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Applications of Generation of ZZ Transform for Solving Linear Volterra Integral Equations of Second Type

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Abstract: In this paper, we presented the applications of the generalization of ZZ transform for finding the exact solution to Volterra integral equations of the second type. The convolution theorem for this transform was demonstrated and applied in three examples. It was shown that this transform facilitated the mathematical operations in finding the exact solution.

Keywords: Integral Equations, ZZ transform, Inverse ZZ transform, Generalization of ZZ transform, Inverse of generalization of ZZ transform, Convolution theorem.

1. Introduction

Many scientists in various fields of science, engineering and technology face problems in obtaining accurate and precise solutions, which led many of them to resort to using integral transforms, which are considered one of the simplest, most useful and fastest techniques at the present time, starting from Laplace and Fourier transforms and reaching new improved and developed types of these transforms, for example, Mahgoub, Kamal, ElZaki, Sumudu, SEE, complex SEE, Emad-Sara ,AL-zughair ,KAJ ,AL-Tememe , Sadik ,and complex Sadik transforms... etc. [5-11,13-15]. Which were used to solve real and realistic problems such as the population growth equation, the control problem of storekeepers [1], and solving the telegraph problem [3]. Also, the mathematical models found in health sciences were solved using Soham [4]...etc. Therefore, we can say that using integral transforms to solve problems of great importance in many fields enables us to reach an accurate solution in record time and without much effort.

In this paper we will discuss the definition of the integral equation, the ZZ transform, the generalization of the ZZ transform, the convolution theorem for the generalization of the ZZ transform, and apply of the generalization of this transform in finding the exact solution of Volterra integral equations of second type.

Definition (1.1) [16]: The integral equation which is an equation that the unknown function exists under the integral sign such that:

$$f(x) = g(x) + \lambda \int_{n(x)}^{r(x)} M(x,t) f(t)\, dt. \qquad (1.1)$$

Where:

is the unknown function. f(x)

g(x) is given function called the driving term.

Is a constant parameter. λ

M(x,t)) is the kernel of integral equation and it’s the known function of two variables x and t

r(x),n(x) are the limits of integration.

Definition (1.2) [2]: If f(t) be a function definition for all t ≥0, then the ZZ transform of this function is given by:

$$z(u,s) = H\{f(t)\} = s \int_0^{\infty} f(ut)\ e^{-st} dt\ . \qquad (1.2)$$

Where z(u,s) is the ZZ transform of the function f(t) and

$f(t) = H^{-1}\{z(u,s)\}$; $H^{-1}\{z(u,s)\}$is the inverse transform of z(u,s).

Definition (1.3) [12]: let k(t) be a function for all t≥0, the general ZZ transform of k(t) is given by:

$$H_g\{k(t)\} = q(s,v) \int_{t=0}^{\infty} e^{-p(s,v)t}\, k(t)\, dt. \qquad (1.3)$$

$$C = \{k(t): \exists\, N, S_1, S_2\ > 0\,, |k(t)|\ > N\, e^{sj|t|}, t \in (-1)^{j-1} \times (-\infty, 0], j = 1,2$$

Such that N is arbitrary constant must be a finite and S_1, S_2 can be finite or infinite constant, while p and q are functions of parameters s and v.

Table (1.1) [12]: Generalization of ZZ transform of some elementary functions:

No.	Function $f(t) = H_g^{-1}\{k(t)\}$	$H_g\{k(t)\}$
1	k , where k is a constant	$\dfrac{k\, q(s,v)}{p(s,v)}\ , p(s,v) \neq 0$
2	t	$\dfrac{q(s,v)}{(p(s,v))^2}\ ; p(s,v) \neq 0$

3	t^n	$\frac{n!\,q(s,v)}{\left(p(s,v)\right)^{n+1}}$; $p(s,v) \neq 0$, $n \in z^+$
4	e^{at}, where a is a constant	$\frac{q(s,v)}{p(s,v)-a}$; $p(s,v) > a$
5	$\cos(at)$, where a is a constant	$\frac{p(s,v).\,q(s,v)}{[p(s,v)]^2+a^2}$
6	$\sin(at)$, where a is a constant	$\frac{a\,q(s,v)}{(p(s,v))^2+a^2}$
7	$\sinh(at)$, where a is a constant	$\frac{a\,q(s,v)}{(p(s,v))^2-a^2}$, $p(s,v) > a$
8	$\cosh(at)$, where a is a constant	$\frac{p(s,v).\,q(s,v)}{[p(s,v)]^2-a^2}$; $p(s,v) > a$

2. Convolution Theorem of Generalization of ZZ Transform:

Let $k_1(t)$and $k_2(t)$ have new integral transform $H_1(s,v)$ and $H_2(s,v)$Then the general integral transform of the convolution of k_1 and k_2 is:

$$H\{k_1(t) * k_2(t)\} = g(s,v)\int_0^{\infty} e^{-p(s,v)t}\,(k_1(t) * k_2(t))dt$$

$$= g(s,v)\int_{t=0}^{\infty} e^{-p(s,v)t}\,dt\,\int_{\tau=0}^{t} k_1(t-\tau)k_2(\tau)d\tau\,,$$

$$= g(s,v)\,\int_{\tau=0}^{\infty} k_2(\tau)\,d\tau\,\int_{t=\tau}^{\infty} e^{-p(s,v)t}\,k_1(t-\tau)dt.$$

This is by the change of variable $t-\tau = x$, $dt = dx$.

$$H\{k_1(t) * k_2(t)\} = g(s,v)\int_{\tau=0}^{\infty} k_2(\tau)d\tau\,\int_{x=0}^{\infty} e^{-p(s,v)\,(x+\tau)}\,k_1(x)dx,$$

$$= g(s,v)\int_{\tau=0}^{\infty} e^{-p(s,v)\tau}\,k_2(\tau)\,d\tau\,.\,\int_{x=0}^{\infty} e^{-p(s,v)x}\,k_1(x)dx\,,$$

$$= H_2\big(p(s,v), g(s,v)\big).\frac{g(s,v)}{g(s,v)}\int_{x=0}^{\infty} e^{-p(s,v)x}\,k_1(x)\,dx,$$

$$= H_2\,\big(p(s,v), g(s,v)\big).\frac{1}{g(s,v)}\,H_1(p(s,v), g(s,v))$$

$$= \frac{1}{g(s,v)}\,H_2\,.\,H_1$$

Then $H\,\{k_1(t) * k_2(t)\} = \frac{1}{g(s,v)}\,H_2 H_1$.

Where $H_1 = H\{k_1(t)\}$ and $H_2 = H\{k_2(t)\}$

3. Applications for Solving Volttera Integral Equations Using Generalization of ZZ Integral TransformObjectives

Example (3.1): Solve $\varphi(y) = \cos y + \int_0^y \sin(y-t)\ \varphi(t)\ dt$.

By taking a generalization of the ZZ transform to both sides of equation:

$$H\{\varphi(y)\} = H\{\cos y\} + H\{\int_0^y \sin(y-t)\varphi(t)\ dt\},$$

$$\varphi\big(p(s,v),q(s,v)\big) = \frac{p(s,v).q(s,v)}{[p(s,v)]^2+1} + \frac{q(s,v)}{(p(s,v))^2+1}.\frac{1}{q(s,v)}\ .\varphi\big(p(s,v),q(s,v)\big),$$

$$= \frac{p(s,v)q(s,v)}{[p(s,v)]^2+1} + \frac{1}{(p(s,v))^2+1}\ .\varphi\big(p(s,v),q(s,v)\big),$$

$$\varphi\big(p(s,v),q(s,v)\big)\,[p(s,v)]^2 = p(s,v)q(s,v).$$

Then $\varphi\big(p(s,v),q(s,v)\big) = \frac{q(s,v)}{p(s,v)}\ , p(s,v) \neq 0.$

$$H^{-1}\big[\varphi\big(p(s,v),q(s,v)\big)\big] = H^{-1}\left[\frac{q(s,v)}{p(s,v)}\right].$$

So $\varphi(y) = 1.$

Example (3.2): Solve $y(t) = 1 - \int_0^x (t-\tau)y(t)\ dt$.

By taking a generalization of the ZZ transform to both sides of equation:

$$H\{y(t)\} = H\{1 - \int_0^x (t-\tau)y(t)\ dt\},$$

$$y\big(p(s,v),q(s,v)\big) = \frac{q(s,v)}{p(s,v)} - \frac{q(s,v)}{\big(p(s,v)\big)^2}\ .\frac{1}{q(s,v)}\ y\big(p(s,v),q(s,v)\big),$$

$$y\big(p(s,v),q(s,v)\big)\left(\frac{\big(p(s,v)\big)^2+1}{\big(p(s,v)\big)^2}\right) = \frac{q(s,v)}{p(s,v)},$$

$$y\big(p(s,v),q(s,v)\big) = \frac{q(s,v)p(s,v)}{(p(s,v))^2+1},$$

$$H^{-1}[y(p(s,v),q(s,v)] = H^{-1}[\frac{q(s,v)p(s,v)}{\big(p(s,v)\big)^2+1}]$$

$So \quad y(t) = \cos t$

Example (3.3): Solve $\varphi(y) = y + \int_0^y \sin(y-t)\ \varphi(t)\ dt$.

By taking a generalization of the ZZ transform to both sides of above equation and the inverse of this transform, we get:

$$\varphi(y) = y + \frac{1}{6}y^3.$$

4. Conclusion:

In this article, applications of the generalization of ZZ transform technique were presented in finding exact solutions to the Volterra integral equations of second type using the convolution theorem for this technique .It was shown through examples that this transform is simpler and easier to perform mathematical operations and that it is more general than many two parameter integral transforms

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