(\mathfrak{w})}{\mathscr{J}_{\lambda,\mu}^{s}\mathcal{N}(\mathfrak{w})} \prec q(\mathfrak{w}).$$

Hence the theorem is proved.

3. Conclusion

In the study carried out, a new operator was described via Subordination. By using this operator, a new class was presented and studied. For the further studies, it is planned to describe novel subclass of m-fold symmetric bi-univalent mappings. It is also planned to present upper bounds for initial Taylor coefficients, Fekete-Szego and Hankel determinant inequalities for functions in the defined classes.

By the following neutrosophic logic, we expect that our work will provide a stepping stone to the study of various kinds of probability distributions. It is conceivable to investigate the Hankel determinant for this distribution in the future. Numerous branches of science, technology and mathematics are anticipated to benefit from the application of Caputo's derivative operator.

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